LMIs & zeros of their determinants

(with Bill Helton, Igor Klep, Scott McCullough)

(over $\mathbb{C}$) $A_1, \ldots, A_g \in M_d \rightsquigarrow L = L_A = I + A_1 x_1 + \ldots + A_g x_g$

<u>monic</u> matrix pencil

$X \in M_n^g$ : $L(X) = I \otimes I + A_1 \otimes X_1 + \ldots + A_g \otimes X_g \in M_{dn}$

matrix point evaluation of $L$

<u>Free locus</u> of $L$ : $\mathcal{Z}(L) = \bigsqcup_{n \in \mathbb{N}} \mathcal{Z}_n(L)$, where

(family of hypersurfaces) $\mathcal{Z}_n(L) = \{X \in M_d^g : \det L(X) = 0\}$

free .... "free of choice of $n$", as in free analysis,
free real alg. geom.

Why free loci?

(1) $A_j^* = A_j \rightsquigarrow L$ hermitian pencil

$\mathcal{D}(L) = \bigsqcup_n \{X = X^* \in M_n^g : L(X) \text{ is psd}\}$

<u>LMI domain</u> (free <u>spectrahedron</u>) of $L$

(important in convexity, optimization, QIT ..)

$\mathcal{Z}(L)$ .... Zariski closure of the boundary of $\mathcal{D}(L)$

(2) domains of noncommutative rat. fun. = complements of free loci

(3) geometric counterpart of factorization in the free algebra $\mathbb{C}\langle x_1, \ldots, x_g \rangle$

## Inclusion problem for free loci

$\mathcal{A} \subseteq M_d$ ... the $\mathbb{C}$-algebra generated by $A_1, \dots, A_g$

$$(A_1, \dots, A_g) \xrightarrow[\text{invariant subspaces}]{\text{basis change}} \begin{pmatrix} \boxed{\star} & & * \\ & \ddots & \\ 0 & & \boxed{\star} \end{pmatrix},$$

$\star = 0$, or submatrices $\star$ generate the full block

$$\underbrace{\mathcal{A}/\operatorname{rad}\mathcal{A}}_{} \cong \mathcal{A}_1 \times \dots \times \mathcal{A}_s$$

↳ $*$; nilradical, Jacobson radical

simple algebras; appear on the diagonal

Clearly: $Z(L_A) = Z(L_{A^1}) \cup \dots \cup Z(L_{A^s})$

$L_A$ is <u>irreducible</u> if $\mathcal{A} = M_d$. (no inv. subspaces)

<u>Thm.</u> (a) $Z(L_A) = \emptyset \iff A_1, \dots, A_g$ jointly nilpotent

(b) $Z(L_A) \subseteq Z(L_B) \iff$ the map $B_j \mapsto A_j$ induces a (surj) homomorphism

$$\mathcal{B}/\operatorname{rad}\mathcal{B} \to \mathcal{A}/\operatorname{rad}\mathcal{A}$$

(c) $L_A, L_B$ irr.

$Z(L_A) = Z(L_B) \iff$ of the same size $d$ & $B_j = P A_j P^{-1}$ for $P \in GL_d$

Main ingredients:

- $\det L(X)$ for $X \in M_n^g$ is a polynomial invariant for $GL_n \curvearrowright M_n^g$ by simultaneous conjugation:

$$P \cdot X = PXP^{-1} = (PX_1P^{-1}, \dots, PX_gP^{-1})$$

Procesi, Artin, ... : generators, relations, closed orbits

- linearization trick:

$$I + A\otimes X + B\otimes Y + AB\otimes Z \text{ sing} \iff \begin{pmatrix} I & B\otimes Z \\ -A\otimes I & 1 + A\otimes X + B\otimes Y \end{pmatrix} \text{ sing}$$

$$\iff I + A\otimes\begin{pmatrix} 0 & 0 \\ -I & X \end{pmatrix} + B\otimes\begin{pmatrix} 0 & Z \\ 0 & Y \end{pmatrix} \text{ sing}$$

Moral: to take care of $AB$, double the input size.

