Time and band limiting: a bouquet of commuting miracles (some old, some very recent), and a new motivation to look for more of them

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A long term obsession with the problem of finding

BANDED matrices that commute with

NATURALLY APPEARING FULL MATRICES.

Full matrix = integral operator

Banded matrix = differential operator.

My "new motivation" comes from looking at two papers:

"Maximal violations of Bell inequalities by position measurements", J. Kiukas and R. Werner, 2009.

and

"Properties of the entanglement Hamiltonian for finite free-fermion chains", V. Eisler and I. Peschel, 2018.

Both of these papers exploit the commutativity property mentioned in the title of this talk in some physically important cases.

Here we let $\Delta_i \neq \emptyset$ be a compact interval for $i = 1, 2$. As already mentioned, ... $P_1(\mathcal{H}) = L^2(\Delta_1) \subset \mathcal{H}_0^{\perp}$ (see e.g. [5]), so the relevant subspace $\mathcal{K}$ is just $L^2(\Delta_1)$. It is convenient to choose the length scales as $l_i := d_i/2$, with d_i the length of Δ_i; passing to the units where l_1 is 1 as discussed above, we see that the relevant operator H is unitarily equivalent to

$$H_u := \chi_{[-1,1]}(Q)\chi_{[-u,u]}(P)\chi_{[-1,1]}(Q) \in \mathcal{B}\big(L^2([-1,1])\big),$$

where u is given by (19), i.e. $u = md_1d_2/(4t\hbar)$. (This equivalence can be seen easily by first applying the usual translation and "velocity boost" unitaries with appropriate shifts to center the intervals to the origin, and then dilating by $d_1/2$.)

The structure of H_u has been extensively studied because of its relevance in band- and timelimiting of signals (see, for instance [8, pp. 21-23], [11, pp. 121-132], [20], or the original papers by Landau, Pollack and Slepian [13, 17, 20]). We summarize the relevant mathematical facts briefly in the following paragraph.

The operator H_u is explicitly given by

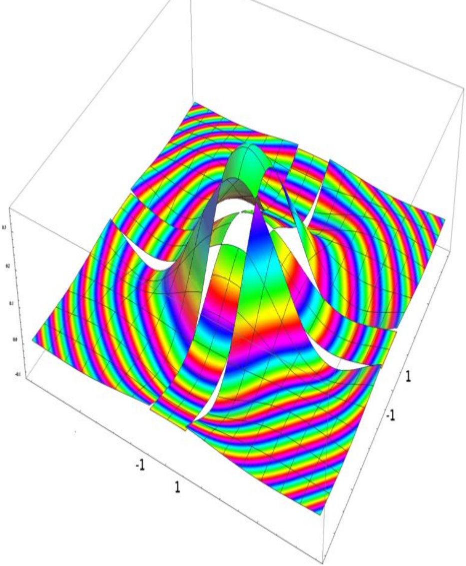
$$(H_u\varphi)(v) = \int_{-1}^{1} \frac{\sin(u(v-w))}{\pi(v-w)}\,\varphi(w)\,dw,\ \varphi \in L^2([-1,1]),$$

from which it follows that H_u commutes with the differential operator $\frac{d}{dv}\left[(1-v^2)\frac{d}{dv}\right] - u^2v^2$ that determines the angular part of the wave equation in prolate spheroidal coordinates. This differential operator has a complete orthonormal set of eigenfunctions $\psi_n^u \in L^2([-1,1])$, $n = 0, 1, \ldots$, called *angular prolate spheroidal wave functions*. In the notation of [19] we have $\psi_n^u(v) = \sqrt{n + \frac{1}{2}}\mathrm{Ps}_n(v,u)$. The corresponding eigenvalues $\lambda_n(u)$ of H_u are

$$\lambda_n(u) = 2u\pi^{-1}S_n^{(1)}(1,u)^2 \in (0,1),\ \ n = 0, 1, 2, \ldots, \tag{20}$$

where $S_n^{(1)}(\cdot, u)$ is the *radial* prolate spheroidal wave function of the first kind. In particular, $\lambda_n(u)$ depends continuously on u. In addition, we have $1 > \lambda_n(u) > \lambda_{n+1}(u) > 0$ for all n and u.

