Truncated Moment Problems: An Introductory Survey

(based on joint work with L.A. Fialkow, S.H. Lee, H.M. Möller, S. Yoo and J. Yoon)

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Overview

Introd.: The Classical Fibonacci Sequence

Consider the classical Fibonacci sequence

$$1, 1, 2, 3, 5, 8, 11, 19, \cdots$$

and the need to represent it concisely. If we let $\{a_n\}_{n\geq 0}$ denote this sequence, we know that

$$a_{n+2} = a_{n+1} + a_n, \quad \text{with } a_0 = 1 \quad \text{and } a_1 = 1.$$

We can organize this matricially as follows:

$$H_a := \begin{pmatrix} 1 & 1 & 2 & 3 & \cdots \\ 1 & 2 & 3 & 5 & \cdots \\ 2 & 3 & 5 & 8 & \cdots \\ 3 & 5 & 8 & 11 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

If we label the columns $1, S, S^2, S^3, \cdots$, we can represent the 2-step recursion as

$$S^2 = S + 1.$$

One can then consider the polynomial $g \in \mathbb{C}[s]$ given by

$$g(s) := s^2 - (s + 1),$$

whose zeros are $s_0 = \frac{1-\sqrt{5}}{2} \cong -0.618$ and $s_1 = \frac{1+\sqrt{5}}{2} \cong 1.618$ and satisfy the equations $s_0 + s_1 = -1,\ s_0 s_1 = -1$. We now define a linear functional on the space of polynomials, given as

$$L_a(p) := \rho_0 \delta_{s_0} + \rho_1 \delta_{s_1} \ \ (p \in \mathbb{C}[s]),$$

where $\rho_0, \rho_1 \in \mathbb{R}$ and δ_z denotes the evaluation at z. We wish L_a to *represent* a. This requires $L_a(s^n) = a_n$ $(n \geq 0)$, that is,

$$
\begin{array}{ccc}
L_a(1) & = & a_0 \\
L_a(s) & = & a_1 \\
L_a(s^2) & = & a_2 \\
\cdots & \cdots & \cdots \\
L_a(s^n) & = & a_n \\
\cdots & \cdots & \cdots \,.
\end{array}
$$

In particular,

$$\rho_0 + \rho_1 = a_0$$

and

$$\rho_0 s_0 + \rho_1 s_1 = a_1.$$

Then

$$\rho_0 + \rho_1 = 1$$

and

$$\rho_0 s_0 + \rho_1 s_1 = 1.$$

It follows that

$$\rho_0 = \frac{5-\sqrt{5}}{10} \cong 0.276$$

and

$$\rho_1 = \frac{5+\sqrt{5}}{10} \cong 0.724.$$

Thus,

$$L_a(p) = \rho_0 p(s_0) + \rho_1 p(s_1) \ (p \in \mathbb{C}[s]).$$

This can also be interpreted as integration of p with respect to the positive 2-atomic Borel measure

$$\begin{aligned} \mu \quad &:= \qquad \rho_0\delta_0 + \rho_1\delta_1 \\ &= \quad \tfrac{5-\sqrt{5}}{10}\delta_{s_0} + \tfrac{5+\sqrt{5}}{10}\delta_{s_1}. \end{aligned}$$

As a result, the Hankel matrix H_a, thought as an operator on the Hilbert space $\ell^2(\mathbb{Z}_+)$, has μ as spectral measure; this is also the spectral measure of the operator M_s of multiplication by the independent variable in the space $L^2(\mu)$. When the initial sequence corresponds to the moments of the weight sequence of a subnormal unilateral weighted shift W acting on $\ell^2(\mathbb{Z}_+)$, the measure μ is also the Berger measure of W. This is not the case of the Fibonacci sequence (a_n), because the resulting unilateral weighted shift is not even hyponormal, much less subnormal.

The expressions $\int s^n \, d\mu$ $(n \geq 0)$ are the *moments* of μ. For every $n \geq 0$, the matrix

$$H_a(n) := \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \cdots & a_n \\ a_1 & a_2 & a_3 & a_4 & \cdots & a_{n+1} \\ a_2 & a_3 & a_4 & a_5 & \cdots & a_{n+2} \\ a_3 & a_4 & a_5 & a_6 & \cdots & a_{n+3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ a_n & a_{n+1} & a_{n+2} & a_{n+3} & \cdots & a_{2n} \end{pmatrix}.$$

is called the *moment matrix* for the finite collection $a_0, \cdots, a_{2n}$. It is not hard to see that $H_a(n) \geq 0$ $(n \geq 0)$ (in the Hilbert space sense) if and only if $L_a \geq 0$, that is, $L_a(p) \geq 0$ for all $p \geq 0$.

Introduction: Truncated Hankel Matrices

The matrices $H_a(n)$ ($n \geq 0$) are the truncated matrices of H_a. In view of the 2-step recursive relation

$$S^2 = S + 1,$$

we have

$$\begin{aligned} \operatorname{rank} H_a(0) &= 1 \\ \operatorname{rank} H_a(1) &= 2 \\ \operatorname{rank} H_a(2) &= 2 \end{aligned}$$

and

$$\operatorname{rank} H_a(n) = 2 \text{ (all } n \geq 3).$$

We will say that $H_a(2)$ is a *flat extension* of $H_a(1)$. Also, H_a is a flat extension of $H_a(1)$.

More generally, if A and M are positive semidefinite matrices such that

$$M \equiv \begin{pmatrix} A & B \\ B^* & C \end{pmatrix},$$

and $\operatorname{rank} M = \operatorname{rank} A$, we will say that M is a flat extension of A.

Also, if Q is an infinite square matrix, and Q_n are its finite truncations of size n, it is true that

$$Q \geq 0 \Longrightarrow \det Q_n \geq 0 \ \ (\text{all } n \geq 0).$$

However, the converse is false.

In joint work with Lawrence A. Fialkow (SUNY at New Paltz), several years ago we initiated the study of truncated moment problems in one or several real or complex variables. A central result in the theory is the so-called Flat Extension Theorem. In this talk we plan to discuss this result, and some applications to numerical analysis (quadratures) will be presented. Motivated by the Fibonacci example, We use the support of a representing measure for this, and this is the common zero set of one or more polynomials. As in the case of quadratures, one needs to allow for non-positive densities, while keeping everything within the real numbers. Solution of TMP involves finding properties of structured matrices that are necessary and sufficient conditions for the existence of representing measures.

Introduction: Numerical Integration

A) Low-order polynomial approx. on subintervals of decreasing size

Commonly used Newton-Cotes formulas

T $\quad n=1 \quad \int_a^b f(x)\,dx = \frac{h}{2}[f(a)+f(b)] - \frac{h^3}{12} f''(\xi)$

S $\quad n=2 \quad \int_a^b f(x)\,dx = \frac{h}{3}[f(a)+4f(\frac{a+b}{2})+f(b)] - \frac{h^5}{90} f^{(4)}(\xi)$

$\frac{3}{8} \quad n=3 \quad \int_a^b f(x)\,dx = \begin{cases} \frac{3h}{8}[f(a)+3f(a+h)+3f(b-h)+f(b)] \\ -\frac{3h^5}{80} f^{(4)}(\xi) \end{cases}$

$n=4 \quad \int_a^b f(x)\,dx = \begin{cases} \frac{2h}{45}[7f(a)+32f(a+h)+12f(\frac{a+b}{2}) \\ +32f(b-h)+7f(b)] - \frac{8h^7}{945} f^{(6)}(\xi) \end{cases}$

B) Polynomial approximation of increasing degree, using fewer, strategically-placed nodes

Definition

A quadrature (or cubature) rule of size p and precision m is a numerical integration formula which uses p nodes, is exact for all polynomials of degree at most m, and fails to recover the integral of some polynomial of degree $m+1$.

Gaussian Quadrature (size n, precision $2n-1$)

$\int_{-1}^{1} f(t)\,dt = \sum_{j=0}^{n-1} \rho_j f(t_j^{(n)})$ for every polynomial $f \in \mathbf{R}_{2n-1}[t]$

(Gaussian means minimum number of nodes possible)

Interpolating Equations:

$$\sum_{j=0}^{n-1} \rho_j t_j^k = \int_{-1}^{1} t^k \, dt = \begin{cases} 0 & k = 1, 3, ..., 2n-1 \\ \\ \frac{2}{k+1} & k = 0, 2, ..., 2n-2 \end{cases}$$

Example: $n = 2$

$$\begin{cases} \rho_0 + \rho_1 & = & 2 \\ \rho_0 t_0 + \rho_1 t_1 & = & 0 \\ \rho_0 t_0^2 + \rho_1 t_1^2 & = & \frac{2}{3} \\ \rho_0 t_0^3 + \rho_1 t_1^3 & = & 0 \end{cases}$$

$\rho_0 = \rho_1 = 1;\ t_0 = -\frac{\sqrt{3}}{3},\ t_1 = \frac{\sqrt{3}}{3}.$

$$\int_{-1}^{1} \sum_{k=0}^{3} a_k t^k = \sum_{j=0}^{1} \rho_j \sum_{k=0}^{3} a_k t_j^k$$

NA textbooks prove this by using orthogonal Legendre polynomials

($t_0 < \ldots < t_{n-1}$ are the zeros of the nth Legendre polynomial)

(RC-L. Fialkow, 1990) Can do this as follows:

$\gamma_0 := 2,\ \gamma_1 := 0,\ \gamma_2 := \frac{2}{3},\ \gamma_3 := 0,\ \gamma_4 := \frac{2}{5}$, etc.