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## Irreducibility

If $L$ is irreducible, is $Z_n(L)$ an irreducible hypersurface?

No. Ex: $L = I + \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} x_1 + \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} x_2$.

$L$ irr, $Z_1(L) = \{(1 + x_1 - x_2)(1 + x_1 + x_2) = 0\}$
union of two lines,

$n \geq 2$: $Z_n(L)$ is irr. of degree $2n$

(related: $(1+x_1)^2 - x_2^2$ irr in $\mathbb{C}\langle x_1, x_2\rangle$, but not in $\mathbb{C}[x_1, x_2]$)

**Thm.** $L$ irreducible $\Longrightarrow$ $f_n = \det L(X)$ as a polynomial map on $M_n^g$ is irreducible for $\underbrace{n \gg 0}$.

"all $n$ larger than some $n_0$"

About the proof:

(1) $L$ of size $d \Rightarrow \exists \tilde{d} \leq d$ s.t. $\deg f_n = \tilde{d} n \quad \forall n >> 0$

(universal division algebras ...)

(2) $f_n = \det L_A(X)$ ← $g$ tuple

$\tilde{f}_n = \det L_{A;A_1A_2}(X;Y)$ ← $g+1$ tuple

Then for $n >> 0$: $\tilde{f}_n$ irr. $\Rightarrow f_{2n}$ irr.

(invariant thy & (1) & linearization)

(3) determinant itself is an irreducible polynomial

Put together:

$A_1, \dots, A_g$ generate $M_d \Rightarrow$ there exist words (products)

$w_{g+1}(A), \dots, w_{d^2}(A)$ such that $A_1, \dots, A_g, w_{g+1}(A), \dots w_{d^2}(A)$

is a basis for $M_d$ (assuming $A_j$ lin. indep.)

$\tilde{L} = I + A_1 x_1 + \dots + A_g x_g + w_{g+1}(A)\, x_{g+1} + \dots + w_{d^2}(A)\, x_{d^2}$,

pencil of size $d$ in $d^2$ variables

$\tilde{L}(\underbrace{\xi_1, \dots, \xi_{d^2}}_{\text{commuting variables}}) \quad \overbrace{\hphantom{xxxxxx}}^{\text{affine change of coordinates}} \quad \begin{pmatrix} \xi_1 & \cdots \\ \vdots & \ddots \\ \cdots & & \xi_{d^2} \end{pmatrix} \quad d \times d$

$\Rightarrow \det \tilde{L}(\xi_1, \dots, \xi_{d^2})$ is irreducible.

Now apply (2) to get rid of $w_i(A)$.

Corollary: The "persistent" components of $\mathcal{Z}(L)$ correspond to irreducible blocks in $L$.

## Smooth points

$\mathcal{Z}_n^1(L) = \{ X \in M_n^g : \dim \ker L(X) = 1 \}$ (geo mltp = 1)

$\mathcal{Z}_n^{1a}(L) = \{ X \in M_n^g : \dim \ker L(X)^2 = 1 \}$ (alg mltp = 1)

Prop. $L$ irr. Then for $n >> 0$:

$$\mathcal{Z}_n^{1a}(L) \subseteq \{ \text{smooth points of } \mathcal{Z}_n(L) \} \subseteq \mathcal{Z}_n^1(L)$$

## Application to LMIs

$L$ hermitian ($A_j^* = A_j$). Recall: $\mathcal{D}(L) = \{ X = X^* : L(X) \succeq 0 \}$

Distinguished boundary:

$$\partial^1 \mathcal{D}(L) = \{ X = X^* : L(X) \succeq 0 \ \& \ \dim \ker L(X) = 1 \}$$

Very important for free analytic maps between LMIs.

Cor. Let $L$ be hermitian and minimal for its LMI domain. Then for $n >> 0$,

$\partial^1 \mathcal{D}(L)$ is Zariski dense in $\mathcal{Z}_n(L)$.

(Kippenhahn conj: does it work for $n = 1$?
No; $8 \times 8$ counterexample by Laffey)