Now $\frac{1}{2} \in \sigma(H)$ exactly when u is chosen so as to make $\lambda_n(u) = \frac{1}{2}$ for some n. Since $\lim_{u\to\infty}\lambda_n(u) = 1$, and $\lim_{u\to\infty}\lambda_n(u) = 0$ for fixed n (see [20]), it follows by continuity that for each n we get at least one value $u_n \in (0,1)$ with $\lambda_n(u_n) = \frac{1}{2}$. On the other hand, $H_u \leq H_{u'}$ if $u \leq u'$, so each $\lambda_n(u)$ is an increasing function of u, and u_n is thus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The paper by Kiukas and Werner refers to work by Correggi and Morchio, where they consider not a free particle (Fourier analysis) but have an external potential.

The examples touched upon by Correggi-Morchio involve the harmonic oscillator, i.e. one should deal with Hermite functions and for them I showed a long time ago that the "time and band limiting" miracle holds. Moral: one could in principle obtain plots analogous to the one on Kiukas-Werner.

At the very end of the talk I will come back to the question of "trajectories" for quantum walks. **JOINT WORK WITH LUIS VELAZQUEZ AND JON WILKENING**

This is not meant to be controversial, but to show that the notion of "monitored evolution" allows one IN THE QUANTUM case to pose and answer some time honored questions for classical coins (where you have trajectories).

With this motivation, back into history

The problem of double concentration, i.e. localizing a function both in physical and frequency space, cuts across several areas of mathematics, physics and engineering.

This topic arises in harmonic analysis, signal processing and quantum mechanics and one is trying to find a good compromise between these two competing goals.

In some instances this issue gives rise to a sharply posed question, as was done (at least implicitly) by Claude Shannon:

if you know the frequency components over a band $[-\kappa, \kappa]$ for an unknown signal of finite support in $[-\tau, \tau]$, what is the best use you can make of this (usually noisy) data?

It is natural to look for the coefficients of an expansion of the unknown signal in terms of the **singular functions** of the problem.

However, one faces a serious computational difficulty: these singular functions are given naturally as the eigenfunctions of an integral operator with most of its eigenvalues crowded together.

As a consequence, one is dealing with huge matrices and the problem of computing the eigenvectors becomes extremely ill-conditioned. One gets total garbage even when using the best numerical packages.

In a remarkable series of papers written at Bell Labs in the 1960s, motivated by Shannon's question, a **mathematical miracle** was uncovered and exploited very successfully. We refer to it as the "**time-band limiting phenomenon**".

Surprising fact: certain naturally appearing integral operators admit second-order commuting differential ones. The spectrum of the differential operator is simple and thus any of its eigenfunctions is an eigenfunction of the integral operator.

The numerical computation of the eigenfunctions of this differential operator is trivial compared to trying to do this for the integral one: we can discretize to a **tridiagonal matrix** (and avoid 97 percent of the work when calling QR) and the eigenvalues are very **separated**. We are in true numerical paradise. **THERE ARE ALSO CONCEPTUAL CONSEQUENCES OF THIS MIRACLE**.

The three classical cases

$$(Ef)(z) = \int_{-\tau}^{\tau} e^{izx} f(x)dx, \quad z \in [-\kappa, \kappa].$$

The central issue is the effective computation of the eigenvectors of the integral operator

$$(EE^* f)(z) = 2 \int_{-\kappa}^{\kappa} \frac{\sin \tau(z - w)}{z - w} f(w)dw, \quad z \in [-\kappa, \kappa]. \qquad (1)$$

This problem was beautifully solved by Landau, Pollak and Slepian in the early 1960's by showing that the integral operator above commutes with the differential operator

$$R(z, \partial_z) = \partial_z(\kappa^2 - z^2)\partial_z - \tau^2,$$

from which they were able to describe their common eigenfunctions by using the differential operator. Note that $R(z, \partial_z)$ is the "radial part" of the Laplacian in prolate-spheroidal coordinates.

Consider the Bessel functions $f_\nu(x,k) = \sqrt{xk}J_\nu(xk)$.
The kernel

$$K_T(k_1,k_2) = \int_0^T f_\nu(x_1k_1)f_\nu(x_1k_2)dx$$

acting on $L^2(-G,G)$ was found by D. Slepian to admit a commuting differential operator, namely

$$A_\nu = -D_{k_1}(G^2 - k_1^2)D_{k_1} + k_1^2T^2 + G^2(\nu^2 - 1/4)/k_1^2.$$

The case of $\nu = 1/2$ (essentially) is very close to the original "prolate spheroidal" result.