Assume n even, and form the Hankel matrix

$$H(n) := \begin{pmatrix} 2 & 0 & \frac{2}{3} & \cdots & 0 & \vdots & \frac{2}{n+1} \\ 0 & \frac{2}{3} & 0 & \cdots & \frac{2}{n+1} & \vdots & 0 \\ \frac{2}{3} & 0 & \frac{2}{5} & \cdots & 0 & \vdots & \frac{2}{n+3} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \cdots \\ 0 & \frac{2}{n+1} & 0 & \cdots & \frac{2}{2n-1} & \vdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \cdots \\ \frac{2}{n+1} & 0 & \frac{2}{n+3} & \cdots & 0 & \vdots & \textbf{NEW MOMENT} \end{pmatrix},$$

label the columns $\mathit{1}, T, T^2, ..., T^n$, require that $T^n = \varphi_0 \mathit{1} + ... + \varphi_{n-1} T^{n-1}$,

build the polynomial $g(t) := t^n - (\varphi_0 + ... + \varphi_{n-1} t^{n-1})$,

(this produces a non-iterative construction of Legendre polynomials) find its zeros ($t_0 < ... < t_{n-1}$),

and

compute the densities using the Vandermonde system

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ t_0 & t_1 & \cdots & t_{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ t_0^{n-1} & t_1^{n-1} & \cdots & t_{n-1}^{n-1} \end{pmatrix} \begin{pmatrix} \rho_0 \\ \rho_1 \\ \cdots \\ \rho_{n-1} \end{pmatrix} = \begin{pmatrix} \gamma_0 \\ \gamma_1 \\ \cdots \\ \gamma_{n-1} \end{pmatrix}.$$

To solve the Gaussian quadrature problem, RC and Fialkow's basic idea was to augment the original Hankel matrix by one row and one column at a time, preserving the rank (which a fortiori preserves positivity):

$$H(n) \prec H(n+1) \prec ...H(\infty)$$

Then define

$$\langle p, q \rangle_{H(\infty)} := (H(\infty)\widehat{p}, \widehat{q})_{\ell_2},$$

and show that

$$\langle p, q \rangle_{H(\infty)} = \int p\bar{q}\, d\mu$$

for some finitely atomic rep. meas., with supp $\mu = \mathcal{Z}(g)$.

The Truncated Real Moment Problem

Given a family of real numbers β: $\beta_0, \beta_1, \ldots, \beta_{2n}$ with $\beta_0 > 0$, the **TMP** entails finding a positive Borel measure μ supported in the real line $\mathbb{R}$ such that

$$\beta_i = \int t^i \, d\mu \qquad (0 \leq i \leq 2n);$$

μ is called a **representing measure** for β.

Theorem

*FULL MP (***Hamburger, 1920***)*

$$\exists \mu \Leftrightarrow A(n) := (\beta_{i+j})_{i,j=0}^{n} \equiv \begin{pmatrix} \beta_0 & \beta_1 & \beta_2 & \beta_3 & \cdots \\ \beta_1 & \beta_2 & \beta_3 & \ddots & \cdots \\ \beta_2 & \beta_3 & \ddots & \ddots & \cdots \\ \beta_3 & \ddots & \ddots & \ddots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \geq 0 \; \forall \quad n \geq 0.$$

Theorem

FULL MP (**Stieltjes, 1894**)

$$\exists \mu \textit{ with } \operatorname{supp} \mu \subseteq [0, +\infty)$$

$$\Leftrightarrow (\beta_{i+j})_{i,j=0}^{n} \geq 0 \textit{ and } (\beta_{i+j+1})_{i,j=0}^{n} \geq 0 \ \forall \ n \geq 0.$$

$$\begin{pmatrix} \beta_0 & \beta_1 & \beta_2 & \beta_3 & \cdots \\ \beta_1 & \beta_2 & \beta_3 & \ddots & \cdots \\ \beta_2 & \beta_3 & \ddots & \ddots & \cdots \\ \beta_3 & \ddots & \ddots & \ddots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \geq 0 \text{ and } \begin{pmatrix} \beta_1 & \beta_2 & \beta_3 & \beta_4 & \cdots \\ \beta_2 & \beta_3 & \beta_4 & \ddots & \cdots \\ \beta_3 & \beta_4 & \ddots & \ddots & \cdots \\ \beta_4 & \ddots & \ddots & \ddots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ & \multicolumn{3}{c}{\textbf{(localizing matrix)}} & \end{pmatrix} \geq 0$$

The positivity of the second matrix guarantees that supp $\mu \subseteq [0, +\infty)$.

The Truncated Complex Moment Problem

- Given $\gamma : \gamma_{00}, \gamma_{01}, \gamma_{10}, \ldots, \gamma_{0,2n}, \ldots, \gamma_{2n,0}$, with $\gamma_{00} > 0$ and $\gamma_{ji} = \bar{\gamma}_{ij}$, the **TCMP** entails finding a positive Borel measure μ supported in the complex plane $\mathbb{C}$ such that

$$\gamma_{ij} = \int \bar{z}^i z^j \, d\mu \qquad (0 \leq i + j \leq 2n);$$

  μ is called a **rep. meas.** for γ.

  In earlier joint work with L. Fialkow,

- We have introduced an approach based on matrix positivity and extension, combined with a new "functional calculus" for the columns of the associated **moment matrix**.

- We have shown that when the TCMP is of **flat data type**, a solution always exists; this is compatible with our previous results for

$$\begin{array}{ll} \text{supp } \mu \subseteq \mathbb{R} & \text{(Hamburger TMP)} \\ \text{supp } \mu \subseteq [0, \infty) & \text{(Stieltjes TMP)} \\ \text{supp } \mu \subseteq [a, b] & \text{(Hausdorff TMP)} \\ \text{supp } \mu \subseteq \mathbb{T} & \text{(Toeplitz TMP)} \end{array}$$

- Along the way we have developed new machinery for analyzing TMP's in **one or several real or complex variables**. For simplicity, in this talk we focus on **one complex variable or two real variables**, although several results have multivariable versions.

- Our techniques also give concrete algorithms to provide finitely-atomic rep. meas. whose atoms and densities can be explicitly computed.
- We obtain applications to quadrature problems in numerical analysis.
- We have obtained a duality proof of a generalized form of the Tchakaloff-Putinar Theorem on the existence of quadrature rules for positive Borel measures on $\mathbb{R}^d$.

Some Applications

- Subnormal Operator Theory (unilateral weighted shifts) (subnormal means the restriction of a normal operator to an invariant subspace.)

  For $\alpha_0 \le \alpha_1 \le \alpha_2 \le \cdots$, the weighted shift W_α is subnormal if and only if the moment problem $\alpha_0^2 \alpha_1^2 \cdots \alpha_{k-1}^2 = \int s^k d\mu(s)$ is soluble.

- Physics (determination of contours, QM, QFT)

- Computer Science (image recognition and reconstruction)

- Geography (location of proposed distribution centers)

- Probability (reconstruction of p.d.f.'s)

- Environmental Science (oil spills, via quadrature domains)
- Engineering (tomography)
- Optimization (finding the global minimum of a real polynomial in several real variables - J. Lasserre)
- Function Theory (a dilation-type structure theorem in Fejér-Riesz factorization theory - S. McCullough)
- Geophysics (inverse problems, cross sections)

Typical Problem: Given a 3-D body, let X-rays act on the body at different angles, collecting the information on a screen. One then seeks to obtain a constructive, optimal way to approximate the body, or in some cases to reconstruct the body.

Basic Positivity Condition

$\mathcal{P}_n$: polynomials p in z and $\overline{z}$, deg $p \leq n$

Given $p \in \mathcal{P}_n$, $\;p(z, \overline{z}) \equiv \sum_{0 \leq i+j \leq n} a_{ij} \overline{z}^i z^j$,

$$0 \leq \int \mid p(z, \overline{z}) \mid^2 d\mu(z, \overline{z})$$

$$= \sum_{ijk\ell} a_{ij} \bar{a}_{k\ell} \int \overline{z}^{i+\ell} z^{j+k} d\mu(z, \overline{z})$$

$$= \sum_{ijk\ell} a_{ij} \bar{a}_{k\ell} \gamma_{i+\ell, j+k}.$$

- To understand this **"matricial" positivity**, we introduce the following lexicographic order on the rows and columns of $M(n)$:

$$1, Z, \bar{Z}, Z^2, \bar{Z}Z, \bar{Z}^2, \ldots$$

Define $M[i,j]$ as in

$$M[3,2] := \begin{pmatrix} \gamma_{32} & \gamma_{41} & \gamma_{50} \\ \gamma_{23} & \gamma_{32} & \gamma_{41} \\ \gamma_{14} & \gamma_{23} & \gamma_{32} \\ \gamma_{05} & \gamma_{14} & \gamma_{23} \end{pmatrix}$$

Then

$$(\textbf{“matricial” positivity}) \ \ \sum_{ijk\ell} a_{ij} \bar{a}_{k\ell} \gamma_{i+\ell, j+k} \geq 0$$

$$\Leftrightarrow M(n) \equiv M(n)(\gamma) := \begin{pmatrix} M[0,0] & M[0,1] & ... & M[0,n] \\ M[1,0] & M[1,1] & ... & M[1,n] \\ ... & ... & ... & \dots \\ M[n,0] & M[n,1] & \dots & M[n,n] \end{pmatrix} \geq 0.$$

For example,

- $$M(1) = \begin{pmatrix} \gamma_{00} & \gamma_{01} & \gamma_{10} \\ \gamma_{10} & \gamma_{11} & \gamma_{20} \\ \gamma_{01} & \gamma_{02} & \gamma_{11} \end{pmatrix},$$

- $$M(2) = \begin{pmatrix} \gamma_{00} & \gamma_{01} & \gamma_{10} & \gamma_{02} & \gamma_{11} & \gamma_{20} \\ \gamma_{10} & \gamma_{11} & \gamma_{20} & \gamma_{12} & \gamma_{21} & \gamma_{30} \\ \gamma_{01} & \gamma_{02} & \gamma_{11} & \gamma_{03} & \gamma_{12} & \gamma_{21} \\ \gamma_{20} & \gamma_{21} & \gamma_{12} & \gamma_{22} & \gamma_{31} & \gamma_{40} \\ \gamma_{11} & \gamma_{12} & \gamma_{21} & \gamma_{13} & \gamma_{22} & \gamma_{31} \\ \gamma_{02} & \gamma_{03} & \gamma_{12} & \gamma_{04} & \gamma_{13} & \gamma_{22} \end{pmatrix}.$$

In general,

$$M(n+1) = \begin{pmatrix} M(n) & B \\ B^* & C \end{pmatrix}$$

Similarly, one can build $M(\infty)$.

Positivity Condition is not sufficient:

By modifying an example of K. Schmüdgen, we have built a family $\gamma_{00}, \gamma_{01}, \gamma_{10}, ..., \gamma_{06}, ..., \gamma_{60}$ with positive invertible moment matrix $M(3)$ but **no** rep. meas. But this can also be done for $n = 2$.

For the Real TMP, given $\beta : \beta_{00}, \beta_{01}, \beta_{10}, \cdots, \beta_{0,2n}, \cdots, \beta_{2n,0}$, with $\beta_{00} > 0$, we seek a positive Borel measure μ supported in $\mathbb{R}^2$. In this case, we let

$$\mathcal{M}(n)_{ij} := \gamma_{i+j}, \ \ i, j \in \mathbb{Z}_+^2.$$

The TCMP and TRMP are structurally equivalent, meaning that there is a bijection linking TCMP in d variables with TRMP in $2d$ variables, via the map $z \equiv x + iy$. Moreover, it is possible to modify a TRMP and obtain an equivalent TRMP using degree-one transformations of the form

$$\varphi(x, y) := (ax + by + e, cx + dy + f),$$

where $ad - bc \neq 0$.

For moment problems in $\mathbb{C}$,

$$M(3) = \begin{array}{c}
\begin{array}{ccccccccccc} 1 & Z & \bar{Z} & Z^2 & \bar{Z}Z & \bar{Z}^2 & \vdots & Z^3 & \bar{Z}Z^2 & \bar{Z}^2Z & \bar{Z}^3 \end{array} \\
\begin{pmatrix}
\gamma_{00} & \gamma_{01} & \gamma_{10} & \gamma_{02} & \gamma_{11} & \gamma_{20} & \vdots & \gamma_{03} & \gamma_{12} & \gamma_{21} & \gamma_{30} \\
\gamma_{10} & \gamma_{11} & \gamma_{20} & \gamma_{12} & \gamma_{21} & \gamma_{30} & \vdots & \gamma_{13} & \gamma_{22} & \gamma_{31} & \gamma_{40} \\
\gamma_{01} & \gamma_{02} & \gamma_{11} & \gamma_{03} & \gamma_{12} & \gamma_{21} & \vdots & \gamma_{04} & \gamma_{13} & \gamma_{22} & \gamma_{31} \\
\gamma_{20} & \gamma_{21} & \gamma_{30} & \gamma_{22} & \gamma_{31} & \gamma_{40} & \vdots & \gamma_{23} & \gamma_{32} & \gamma_{41} & \gamma_{50} \\
\gamma_{11} & \gamma_{12} & \gamma_{21} & \gamma_{13} & \gamma_{22} & \gamma_{31} & \vdots & \gamma_{14} & \gamma_{23} & \gamma_{32} & \gamma_{41} \\
\gamma_{02} & \gamma_{03} & \gamma_{12} & \gamma_{04} & \gamma_{13} & \gamma_{22} & \vdots & \gamma_{05} & \gamma_{14} & \gamma_{23} & \gamma_{32} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\gamma_{30} & \gamma_{31} & \gamma_{40} & \gamma_{32} & \gamma_{41} & \gamma_{50} & \vdots & \gamma_{33} & \gamma_{42} & \gamma_{51} & \gamma_{60} \\
\gamma_{21} & \gamma_{22} & \gamma_{31} & \gamma_{23} & \gamma_{32} & \gamma_{41} & \vdots & \gamma_{24} & \gamma_{33} & \gamma_{42} & \gamma_{51} \\
\gamma_{12} & \gamma_{13} & \gamma_{22} & \gamma_{14} & \gamma_{23} & \gamma_{32} & \vdots & \gamma_{15} & \gamma_{24} & \gamma_{33} & \gamma_{42} \\
\gamma_{03} & \gamma_{04} & \gamma_{13} & \gamma_{05} & \gamma_{14} & \gamma_{23} & \vdots & \gamma_{06} & \gamma_{15} & \gamma_{24} & \gamma_{33}
\end{pmatrix}
\end{array}.$$

For moment problems in $\mathbb{R}^2$, the moment matrix $\mathcal{M}(3)$ is given by

$$\begin{array}{cc}
 & \begin{array}{ccccccccccc} 1 & X & Y & X^2 & XY & Y^2 & \vdots & X^3 & X^2Y & XY^2 & Y^3 \end{array} \\
\begin{array}{c} 1 \\ X \\ Y \\ X^2 \\ XY \\ Y^2 \\ \cdots \\ X^3 \\ X^2Y \\ XY^2 \\ Y^3 \end{array} &
\begin{pmatrix}
\beta_{00} & \beta_{01} & \beta_{10} & \beta_{02} & \beta_{11} & \beta_{20} & \vdots & \beta_{03} & \beta_{12} & \beta_{21} & \beta_{30} \\
\beta_{01} & \beta_{02} & \beta_{11} & \beta_{03} & \beta_{12} & \beta_{21} & \vdots & \beta_{04} & \beta_{13} & \beta_{22} & \beta_{31} \\
\beta_{10} & \beta_{11} & \beta_{20} & \beta_{12} & \beta_{21} & \beta_{30} & \vdots & \beta_{13} & \beta_{22} & \beta_{31} & \beta_{40} \\
\beta_{02} & \beta_{03} & \beta_{12} & \beta_{04} & \beta_{13} & \beta_{22} & \vdots & \beta_{05} & \beta_{14} & \beta_{23} & \beta_{32} \\
\beta_{11} & \beta_{12} & \beta_{21} & \beta_{13} & \beta_{22} & \beta_{31} & \vdots & \beta_{14} & \beta_{23} & \beta_{32} & \beta_{41} \\
\beta_{20} & \beta_{21} & \beta_{30} & \beta_{22} & \beta_{31} & \beta_{40} & \vdots & \beta_{23} & \beta_{32} & \beta_{41} & \beta_{50} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\beta_{03} & \beta_{04} & \beta_{13} & \beta_{05} & \beta_{14} & \beta_{23} & \vdots & \beta_{06} & \beta_{15} & \beta_{24} & \beta_{33} \\
\beta_{12} & \beta_{13} & \beta_{22} & \beta_{14} & \beta_{23} & \beta_{32} & \vdots & \beta_{15} & \beta_{24} & \beta_{33} & \beta_{42} \\
\beta_{21} & \beta_{22} & \beta_{31} & \beta_{23} & \beta_{32} & \beta_{41} & \vdots & \beta_{24} & \beta_{33} & \beta_{42} & \beta_{51} \\
\beta_{30} & \beta_{31} & \beta_{40} & \beta_{32} & \beta_{41} & \beta_{50} & \vdots & \beta_{33} & \beta_{42} & \beta_{51} & \beta_{60}
\end{pmatrix}
\end{array}.$$

Moment Problems and Nonnegative Polynomials (FULL MP Case)

- $\mathcal{M} := \{\gamma \equiv \gamma^{(\infty)} : \gamma$ admits a rep. meas. $\mu\}$
- $\mathcal{P}_+$: nonnegative poly's

Duality

For C a cone in $\mathbb{R}^{\mathbb{Z}_+^2}$, we let

$$C^* := \{\xi \in \mathbb{R}^{\mathbb{Z}_+^2} : \text{supp}(\xi) \text{ is finite and } \langle p, \xi \rangle \geq 0 \text{ for all } p \in C\}.$$

- (Riesz-Haviland) $\mathcal{P}_+^* = \mathcal{M}$

  For, consider the Riesz functional $\Lambda_\gamma(p) := p(\gamma) \equiv \langle p, \gamma \rangle$, which induces a map $\mathcal{M} \to \mathcal{P}_+^*$ $(\gamma \mapsto \Lambda_\gamma)$; Haviland's Theorem says that this maps is onto, that is, there exists μ r.m. for γ if and only if $\Lambda_\gamma \geq 0$ on $\mathcal{P}_+$.