In the early 1990's Tracy and Widom discovered one more remarkable commuting pair of integral and differential operators associated to the Airy kernel. They effectively used this pair and the one for the Bessel kernel in their study of the asymptotics of the level spacing distribution functions of the edge scaling limits of the Gaussian Unitary Ensemble and the Laguerre and Jacobi Ensembles. More precisely, Tracy and Widom proved that the integral operator with the Airy kernel

$$\frac{A(z)A'(w) - A'(z)A(w)}{z - w}$$

acting on $L^2(\tau, +\infty; dw)$ admits the commuting differential operator

$$\partial_z(\tau - z)\partial_z - z(\tau - z),$$

where $A(z)$ denotes the Airy function.

This phenomenon, featuring a pair of commuting integral and differential operators plays an important role in at least three areas of applied mathematics:

the problem of time-and-band limiting in signal processing, studied by Slepian, Landau and Pollak;

the problem of limited angle X-ray tomography (this is how I got into the problem): the gantry cannot go all the way around the patient.

and finally in Random Matrix Theory (Mehta 1967, Tracy and Widom 1994).

The three Fourier based cases

PSF:

$$\text{(1)} \qquad \int_{-T}^{T} \frac{\sin \sigma(t-\tau)}{\pi(t-\tau)} \phi_K(\tau)\, d\tau = \lambda_K \phi_K(t), \qquad K = 0, 1, \cdots.$$

DPSS:

$$\text{(2)} \qquad \sum_{n=-M}^{M} \frac{\sin \sigma(m-n)}{\pi(m-n)} \phi_K[n] = \lambda_K \phi_K[m], \qquad K = 0, \cdots, 2M.$$

P-DPSS:

$$\text{(3)} \qquad \sum_{n=0}^{M} \frac{\sin((2K+1)(m-n)\pi/N)}{N \sin((m-n)\pi/N)} \phi_i[n] = \lambda_i \phi_i[m], \qquad i = 0, \cdots, M.$$

My interaction with Slepian and Landau ....or what happens when you have to change to a smaller office after collecting junk for forty years

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April 25, 1979

Professor F. Alberto Grünbaum
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Dear Alberto:

It was a pleasure talking with you today. I'm sure we'll be communicating again in the future.

The reprint "Estimation, etc." enclosed details how I first solved the $\frac{\sin x}{x}$ integral equation. See Appendix 2. The reprint "Group Codes, etc." might be of interest to you as it poses some nice problems about finding point configurations spread out on the sphere.

In my recent paper on discrete prolates, the eigenvalues of

$$a_{ij} = \frac{\sin 2\pi W(i-j)}{\pi(i-j)}, \quad 0 < W < \tfrac{1}{2}$$

$$i,j = 0,1,\ldots,N-1$$

are considered. In dealing with the discrete Fourier transform as you proposed here, the matrix of interest is

$$\hat{a}_{ij} = \frac{\sin 2\pi \frac{M}{N}(i-j)}{\sin 2\pi \frac{(i-j)}{N}}, \qquad M < N$$

$$i,j = 1,2,\ldots,M' < N .$$

I haven't worked at this one ever.

The Russian paper I was trying to recall is Y. I. Khurgin and V. P. Yakovlev, "Progress in the Soviet Union on the Theory and Applications of Bandlimited Functions", Proceedings of the IEEE, Vol. 65, No. 7, pp. 1005-1029, July 1977. They touch on measurement and there is a large bibliography of the Russian literature.

Best regards,

MH-1218-DS-js

David Slepian

Enclosures

When doing expansions in terms of "classical orthogonal polynomials", Hermite, Jacobi and Laguerre

the "time-band limiting miracle" holds.

It also holds for functions on the sphere and expansions in spherical harmonics. (noncommutative Fourier)

One more, rather classical, source of examples

9. Hyperbolic spaces: Minkowski space. Another interesting class of examples is given by the hyperbolic spaces H_n of dimension n; see [8].

Just as the spheres S^{n-1} were obtained by factoring the group $SO(n)$ by $SO(n-1)$, one has

$$H_n = SO_0(n,1)/SO(n).$$

Here $SO(n,1)$ denotes the group of matrices acting on R^{n+1} that keep invariant the form $x_1^2+x_2^2+\cdots+x_n^2-x_{n+1}^2$. The subscript 0 refers to the component of $SO(n,1)$ which contains the identity.

The simplest examples are given by $n=2$ and $n=3$.