There exists a version of Riesz-Haviland for TMP, as we will see shortly.

The link between TMP and FMP is provided by another result of Stochel (2001):

Theorem

$\beta^{(\infty)}$ has a rep. meas. supported in a closed set $K \subseteq \mathbb{R}^2$ if and only if, for each n, $\beta^{(2n)}$ has a rep. meas. supported in K.

Positivity of Block Matrices

Theorem

(Smul'jan, 1959)

$$\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \geq 0 \Leftrightarrow \begin{cases} A \geq 0 \\ B = AW \\ C \geq W^*AW \end{cases}.$$

Moreover, $\operatorname{rank} \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} = \operatorname{rank} A \Leftrightarrow C = W^*AW.$

Corollary

Assume $\operatorname{rank} \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} = \operatorname{rank} A$. *Then*

$$A \geq 0 \Leftrightarrow \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \geq 0.$$

We say that

$$\begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$$

is a flat extension of A. Observe that

$$\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} = \begin{pmatrix} A & AW \\ W^*A & W^*AW \end{pmatrix}.$$

Corollary

Assume that

$$\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \geq 0.$$

Then

$$\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} = \begin{pmatrix} A & AW \\ W^*A & W^*AW \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & C - W^*AW \end{pmatrix}$$

Schur complement $\nearrow$

$$= \begin{pmatrix} \sqrt{A} & \sqrt{A}W \end{pmatrix}^* \begin{pmatrix} \sqrt{A} & \sqrt{A}W \end{pmatrix} + \begin{pmatrix} 0 & \sqrt{C - W^*AW} \end{pmatrix}^* \begin{pmatrix} 0 & \sqrt{C - W^*AW} \end{pmatrix}$$

(sum-of-squares representation).

Functional Calculus

For $p \in \mathcal{P}_n$, $p(z, \bar{z}) \equiv \sum_{0 \le i+j \le n} a_{ij} \bar{z}^i z^j$, let $\widehat{p}$ denote the vector of coefficients and define

$$p(Z, \bar{Z}) := \sum a_{ij} \bar{Z}^i Z^j \equiv M(n)\widehat{p}.$$

If there exists a rep. meas. μ, then

$$p(Z, \bar{Z}) = 0 \Leftrightarrow \text{supp } \mu \subseteq \mathcal{Z}(p).$$

The following is our analogue of recursiveness for the TCMP

(Recursiveness) If $p, q, pq \in \mathcal{P}_n$, **and** $p(Z, \bar{Z}) = 0$,

then $(pq)(Z, \bar{Z}) = 0.$

Singular TMP; Real Case

- Given a finite family of moments, build the relevant moment matrix.
- Label the columns, $1, X, Y, X^2, XY, Y^2, \cdots$.
- Identify column relations, as $p(X, Y) = 0$.
- Observe that $p(X, Y) = 0$ is equivalent to $\mathcal{M}(n)\widehat{p} = 0$.
- Build algebraic variety

$\mathcal{V} := \bigcap_{p \in \mathcal{P}_n,\ \widehat{p} \in \ker \mathcal{M}(n)} \ \mathcal{Z}_p.$

- Always true: in the presence of a measure,

$\operatorname{supp} \ \mu \subseteq \mathcal{V}.$

Therefore,

$$r := \operatorname{rank} \mathcal{M}(n) \leq \text{card supp } \mu \leq v := \operatorname{card} \mathcal{V}.$$

It follows that if $r > v$ then $\mathcal{M}(n)$ has no representing measure.

If the variety is finite there's a natural candidate for supp μ, i.e.,

supp $\mu = \mathcal{V}$

(**However, it is possible for the inclusion** supp $\mu \subseteq \mathcal{V}$ **to be proper**.)

A new notion, of core variety $\mathcal{V}_{core}$, has recently been introduced by G. Blekherman and L. Fialkow. When the TMP is soluble, supp $\mu = \mathcal{V}_{core}$.

General Strategy for Solving the Bivariate Truncated Moment Problem

	Invertible $M(n)$	**Singular** $M(n)$
$n = 1$	$r = 3$; there exists a flat extension $M(2)$.	$r \leq 2$; there exists a flat extension $M(2)$.
$n = 2$	$r = 6$; there exists a flat extension $M(3)$.	$r \leq 5$; for $r \leq 4$, there exists a flat extension $M(3)$; for $r = 5$, there exists a measure μ with $\operatorname{card}\operatorname{supp}\mu \leq 6$.

	Invertible $M(n)$	**Singular** $M(n)$
$n=3$	$r=10$; there exists $M(3)$ with no representing measure.	$r \leq 9$; need to distinguish between finite and infinite algebraic varieties.
$n=4$	$r=15$; open problem	partial results are known
$n=5$	$r=21$; open problem. there exists $M(5)$ with 22-atomic representing measure, but no 21-atomic representing measure. This was proved by J.E. McCarthy via a topological dimension argument that uses the Open Mapping Theorem.	open problem

First Existence Criterion for TCMP

Theorem

(RC-L. Fialkow, 1998) Let γ *be a truncated moment sequence. TFAE:*

(i) γ *has a rep. meas.;*

(ii) γ *has a finitely atomic rep. meas. (with at most* $(n+2)(2n+3)$ *atoms);*

(iii) $M(n) \geq 0$ *and for some* $k \geq 0$ $M(n)$ *admits a positive extension* $M(n+k)$*, which in turn admits a flat extension* $M(n+k+1)$*. (The number of steps* k *satisfies* $k \leq 2n^2+6n+6$ *)).*

Case of Flat Data

Recall: If μ is a rep. meas. for $M(n)$, then rank $M(n) \leq$ card supp μ.

γ is flat if $M(n) = \begin{pmatrix} M(n-1) & M(n-1)W \\ W^* M(n-1) & W^* M(n-1)W \end{pmatrix}$.

Theorem

(RC-L. Fialkow, 1996) If γ is flat and $M(n) \geq 0$, then $M(n)$ admits a unique flat extension of the form $M(n+1)$.

Theorem

(RC-L. Fialkow, 1996) The truncated moment sequence γ has a rank $M(n)$*-atomic rep. meas. if and only if $M(n) \geq 0$ and $M(n)$ admits a flat extension $M(n+1)$.*

To find μ concretely, let $r :=$rank $M(n)$ and look for the analytic column relation

$$Z^r = c_0 1 + c_1 Z + ... + c_{r-1} Z^{r-1}.$$

We then define

$$p(z) := z^r - (c_0 + ... + c_{r-1} z^{r-1})$$

and solve the Vandermonde equation

$$\begin{pmatrix} 1 & \cdots & 1 \\ z_0 & \cdots & z_{r-1} \\ \cdots & \cdots & \cdots \\ z_0^{r-1} & \cdots & z_{r-1}^{r-1} \end{pmatrix} \begin{pmatrix} \rho_0 \\ \rho_1 \\ \cdots \\ \rho_{r-1} \end{pmatrix} = \begin{pmatrix} \gamma_{00} \\ \gamma_{01} \\ \cdots \\ \gamma_{0r-1} \end{pmatrix}.$$

Then

$$\mu = \sum_{j=0}^{r-1} \rho_j \delta_{z_j}.$$

An Application to Optimization

Consider the problem

$$p^* := inf\ p(x)\ (x \in \mathbb{R}^n) \text{ subject to } \ h_1 \geq 0, \cdots, h_m \geq 0;$$

that is, we try to minimize the values of the polynomial p over the semialgebraic set F determined by the polynomials $h_1, \cdots, h_m$.
Let $d_0 := [(degp)/2]$ and $d_i := [(degh_i)/2]$. For $t \geq max\{d_0, d_1, \cdots, d_m\}$, consider the associated optimization problem

An Application to Optimization, cont.

$$p_t^* := inf\ p^T\beta \ \ (t \in \mathbb{Z}_+)$$

subject to

$$\beta_0 = 1,\ M(t)[\beta] \geq 0 \ \text{ and } \ M_{h_j}(t - d_j)[\beta] \geq 0\ (j = 1, \cdots, m).$$

This is a semidefinite program. One proves that

$$p_t^* \leq p_{t+1}^* \leq p^*.$$

That is, the sequence $(p_t^*)_t$ approximates the absolute minimum p^* from below.

An Application to Optimization, cont.

J. Lasserre was able to use the Flat Extension Theorem to prove that the sequence converges to p^* when the semialgebraic set F is compact. Hence, the above mentioned semidefinite program can be used to approximate the minimum value of p over F.

Moreover, in a few cases Lasserre was able to prove finite convergence.

The significant outcome of this is that for certain optimization problems, the Flat Extension Theorem allows one to establish finite stopping times for suitable algorithms.

Localizing Matrices

Consider the **full, complex** MP

$$\int \bar{z}^i z^j \, d\mu = \gamma_{ij} \;\; (i, j \geq 0),$$

where supp $\mu \subseteq K$, for K a closed subset of $\mathbb{C}$.

- The **Riesz functional** is given by

$$\Lambda_\gamma(\bar{z}^i z^j) := \gamma_{ij} \;\; (i, j \geq 0).$$

- Riesz-Haviland:

  There exists μ with supp $\mu \subseteq K \Leftrightarrow \Lambda_\gamma(p) \geq 0$ for all p such that $p|_K \geq 0$.