In the first case, H_2 is just the celebrated "upper half plane with hyperbolic metric" with a Laplace–Beltrami operator given by

$$\nabla^2 = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$

One usually thinks of this space as $SL(2,R)/SO(2)$.

In the second case, H_3 is the space of relativity theory, obtained by dividing the proper homogeneous Lorentz group by ordinary spatial rotations.

One can also think of this group as $SL(2,C)/SU(2)$.

We prefer to discuss only H_3 since the analysis is simpler in this instance; see [16].

We simplify even further by considering only radial functions in H_3. In this case the Laplace–Beltrami operator takes the form

$$\nabla^2 \equiv \frac{1}{\sinh^2 r}\frac{d}{dr}\left(\sinh^2 r \frac{d}{dr}\right),$$

and it gives a selfadjoint operator in $L^2((0,\infty), \sinh^2 r\, dr)$. A complete set of eigenfunctions is given by

$$\phi_\lambda(r) = \frac{\sin r\lambda}{\lambda \sinh r}, \qquad \lambda \geqq 0,$$

and we have

$$\nabla^2 \phi_\lambda(r) = -(1+\lambda^2)\phi_\lambda(r).$$

The operator $K_L(r_1, r_2)$ is obtained by

$$K_L(r_1, r_2) = \int_0^L \phi_\lambda(r_1)\phi_\lambda(r_2)\lambda^2\, d\lambda,$$

and one can easily see that

$$K_L(r_1, r_2) = \frac{\dfrac{\sin(r_1 - r_2)L}{r_1 - r_2} - \dfrac{\sin(r_1 + r_2)L}{r_1 + r_2}}{\sinh r_1 \sinh r_2}.$$

The analysis proceeds as in the Euclidean or spherical cases. Pick a ball of radius R in H_3 centered at $r = 0$.

At issue now is the existence of a second order differential operator D_r with leading coefficient vanishing at $r = R$ and satisfying

$$(D_{r_1} - D_{r_2})K_L(r_1, r_2) = 0.$$

One can check that the operator given by

$$D_r \equiv \frac{1}{\sinh^2 r}\frac{d}{dr}\left((R^2 - r^2)\sinh^2 r \frac{d}{dr}\right) + \left(2r - \frac{2re^r}{\sinh r} - r^2\right) + L^2(R^2 - r^2)$$

satisfies the desired relation.

The true reasons behind this remarkable algebraic "accident", deserve further study.

It appears that what is behind this miracle (in the **exceptional** cases when it holds) is the notion of

BISPECTRALITY

that will be introduced below.

Duistermaat, J. J., Grünbaum, F. A. "Differential equations in the spectral parameter". *Commun. Math. Phys.* **103** (1986), 177–240.

CRM Lectures notes
The bispectral problem

The bispectral problem

The problem as posed and solved with H. Duistermaat is as follows: Find all nontrivial instances where a function $\varphi(x,k)$ satisfies

$$L\left(x, \frac{d}{dx}\right)\varphi(x,k) \equiv (-D^2 + V(x))\varphi(x,k) = k\varphi(x,k)$$

as well as

$$B\left(k, \frac{d}{dk}\right)\varphi(x,k) \equiv \left(\sum_{i=0}^{M} b_i(k)\left(\frac{d}{dk}\right)^i\right)\varphi(x,k) = \Theta(x)\varphi(x,k).$$

All the functions $V(x), b_i(k), \Theta(x)$ are, in principle, arbitrary except for smoothness assumptions. Notice that here M is arbitrary (finite). The operator L could be of higher order, but I will stick now to order two.

The complete solution (when L has order two) is given as follows:

Theorem *If $M = 2$, then $V(x)$ is (except for translation) either c/x^2 or ax, i.e. we have a Bessel or an Airy case. If $M > 2$, there are two families of solutions*

a) *L is obtained from $L_0 = -D^2$ by a finite number of Darboux transformations ($L = AA^* \to \tilde{L} = A^*A$). In this case V is a rational solution of* the Korteweg deVries hierarchy of equations.

b) *L is obtained from $L_0 = -D^2 + \frac{1}{4x^2}$ after a finite number of rational Darboux transformations. In this case V is a rational solution of the Virasoro flows, also called the master symmetries of KdV.*

Case A:

$$V(x) = \frac{2}{(x+t^{1/3})^2} + \frac{2}{(x+\omega\cdot t^{1/3})^2} + \frac{2}{(x+\omega^2 t^{1/3})^2}$$
$$= -2\partial_x^2 \log(x^3+t), \quad \omega = e^{2\pi i/3}, \tag{1.38}$$

with the eigenfunction

$$\phi(x,\lambda) = e^{ikx}\frac{x^3 - 3x^2/ik - 3x/k^2 + t}{x^3+t}, \quad k^2 = \lambda, \tag{1.39}$$

satisfying the differential equation

$$\left[\left(-\partial_k^2 + \frac{6}{k^2}\right)^2 - 4ti\partial_k\right]\phi = (x^4 + 4tx)\phi, \tag{1.40}$$

the common solution space of (1.40) and (1.2) being one-dimensional.

Case B:

$$V(x) = -\frac{1}{4}\frac{1}{x^2} + \frac{2}{(x+i\sqrt{t})^2} + \frac{2}{(x-i\sqrt{t})^2}, \tag{1.41}$$

with the eigenfunctions

$$\phi(x,\lambda) = \psi'(kx) - \frac{3x^2 - t}{2kx(t+x^2)} \cdot \psi(kx), \qquad k^2 = \lambda, \tag{1.42}$$

where ψ is any solution of the Bessel equation

$$\left(-\partial_y^2 + \frac{3}{4}\frac{1}{y^2}\right)\psi(y) = \psi(y). \tag{1.43}$$

$\phi(x,\lambda)$ satisfies the differential equation

$$\left[\left(-\partial_k^2 + \frac{15}{4}\frac{1}{k^2}\right)^2 + 2t\left(-\partial_k^2 - \frac{1}{4}\frac{1}{k^2}\right)\right]\phi = (x^4 + 2tx^2)\cdot\phi, \tag{1.44}$$

and the common solution space of (1.44) and (1.2) is two-dimensional.

Extensions of the "classical results" by using bispectrality

From $L_0 = -\partial_x^2 - (1/4x^2)$ by two Darboux transformations, one gets the operator

$$L_2 = -\partial_x^2 - \frac{1}{4x^2} + \frac{4(x^2 - t_1)}{(x^2 + t_1)^2} \ .$$

In this case $\Theta(x) = x^4 + 2t_1 x^2$ and the differential operator in the spectral parameter (given above) is

$$B_2(k, \partial_k) = \left(-\partial_k^2 + \frac{15}{4k^2}\right)^2 + 2t_1 \left(-\partial_k^2 - \frac{1}{4k^2}\right)$$

If one builds the integral kernel out of the eigenfunctions of L_2, namely

$$f^{(1)}(x,k,t) = (1/k)(D_x + (t-3x^2)/(2x(x^2+t)))\sqrt{kx}J_1(kx)$$

then one can see that for each t there exists a **fourth-order** commuting differential operator for this integral one. Moreover, if we call $A(t)$ the differential operator in question, we have that

$$A(t) = 2G^2A_0t + (A_2^2 - 3/2A_2 - \frac{11}{2}G^2T^2).$$

where A_ν are the operators found by D. Slepian for the Bessel cases.

Some **very recent** results involving the Korteweg-deVries solutions.

Let $r \in \mathbb{R}^*$. Consider the function

$$\psi(x, z) = \frac{(x + z^{-1})^3 - z^3 + r}{x^3 + r} e^{-xz},$$

which up to a change of variables is precisely the first nontrivial bispectral function in the paper with H. Duistermaat, given on Eq. (1.39). The integral operator $\mathcal{E}_\psi \mathcal{E}_\psi^*$ has kernel

$$K_\psi(z, w) = \frac{\psi(s, z)\psi_x(s, w) - \psi_x(s, z)\psi(s, w)}{z^2 - w^2}.$$

The commuting differential operator -independent of the parameter r- is given by

$$R_{s,t}(z, \partial_z) = \sum_{m=0}^{3} \partial_z^m f_m(z) \partial_z^m,$$

where

$$\begin{aligned}
f_0(z) &= \frac{z^2(3s^6t^3 - 54s^4t)}{6} + s^6z^5 - \frac{3s^6tz^4}{2} + 12s^4z^3, \\
f_1(z) &= (z-t)\left(3s^4z^4 - 3s^4tz^3 + 12s^2z^2 + 9s^2tz - 9s^2t^2\right), \\
f_2(z) &= (z-t)^2\left(3s^2z^3 - \frac{3s^2tz^2}{2} + 12t\right), \\
f_3(z) &= (z-t)^3z^2.
\end{aligned}$$

Looking for a general method

There is a short and elegant paper by Perline given a constructive way to go from bisectrality to the commuting differential operator in question.