If q is a polynomial in z and $\bar{z}$, and

$$K \equiv K_q := \{z \in \mathbb{C} : q(z, \bar{z}) \geq 0\},$$

then $L_q(p) := L(qp)$ must satisfy $L_q(p\bar{p}) \geq 0$ for μ to exist. For,

$$L_q(p\bar{p}) = \int_{K_q} qp\bar{p}\, d\mu \geq 0 \ \text{ (all } p).$$

- K. Schmüdgen (1991): If K_q is compact, $\Lambda_\gamma(p\bar{p}) \geq 0$ and $L_q(p\bar{p}) \geq 0$ for all p, then there exists μ with supp $\mu \subseteq K_q$.

We will now present a version of this result for TMP.

For $q \in \mathcal{P}_n$, define the localizing matrix M_q by

$$M_q(n)\hat{p} := \Lambda_\gamma(qp) \quad (p \in \mathcal{P}_n).$$

Clearly, $M_1 = M$, and M_z and $M_{\bar{z}}$ are the natural analogues of the shifted matrix in Stieltjes Theorem.

Theorem

*(**Localization of the support**) (RC-L. Fialkow, 2000) Let $M(n) \geq 0$ and suppose* $\deg(q) = 2k$ *or* $2k - 1$ *for some* $k \leq n$. *Then* $\exists\, \mu$ *with* $\operatorname{rank} M(n)$ *atoms and* $\operatorname{supp}\, \mu \subseteq K_q$ *if and only if* $\exists$ *a flat extension* $M(n+1)$ *for which* $M_q(n+k) \geq 0$. *In this case,* $\exists\, \mu$ *with exactly* $\operatorname{rank} M(n) - \operatorname{rank}\, M_q(n+k)$ *atoms in* $\mathcal{Z}(q)$.

Remark

M. Laurent (2005) has found an alternative proof, using ideas from real algebraic geometry.

Actually, M. Laurent was able to use techniques from algebraic geometry

Unilateral Weighted Shifts

- $\alpha \equiv \{\alpha_k\}_{k=0}^{\infty} \in \ell^\infty(\mathbb{Z}_+)$, $\alpha_k > 0$ (all $k \geq 0$)
- $W_\alpha : \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+)$, $\{e_k\}_{k\geq 0}$ ONB of $\ell^2(\mathbb{Z}_+)$

$$W_\alpha e_k := \alpha_k e_{k+1} \quad (k \geq 0)$$

- When $\alpha_k = 1$ (all $k \geq 0$), $W_\alpha = U_+$, the (unweighted) unilateral shift
- In general, $W_\alpha = U_+ D_\alpha$ (polar decomposition)

The moments of α are given as

$$\gamma_k \equiv \gamma_k(\alpha) := \left\{ \begin{array}{cc} 1 & \text{if } k = 0 \\ \alpha_0^2 \cdot \ldots \cdot \alpha_{k-1}^2 & \text{if } k > 0 \end{array} \right\}.$$

Berger Measures

- (Berger; Gellar-Wallen) W_α is subnormal if and only if there exists a positive Borel measure ξ on $[0, \|W_\alpha\|^2]$ such that

$$\gamma_k = \int t^k \, d\xi(t) \ \ (\text{all } k \geq 0).$$

  ξ is the Berger measure of W_α.

- For $0 < a < 1$ we let $S_a := \text{shift}\,(a, 1, 1, \ldots)$.
- The Berger measure of U_+ is δ_1.
- The Berger measure of S_a is $(1 - a^2)\delta_0 + a^2\delta_1$.
- The Berger measure of B_+ (the Bergman shift) is Lebesgue measure on the interval $[0, 1]$; the weights of B_+ are $\alpha_n := \sqrt{\frac{n+1}{n+2}}$ $(n \geq 0)$.

Multivariable Weighted Shifts

$$\alpha_{\mathbf{k}}, \beta_{\mathbf{k}} \in \ell^\infty(\mathbb{Z}_+^2), \quad \mathbf{k} \equiv (k_1, k_2) \in \mathbb{Z}_+^2 := \mathbb{Z}_+ \times \mathbb{Z}_+$$

$$\ell^2(\mathbb{Z}_+^2) \cong \ell^2(\mathbb{Z}_+) \bigotimes \ell^2(\mathbb{Z}_+).$$

We define the 2-variable weighted shift $\mathbf{T} \equiv (T_1, T_2)$ by

$$T_1 e_{\mathbf{k}} := \alpha_{\mathbf{k}} e_{\mathbf{k}+\varepsilon_1} \quad T_2 e_{\mathbf{k}} := \beta_{\mathbf{k}} e_{\mathbf{k}+\varepsilon_2},$$

where $\varepsilon_1 := (1,0)$ and $\varepsilon_2 := (0,1)$. Clearly,

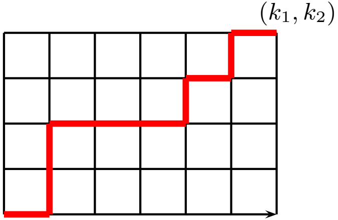
$$T_1 T_2 = T_2 T_1 \Longleftrightarrow \beta_{\mathbf{k}+\varepsilon_1}\alpha_{\mathbf{k}} = \alpha_{\mathbf{k}+\varepsilon_2}\beta_{\mathbf{k}} \ \text{(all } \mathbf{k}\text{)}.$$

$$\begin{tikzcd}
(k_1, k_2+1) \arrow[r, "\alpha_{k_1,k_2+1}"] & (k_1+1, k_2+1) \\
(k_1, k_2) \arrow[u, "\beta_{k_1,k_2}"'] \arrow[r, "\alpha_{k_1,k_2}"] & (k_1+1, k_2) \arrow[u, "\beta_{k_1+1,k_2}"']
\end{tikzcd}$$

We now recall the notion of moment of order $\mathbf{k}$ for a commuting pair (α,β). Given $\mathbf{k}\in\mathbb{Z}_+^2$, the moment of (α,β) of order $\mathbf{k}$ is $\gamma_{\mathbf{k}}\equiv\gamma_{\mathbf{k}}(\alpha,\beta)$

$$:=\begin{cases} 1 & \text{if } \mathbf{k}=0 \\ \alpha_{(0,0)}^2\cdot\ldots\cdot\alpha_{(k_1-1,0)}^2 & \text{if } k_1\geq 1 \text{ and } k_2=0 \\ \beta_{(0,0)}^2\cdot\ldots\cdot\beta_{(0,k_2-1)}^2 & \text{if } k_1=0 \text{ and } k_2\geq 1 \\ \alpha_{(0,0)}^2\cdot\ldots\cdot\alpha_{(k_1-1,0)}^2\cdot\beta_{(k_1,0)}^2\cdot\ldots\cdot\beta_{(k_1,k_2-1)}^2 & \text{if } k_1\geq 1 \text{ and } k_2\geq 1. \end{cases}$$

By commutativity, $\gamma_{\mathbf{k}}$ can be computed using any nondecreasing path from $(0,0)$ to (k_1,k_2).



- (Jewell-Lubin)

$$W_\alpha \text{ is subnormal} \quad \Leftrightarrow \quad \gamma_{\mathbf{k}} := \prod_{i=0}^{k_1-1} \alpha_{(i,0)}^2 \cdot \prod_{j=0}^{k_2-1} \beta_{(k_1-1,j)}^2$$

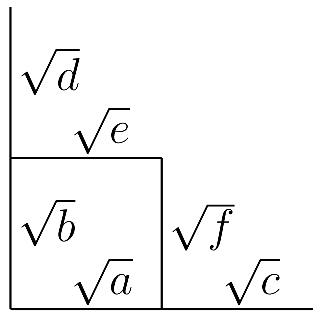
$$= \int t_1^{k_1} t_2^{k_2} \, d\mu(t_1, t_2) \;\; (\text{all } \mathbf{k} \geq \mathbf{0}).$$

Thus, the study of subnormality for multivariable weighted shifts is intimately connected to multivariable real moment problems.

The Subnormal Completion Problem

for 2-variable weighted shifts

Consider the following completion problem: Given



$be = af$ (commutativity)

Figure 1. The initial family of weights Ω_1

we wish to add infinitely many weights and generate a subnormal 2-variable weighted shift, that is, a weighted shift with a Berger measure interpolating the initial family of weights.

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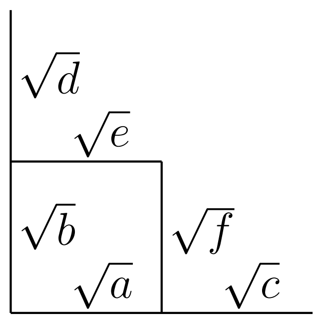

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Figure 1. The initial family of weights Ω_1

we wish to add infinitely many weights and generate a subnormal 2-variable weighted shift, that is, a weighted shift with a Berger measure interpolating the initial family of weights.

The initial family needs to satisfy an obvious necessary condition, that is,

$$\mathcal{M}(\Omega_1) := \begin{pmatrix} \beta_{00} & \beta_{01} & \beta_{10} \\ \beta_{01} & \beta_{02} & \beta_{11} \\ \beta_{10} & \beta_{01} & \beta_{20} \end{pmatrix} \equiv \begin{pmatrix} 1 & a & b \\ a & ac & be \\ b & be & bd \end{pmatrix}. \tag{11.1}$$

We use tools and techniques from the theory of TMP to solve SCP in the foundational case of six prescribed initial weights; these weights give rise to the quadratic moments. For this case, the natural necessary conditions for the existence of a subnormal completion are also sufficient.

To calculate explicitly the associated Berger measure, we compute the algebraic variety of the associated truncated moment problem; it turns out that this algebraic variety is precisely the support of the Berger measure of the subnormal completion.

In this case, solving the SCP consists of finding a probability measure μ supported on $\mathbb{R}_+^2$ such that $\int_{\mathbb{R}_+^2} y^i x^j \, d\mu(x,y) = \gamma_{ij}$ $(i,j \geq 0, \; i+j \leq 2)$. To ensure that the support of μ remains in $\mathbb{R}_+^2$ we use the localizing matrices $\mathcal{M}_x(2)$ and $\mathcal{M}_y(2)$; each of these matrices will need to be positive semidefinite.

Theorem

(RC, S.H. Lee and J. Yoon; 2010) Let Ω_1 be a quadratic, commutative, initial set of positive weights, and assume $\mathcal{M}(\Omega_1) \geq 0$. Then there always exists a quartic commutative extension $\hat{\Omega}_2$ of Ω_1 such that $\mathcal{M}(\hat{\Omega}_2)$ is a flat extension of $\mathcal{M}(\Omega_1)$, and $\mathcal{M}_x(\hat{\Omega}_2) \geq 0$ and $\mathcal{M}_y(\hat{\Omega}_2) \geq 0$. As a consequence, Ω_1 admits a subnormal completion $\mathbf{T}_{\hat{\Omega}_\infty}$.

$$M(2) = \begin{pmatrix} 1 & a & b & ac & be & bd \\ a & ac & be & acp & beq & bdr \\ b & be & bd & beq & bdr & bds \\ ac & acp & beq & & & \\ be & beq & bdr & & & \\ bd & bdr & bds & & & \end{pmatrix} \tag{11.2}$$

(with the lower right-hand 3×3 corner yet undetermined) and

$$M_x(2) = \begin{pmatrix} a & ac & be \\ ac & acp & beq \\ be & beq & bdr \end{pmatrix} \text{ and } M_y(2) = \begin{pmatrix} b & be & bd \\ be & beq & bdr \\ bd & bdr & bds \end{pmatrix}.$$

It is actually possible to provide a concrete description of the Berger measure for the subnormal completion in terms of the initial data.

Remark

Flat extensions may not exist for bigger families of initial weights. That is, one can build an example of a quartic family of initial weights Ω_2 for which the associated moment matrix $\mathcal{M}(2)$ admits a representing measure, but such that $\mathcal{M}(2)$ has no flat extension $\mathcal{M}(3)$.

Related Research

G. Blekherman, Positive Gorenstein ideals

G. Blekherman, Nonnegative polynomials and sums of squares

G. Blekherman and L. Fialkow, The core variety and representing measures in the truncated moment problem

G. Blekherman and J.B. Lasserre, The truncated K-moment problem for closure of open sets

R. Curto, S.H. Lee and J. Yoon, A new approach to the subnormal completion problem (for 2-variable weighted shifts)

Ph. di Dio, The multidimensional truncated moment problem: Gaussian and Log-normal mixtures, their Carathéodory numbers, and set of atoms

Ph. di Dio and M. Kummer, The multidimensional truncated moment problem: Carathéodory numbers from Hilbert functions and shape reconstruction from derivatives of moments

Related Research, cont.

Ph. di Dio and K. Schmüdgen, On the truncated multidimensional moment problem: atoms, determinacy and core variety

Ph. di Dio and K. Schmüdgen, On the truncated multidimensional moment problem: Carathéodory numbers

Ph. di Dio and K. Schmüdgen, The multidimensional truncated moment problem: The moment cone

C. Easwaran and L. Fialkow, Positive linear functionals without representing measures

C. Easwaran, L. Fialkow and S. Petrovic, Can a minimal degree 6 cubature rule for the disk have all points inside?

L. Fialkow, Solution of the truncated moment problem with $y = x^3$

L. Fialkow, The truncated moment problem on parallel lines

Related Research, cont.

L. Fialkow, The core variety of a multisequence in the truncated moment paroblem

L. Fialkow and J. Nie, The truncated moment problem via homogenization and flat extensions

L. Fialkow and J. Nie, On the closure of positive flat moment matrices

M. Ghasemi, S. Kuhlmann and E. Samei, The moment problem for continuous positive semidefinite linear functionals

M. Ghasemi, M. Infusino, S. Kuhlmann and M. Marshall, Moment problems for symmetric algebras of locally convex spaces

J.W. Helton and J. Nie, A semidefinite approach for truncated K-moment problems

D. Henrion, J.B. Lasserre and M. Mevissen, Mean squared error minimization for inverse moment problems

Related Research, cont.

D. Henrion, J.B. Lasserre and C. Savorgnan, Approximate volume and integration for basic semialgebraic sets

M. Infusino, Quasi-analyticity and determinacy of the full moment problem from finite to infinite dimensions

M. Infusino and S. Kuhlmann, Infinite dimensional moment problem: Open questions and applications

D. Kimsey, The cubic complex moment problem

D. Kimsey, The subnormal completion problem in several variables

D. Kimsey and H. Woerdeman, The truncated matrix-valued K-moment problem on $\mathbb{R}^d$, $\mathbb{C}^d$ and $\mathbb{T}^d$

S. Kuhlmann and M. Marshall, Positivity, sums of squares and multidimensional moment problems

Related Research, cont.

T. Kuna, J.L. Lebowitz and E.R. Speer, Necessary and sufficient conditions for realizability of point processes

M. Laurent, Sums of squares, moment matrices and optimization over polynomials

M. Laurent and B. Mourrain, A generalized at extension theorem for moment matrices

J.B. Lasserre, Global optimization with polynomials and the problem of moments

J.B. Lasserre, Existence of Gaussian cubature formulas

B. Mourrain and K. Schmüdgen, Flat extensions in *-algebras

B. Reznick, On Hilbert's construction of positive polynomials

Related Research, cont.

C. Riener and M. Schweighofer, Optimization approaches to quadrature: New characterizations of Gaussian quadrature on the line and quadrature with few nodes on plane algebraic curves, on the plane and higher dimensions

K. Schmüdgen, *The Moment Problem*

M. Schweighofer, An algorithmic approach to Schmüdgen's Positivstellensatz

M. Schweighofer, Optimization of polynomials on compact semialgebraic sets

M. Schweighofer, A Gröbner basis proof of the Flat Extension Theorem for moment matrices

Related Research, cont.

F.-H. Vasilescu, Dimensional stability in truncated moment problems

S. Zagorodnyuk, On the truncated two-dimensional moment problem

S. Zagorodnyuk, The operator approach to the truncated multidimensional moment problem

A Version of Riesz-Haviland for TMP

Given a moment sequence β, the Riesz functional is

$$L_\beta(p) := p(\beta) \ \ (p \in \mathbb{C}[z, \bar{z}]).$$

Recall the Riesz-Haviland Theorem:

$\exists\ \mu$ rep. meas. for $\beta \Leftrightarrow L \equiv L_\beta \geq 0$ on $\mathcal{P}_+$.

For TMP, the natural analogue won't work.

We say that the Riesz functional L is *K-positive* if

$p \in \mathcal{P}$ and $p|K \geq 0 \Rightarrow L(p) \geq 0$.

Consider the case

$d = 1,\ K = \mathbb{R}$, and

$$M(2) := \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix} \geq 0.$$

In this case,

$$\mathrm{L}(\mathrm{a}_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4) := a_0 + a_1 + a_2 + a_3 + 2a_4$$

One proves that L is K-positive, but β has no representing measure.

In TMP, K-positivity is a necessary (but not sufficient) condition for a K-representing measure μ.

Theorem (TMP Version of Riesz-Haviland)

(RC-L. Fialkow, 2007) $\beta \equiv \beta^{(2n)}$ *admits a* K*-representing measure if and only if* L_β *admits a* K*-positive linear extension* $L : \mathcal{P}_{2n+2} \longmapsto \mathbb{R}$.

This Theorem implies the classical Riesz-Haviland, via Stochel's Theorem.

$$\begin{array}{ccc}
\mathcal{P} & & \\
\vdots & & \\
\uparrow & & \\
\mathcal{P}_{2n+2} & & \\
\uparrow & \overset{K\text{-pos.}}{\searrow} & \\
\mathcal{P}_{2n} & \overset{K\text{-pos.}}{\longrightarrow} & \mathbb{R}
\end{array}$$

In general it is quite difficult to directly verify that an extension $\tilde{L} : \mathcal{P}_{2n+2} \longrightarrow \mathbb{R}$ is K-positive.