A very recent pair of papers

Algebraic Heun Operator and Band-Time Limiting
Grünbaum, F.A., Vinet, L., Zhedanov, A.

Bispectrality and Time-Band-Limiting: Matrix-valued polynomials
A.Grünbaum, I. Pacharoni and I. Zurrián

shows that the ideas of Perline can be extended to other scenarios, with extra conditions in the matrix valued case.

See also

The CMV bispectral problem, A. Grünbaum, Luis Velazquez.

A general result for real valued symmetric Toeplitz matrices, rather sad

$$M(i,j) = r_{|i-j|} = \frac{\sin(\alpha)(i-j)}{\sin(\beta)(i-j)},$$

New examples for matrices that are not Toeplitz, but Hankel

$$\begin{pmatrix} 1 & \frac{\sqrt{q}}{q+1} & \frac{q}{q^2+q+1} \\ \frac{\sqrt{q}}{q+1} & \frac{q}{q^2+q+1} & \frac{q^{\frac{3}{2}}}{q^3+q^2+q+1} \\ \frac{q}{q^2+q+1} & \frac{q^{\frac{3}{2}}}{q^3+q^2+q+1} & \frac{q^2}{q^4+q^3+q^2+q+1} \end{pmatrix}$$

$$\begin{pmatrix} \sqrt{q} & \frac{q^3+q}{2\,q^3+q^2+q+2} & 0 \\ \frac{q^3+q}{2\,q^3+q^2+q+2} & 0 & \frac{q^4+q^3+q^2+q+1}{2\,q^3+q^2+q+2} \\ 0 & \frac{q^4+q^3+q^2+q+1}{2\,q^3+q^2+q+2} & -\frac{q^4-2\,q^3+2\,q^2-2\,q+1}{\sqrt{q}\,(2\,q^2-q+2)} \end{pmatrix}$$

$$
\left(\begin{array}{ccccc}
1 & \frac{\sqrt{q}}{q+1} & \frac{q}{q^2+q+1} & \frac{q^{\frac{3}{2}}}{q^3+q^2+q+1} & \frac{q^{2}}{q^4+q^3+\cdots} \\
\frac{\sqrt{q}}{q+1} & \frac{q}{q^2+q+1} & \frac{q^{\frac{3}{2}}}{q^3+q^2+q+1} & \frac{q^2}{q^4+q^3+q^2+q+1} & \frac{q^{\frac{5}{2}}}{q^5+q^4+q^3\cdots} \\
\frac{q}{q^2+q+1} & \frac{q^{\frac{3}{2}}}{q^3+q^2+q+1} & \frac{q^2}{q^4+q^3+q^2+q+1} & \frac{q^{\frac{5}{2}}}{q^5+q^4+q^3+q^2+q+1} & \frac{q^{3}}{q^6+q^5+q^4+\cdots} \\
\frac{q^{\frac{3}{2}}}{q^3+q^2+q+1} & \frac{q^2}{q^4+q^3+q^2+q+1} & \frac{q^{\frac{5}{2}}}{q^5+q^4+q^3+q^2+q+1} & \frac{q^3}{q^6+q^5+q^4+q^3+q^2+q+1} & \frac{q^{\frac{7}{2}}}{q^7+q^6+q^5+q^4\cdots} \\
\frac{q^2}{q^4+q^3+q^2+q+1} & \frac{q^{\frac{5}{2}}}{q^5+q^4+q^3+q^2+q+1} & \frac{q^3}{q^6+q^5+q^4+q^3+q^2+q+1} & \frac{q^{\frac{7}{2}}}{q^7+q^6+q^5+q^4+q^3+q^2+q+1} & \frac{q^{4}}{q^8+q^7+q^6+q^5+\cdots}
\end{array}\right.
$$