The Quartic Moment Problem

Recall the lexicographic order on the rows and columns of $M(2)$:

$$1, Z, \bar{Z}, Z^2, \bar{Z}Z, \bar{Z}^2$$

- $(r = 1)$ $Z = A\,1$ (Dirac measure)
- $(r = 2)$ $\bar{Z} = A\,1 + B\,Z$ (supp $\mu \subseteq$ line)
- $(r = 3)$ $Z^2 = A\,1 + B\,Z + C\,\bar{Z}$ (flat extensions always exist)
- $(r = 4)$ $\bar{Z}Z = A\,1 + B\,Z + C\,\bar{Z} + D\,Z^2$

$$\begin{aligned} D &= 0 \Rightarrow \bar{Z}Z = A\,1 + B\,Z + \bar{B}\,\bar{Z} \text{ and } C = \bar{B} \\ &\Rightarrow (\bar{Z} - B)(Z - \bar{B}) = A + |B|^2 \\ &\Rightarrow \bar{W}W = 1 \text{ (circle), for } W := \frac{Z - \bar{B}}{\sqrt{A + |B|^2}}. \end{aligned}$$

Case $r = 5$

With $x := \mathrm{Re}[z]$ and $y := \mathrm{Im}[z]$, and using the flat data result, one can reduce the study to cases corresponding to the following five real conics:

$$
\begin{array}{lll}
(a) & \bar{W}^2 = -2iW + 2i\bar{W} - W^2 - 2\bar{W}W & \text{parabola; } y = x^2 \\
(b) & \bar{W}^2 = -4i1 + W^2 & \text{hyperbola; } yx = 1 \\
(c) & \bar{W}^2 = W^2 & \text{pair of intersect. lines; } yx = 0 \\
(d) & \bar{W}W = 1 & \text{unit circle; } x^2 + y^2 = 1 \\
(e) & W^2 + 2\bar{W}W + \bar{W}^2 = 2W + 2\bar{W} & \text{two parallel lines; } x(x-1) = 0.
\end{array}
$$

Theorem

(RC-L. Fialkow, 2005) Assume that $M(2) \geq 0$, $M(2)$ *singular, and that* rank $M(2) \leq$ card $\mathcal{V}(\gamma^{(4)})$. *Then* $M(2)$ *admits a representing measure.*

The Case of Invertible $M(2)$

(L. Fialkow and J. Nie, 2010) Consider a quartic moment problem with invertible $M(2)$. Then there exists a representing measure.
The proof is abstract, using convex analysis.

(RC-S. Yoo, 2013) Concrete construction of a representing measure, when $M(2)$ is invertible. Moreover, there exists a 6-atomic representing measure, that is, $M(2)$ admits a flat extension $M(3)$.
The proof uses a new idea: rank reduction

Extremal Real MP; $r = v$

Recall: The *algebraic variety* of β is

$$\mathcal{V} \equiv \mathcal{V}_\beta := \bigcap_{p \in \mathcal{P}_n, \widehat{p} \in \ker \mathcal{M}(n)} \mathcal{Z}_p,$$

where $\mathcal{Z}_p = \{x \in \mathbb{R}^d : p(x) = 0\}$. If β admits a rep. measure μ, then

$$p \in \mathcal{P}_n \text{ satisfies } \widehat{p} \in \ker \mathcal{M}(n) \Leftrightarrow \text{supp } \mu \subseteq \mathcal{Z}_p$$

Thus supp $\mu \subseteq \mathcal{V}$, so $r :=$ rank $\mathcal{M}(n)$ and $v :=$ card $\mathcal{V}$ satisfy

$$r \leq \text{card supp } \mu \leq v.$$

Extension Principle: $\mathcal{M}(n+1)$ rec. gen. extension of $\mathcal{M}(n)$ and $p(X, Y) = 0$ in $\mathcal{M}(n)$, then $p(X, Y) = 0$ in $\mathcal{M}(n+1)$.

Then $\mathcal{V}(n+1) \subseteq \mathcal{V}(n)$ and therefore $r_n \leq r_{n+1} \leq v_{n+1} \leq v_n$.

Basic necessary conditions for the existence of a representing measure

$$\text{(Positivity)}\ M(n) \geq 0$$

$$\text{(Consistency)}\ p \in \mathcal{P}_{2n},\ p|_{\mathcal{V}} \equiv 0 \Longrightarrow \Lambda(p) = 0$$

(where Λ is the Riesz functional associated to $M(n)$)

$$\text{(Variety Condition)}\ r \leq v, \text{ i.e., } \operatorname{rank}\ M(n) \leq \operatorname{card}\ \mathcal{V}.$$

Consistency implies

$$\text{(Recursiveness)}\ p, q, pq \in \mathcal{P}_n,\ \widehat{p} \in \ker M(n) \Longrightarrow \widehat{pq} \in \ker M(n).$$

Consistency is intimately related to J. Stochel's Type B: A polynomial $P \in \mathcal{P}_{2n}$ is type B if $\Phi \geq 0$, linear and $\Phi|_{\mathcal{I}(\mathcal{Z}(P))} \equiv 0 \Rightarrow \Phi(f) = \int f\ d\mu$.

$$\text{(Consistency) } p \in \mathcal{P}_{2n},\ p|_{\mathcal{V}} \equiv 0 \Longrightarrow \Lambda(p) = 0$$

$$\text{(Weak Consistency) } p \in \mathcal{P}_{n},\ p|_{\mathcal{V}} \equiv 0 \Longrightarrow \Lambda(p) = 0$$

Consistency $\Longrightarrow$ Weak Consistency $\Longrightarrow$ Recursively generated

Theorem

(RC, L. Fialkow and M. Möller, 2005) Suppose $\mathcal{M}(3) \geq 0$, recursively generated, $Y = X^3$ and $r \leq v \leq 7$. Then $\mathcal{M}(3)$ has a rep. measure.

Theorem

(RC, L. Fialkow and M. Möller, 2005) There exists a real moment matrix $\mathcal{M}(3) \geq 0$, recursively generated, with $r = v = 8$, $Y = X^3$, and no rep. measure.

Theorem

(L. Fialkow; TAMS, 2011) There exists a real moment matrix $\mathcal{M}(3)$ *which is positive, consistent, with column relation* $Y = X^3$ *and no rep. measure.*

Theorem EXT

(RC, L. Fialkow and M. Möller, 2005) For $\beta \equiv \beta^{(2n)}$ **extremal**, *i.e.,* $r = v$, *the following are equivalent:*

(i) β *has a representing measure;*

(ii) β *has a unique representing measure, which is* rank $M(n)$*-atomic (minimal);*

(iii) There exists $\mathcal{M}(n+1)$ *flat extension of* $\mathcal{M}(n)$*;*

(iv) There exists a unique flat extension of $\mathcal{M}(n)$*;*

(v) $M(n) \geq 0$ *and* β *is consistent.*

Cubic Column Relations

Since we know how to solve the singular Quartic MP, WLOG we will assume $M(2) > 0$, and that $Z^3 = p_2(Z, \bar{Z})$, with deg $p_2 \leq 2$.

First, we would like to focus on the case of harmonic poly's: $q(z, \bar{z}) := f(z) - \overline{g(z)}$, with deg $q = 3$.

Recall that rank $M(n) \leq$ card $\mathcal{Z}(q)$. Of special interest is the case when card $\mathcal{Z}(q) \geq 7$, since otherwise the TMP either admits a flat extension or has no representing measure. In the case when $g(z) \equiv z$, we have

Lemma

(Wilmshurst '98, Sarason-Crofoot, '99, Khavinson-Swiatek, '03)

$$\text{card } \mathcal{Z}(f(z) - \bar{z}) \leq 7.$$

Bézout's Theorem predicts card $\mathcal{Z}(f(z) - \bar{z}) \leq 9$

WLOG, one considers harmonic polynomials of the form
$q_7(z, \bar{z}) := z^3 - itz - u\bar{z}$.

Proposition

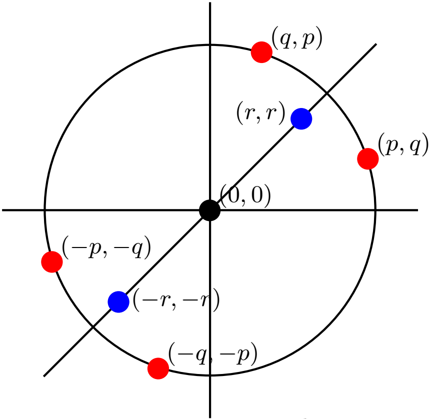
(RC-S. Yoo, 2009) For $(0 < u < -t < 2u)$, *we have* card $\mathcal{Z}(q_7) = 7$. *In fact,*

$$\mathcal{Z}(q_7) = \{0, p + iq, q + ip, -p - iq, -q - ip, r + ir, -r - ir\},$$

where $p, q, r > 0$, $p^2 + q^2 = u$ *and* $r^2 = \frac{|t+u|}{2}$.

Harmonic Polynomials

Consider the harmonic polynomial $q_7(z, \bar{z}) := z^3 - itz - u\bar{z}$, with $(0 < u < -t < 2u)$:



Since rank $M(3) = 7$, there must be another column relation besides $q_7(Z, \bar{Z}) = 0$. Clearly the columns

$$1, Z, \bar{Z}, Z^2, \bar{Z}Z, \bar{Z}^2, \bar{Z}Z^2$$

must be linearly independent (otherwise $M(3)$ would be a flat extension of $M(2)$), so the new column relation must involve $\bar{Z}Z^2$ and $\bar{Z}^2Z$. An analysis using the properties of the functional calculus shows that, **in the presence of a representing measure**, the new column relation must be

$$\bar{Z}^2Z + i\bar{Z}Z^2 - iuZ - u\bar{Z} = 0.$$

Notation

Define

$$
\begin{aligned}
q_{LC}(z,\bar{z}) &:= \bar{z}^2 z + i\bar{z}z^2 - iuz - u\bar{z} \\
&= i(z - i\bar{z})(\bar{z}z - u).
\end{aligned}
$$

Observe that the zero set of q_{LC} is the union of a line and a circle, and that $\mathcal{Z}(q_7) \subset \mathcal{Z}(q_{LC})$.

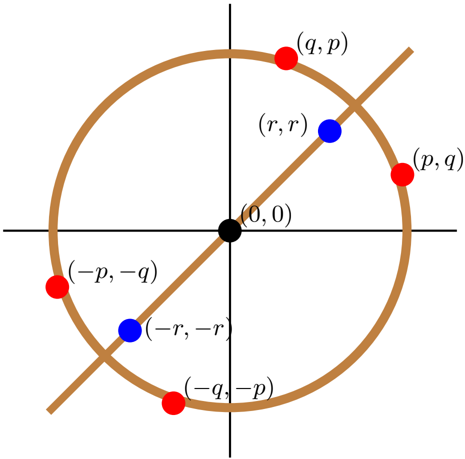


FIGURE 2. The sets $\mathcal{Z}(q_7)$ and $\mathcal{Z}(q_{LC})$

Theorem

(RC-S. Yoo, 2014) Let $M(3) \geq 0$*, with* $M(2) > 0$ *and* $q_7(Z, \bar{Z}) = 0$*.*
There exists a representing measure for $M(3)$ *if and only if*

$$\begin{cases} \Lambda(q_{LC}) &= 0 \\ \Lambda(zq_{LC}) &= 0. \end{cases}$$

where $\Lambda \equiv \Lambda_\beta$ *is the Riesz functional. Equivalently,*

$$\begin{cases} Re\, \gamma_{12} - Im\, \gamma_{12} = u(Re\, \gamma_{01} - Im\, \gamma_{01}) &= 0 \\ \gamma_{22} = (t+u)\gamma_{11} - 2u\, Im\, \gamma_{02} &= 0. \end{cases}$$

Equivalently,

$$q_{LC}(Z, \bar{Z}) = 0$$

Proof uses Consistency Property.

Proposition (Representation of Polynomials)

Let $\mathcal{P}_6 := \{p \in \mathbb{C}_6[z,\bar{z}] : p|_{\mathcal{Z}(q_7)} \equiv 0\}$ *and let*

$\mathcal{I} := \{p \in \mathbb{C}_6[z,\bar{z}] : p = fq_7 + g\bar{q}_7 + hq_{LC}$ *for some* $f, g, h \in \mathbb{C}_3[z,\bar{z}]\}$.

Then $\mathcal{P}_6 = \mathcal{I}$.

The Division Algorithm

Division Algorithm in $\mathbb{R}[x_1, \cdots, x_n]$

Fix a monomial order $>$ on $\mathbb{Z}^n_{\geq 0}$ and let $F = (f_1, \cdots, f_s)$ be an ordered s-tuple of polynomials in $\mathbb{R}[x_1, \cdots, x_n]$. Then every $f \in \mathbb{R}[x_1, \cdots, x_n]$ can be written as

$$f = a_1 f_1 + \cdots + a_s f_s + r,$$

where $a_i, \in \mathbb{R}[x_1, \cdots, x_n]$, and either $r = 0$ or r is a linear combination, with coefficients in $\mathbb{R}$, of monomials, none of which is divisible by any of the leading terms in $f_1, \cdots, f_s$.

Furthermore, if $a_i f_i \neq 0$, then we have

$$multideg(f) \geq multideg(a_i f_i).$$

Key idea: Use the Division Algorithm to establish representation theorems for polynomials vanishing on the algebraic variety of β.

Classification of sextic MP

r_3	v_3	$v_3 - r_3$	MaxExt		Solution Presented in
7	7	0	$\mathcal{M}(4)$	extremal	RC-S. Yoo;JFA(2014),IEOT(2
7	8	1	$\mathcal{M}(5)$	non-extremal	RC-S. Yoo; JFA(2015)
7	9	2	$\mathcal{M}(6)$	non-extremal	RC-S. Yoo; JFA(2015)
7	∞	N/A	N/A	non-extremal	RC-S. Yoo; JFA(2015)
8	8	0	$\mathcal{M}(4)$	extremal	RC-S. Yoo;IEOT(2017)
8	9	1	$\mathcal{M}(5)$	non-extremal	RC-S. Yoo; JFA(2015)
8	∞	N/A	N/A	non-extremal	RC-S. Yoo; JFA(2015)
9	∞	N/A	N/A	non-extremal	L. Fialkow;TAMS(2011)(cas
10	∞	N/A	N/A	non-extremal	open problem

A New Tool: Rank Reduction

Given a point $(a, b) \in \mathbb{R}^2$ we let $\mathbf{v} \equiv \mathbf{v}_{(a,b)}$ denote the row vector

$$(1, a, b, a^2, ab, b^2, a^3, a^2b, ab^2, b^3)$$

We also let $\delta_{(a,b)}$ denote the point mass at (a, b). It is easy to see that the moment matrix associated with $\delta_{(a,b)}$ is $\mathbf{vv}^T$, that is, the matrix whose entries are $\mathcal{M}(3)_{ij} = a^i b^j$. For this moment matrix, $r = 1$ and $\mathcal{V} = \{(a, b)\}$.

Theorem

(RC-S. Yoo, 2015) Assume $\mathcal{M}(3) \geq 0$, $\mathcal{M}(2) > 0$, rank $\mathcal{M}(3) = 7$ and card $\mathcal{V} \geq 8$. Assume also that $\mathcal{M}(3)$ satisfies the Consistency Property. Then $\mathcal{M}(3)$ admits a flat extension $\mathcal{M}(4)$; that is, there exists a representing measure μ with card supp $\mu = 7$.

Sketch of Proof. WLOG, assume

$$\mathcal{V} = \{(x_1, y_1), \dots, (x_8, y_8)\}.$$

Also assume that in $\mathcal{M}(3)$ the first seven columns are linearly independent. Now form the Vandermonde matrix

$$\begin{pmatrix} 1 & x_1 & y_1 & x_1^2 & x_1 y_1 & y_1^2 & x_1^3 & x_1^2 y_1 & x_1 y_1^2 & y_1^3 \\ 1 & x_2 & y_2 & x_2^2 & x_2 y_2 & y_2^2 & x_2^3 & x_2^2 y_2 & x_2 y_2^2 & y_2^3 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & x_8 & y_8 & x_8^2 & x_8 y_8 & y_8^2 & x_8^3 & x_8^2 y_8 & x_8 y_8^2 & y_8^3 \end{pmatrix}.$$

This is an 8×10 matrix, with rank 7. It follows that exactly seven rows are linearly independent, so one of them must be a linear combination of the other seven, say

$$R_j = \sum_{i \neq j} \lambda_i R_i.$$

The row R_j must be associated with a point $(x_j, y_j) \in \mathcal{V}$. To single out this point, we will denote it by (a, b). Now let

$$\mathcal{V}' := \mathcal{V} \setminus \{(a, b)\}.$$

Claim. No conic goes through $\mathcal{V}'$. Proof uses invertibility of $\mathcal{M}(2)$ and Consistency.

We now define

$$\widetilde{\mathcal{M}(3)} := \mathcal{M}(3) - \rho \mathbf{v}\mathbf{v}^T,$$

where $\mathbf{v}$ is the row vector associated with the point (a, b).

We wish to prove that rank $\widetilde{\mathcal{M}(3)} = 6$ for some positive value of ρ. If we do this, then $\widetilde{\mathcal{M}(3)}$ will be a flat extension of $\widetilde{\mathcal{M}(2)}$, and we will have a 6-atomic measure for $\widetilde{\mathcal{M}(3)}$, and therefore a 7-atomic measure for $\mathcal{M}(3)$, since $\mathcal{M}(3) = \widetilde{\mathcal{M}(3)} + \rho \mathbf{v}\mathbf{v}^T$. Moreover, one can show that rank $\widetilde{\mathcal{M}(2)} = 6$, using above Claim. Also, observe that $\widetilde{\mathcal{M}(3)} \geq 0$.

Let λ denote the nonzero eigenvalue of $\mathbf{v}\mathbf{v}^T$, and let $\mathcal{B}$ be the basis of the column space of $\mathcal{M}(3)$. Then

$$\det \widetilde{\mathcal{M}(3)}_{\mathcal{B}} = \det \mathcal{M}(3)_{\mathcal{B}} - \rho\lambda \det(\mathcal{M}(3)_{\mathcal{B}}|_{\{2,3,4,5,6,7\}}).$$

Thus, with

$$\rho := \frac{\det \mathcal{M}(3)_{\mathcal{B}}}{\lambda \det(\mathcal{M}(3)_{\mathcal{B}}|_{\{2,3,4,5,6,7\}})},$$

we successfully reduce the rank. $\square$

$\mathcal{M}(3)$ WITH $r = 8$ AND $v = 9$

We again use a Rank Reduction strategy: In the specific case of $r = 8$ and $v = 9$, one must have the algebraic variety $\mathcal{V}$ of $\mathcal{M}(3)$ as the intersection of two cubics C_1 and C_2 in general position. We then use the Cayley-Bacharach Theorem:

> Assume that two cubics C_1 and C_2 in the projective plane meet in nine (different) points (that is $C_1 \bigcap C_2 = \mathcal{V}$). Then every cubic C that passes through any eight of the points in $\mathcal{V}$ also passes through the ninth point.

We can then generate an Algorithm to determine solubility of the TMP.

Thank you!