$$
\begin{pmatrix}
0 & -\dfrac{\sqrt{q}\left(q^{4}-q^{3}+q^{2}\right)}{2\,q^{7}+q^{6}+2\,q^{5}+q^{4}+q^{3}+2\,q^{2}+q+2} & 0 \dots \\
-\dfrac{\sqrt{q}\left(q^{4}-q^{3}+q^{2}\right)}{2\,q^{7}+q^{t}iny6+2\,q^{5}+q^{4}+q^{3}+2\,q^{2}+q+2} & \dfrac{2\,q^{8}+4\,q^{6}-q^{5}+4\,q^{4}+2\,q^{2}}{2\,q^{10}+q^{9}+6\,q^{8}+q^{7}+8\,q^{6}+8\,q^{4}+q^{3}+6\,q^{2}+q+2} & -\dfrac{\sqrt{q}\left(q^{7}+q^{6}+q^{5}\dots\right.}{2\,q^{9}+q^{8}+4\,q^{7}+2\,q^{6}+3\dots} \\
0 & -\dfrac{\sqrt{q}\left(q^{7}+q^{6}+q^{5}+q^{4}+q^{3}+q^{2}+q\right)}{2\,q^{9}+q^{8}+4\,q^{7}+2\,q^{6}+3\,q^{5}+3\,q^{4}+2\,q^{3}+4\,q^{2}+q+2} & \dfrac{2\,q^{5}-2\,q^{4}+3\dots}{2\,q^{6}-3\,q^{5}+6\,q^{4}-\dots} \\
0 & 0 & -\dfrac{\sqrt{q}\left(\dots\right.}{2\,q^{5}-q^{4}+\dots} \\
0 & 0 & 0 \dots
\end{pmatrix}
$$

$$
\begin{pmatrix}
\frac{4}{31} & \frac{2^{\frac{5}{2}}}{63} & \frac{8}{127} & \frac{2^{\frac{7}{2}}}{255} & \frac{16}{511} & \frac{2^{\frac{9}{2}}}{1023} & \frac{32}{2047} & \frac{2^{\frac{11}{2}}}{4095} \\
\frac{2^{\frac{5}{2}}}{63} & \frac{8}{127} & \frac{2^{\frac{7}{2}}}{255} & \frac{16}{511} & \frac{2^{\frac{9}{2}}}{1023} & \frac{32}{2047} & \frac{2^{\frac{11}{2}}}{4095} & \frac{64}{8191} \\
\frac{8}{127} & \frac{2^{\frac{7}{2}}}{255} & \frac{16}{511} & \frac{2^{\frac{9}{2}}}{1023} & \frac{32}{2047} & \frac{2^{\frac{11}{2}}}{4095} & \frac{64}{8191} & \frac{2^{\frac{13}{2}}}{16383} \\
\frac{2^{\frac{7}{2}}}{255} & \frac{16}{511} & \frac{2^{\frac{9}{2}}}{1023} & \frac{32}{2047} & \frac{2^{\frac{11}{2}}}{4095} & \frac{64}{8191} & \frac{2^{\frac{13}{2}}}{16383} & \frac{128}{32767} \\
\frac{16}{511} & \frac{2^{\frac{9}{2}}}{1023} & \frac{32}{2047} & \frac{2^{\frac{11}{2}}}{4095} & \frac{64}{8191} & \frac{2^{\frac{13}{2}}}{16383} & \frac{128}{32767} & \frac{2^{\frac{15}{2}}}{65535} \\
\frac{2^{\frac{9}{2}}}{1023} & \frac{32}{2047} & \frac{2^{\frac{11}{2}}}{4095} & \frac{64}{8191} & \frac{2^{\frac{13}{2}}}{16383} & \frac{128}{32767} & \frac{2^{\frac{15}{2}}}{65535} & \frac{256}{131071} \\
\frac{32}{2047} & \frac{2^{\frac{11}{2}}}{4095} & \frac{64}{8191} & \frac{2^{\frac{13}{2}}}{16383} & \frac{128}{32767} & \frac{2^{\frac{15}{2}}}{65535} & \frac{256}{131071} & \frac{2^{\frac{17}{2}}}{262143} \\
\frac{2^{\frac{11}{2}}}{4095} & \frac{64}{8191} & \frac{2^{\frac{13}{2}}}{16383} & \frac{128}{32767} & \frac{2^{\frac{15}{2}}}{65535} & \frac{256}{131071} & \frac{2^{\frac{17}{2}}}{262143} & \frac{512}{524287}
\end{pmatrix}
$$

$$
\begin{pmatrix}
0 & -\frac{1040257\, 2^{\frac{7}{2}}}{869528517} & 0 & 0 & 0 & 0 \\
-\frac{1040257\, 2^{\frac{7}{2}}}{869528517} & \frac{31590384}{816829819} & -\frac{65024127\, 2^{\frac{5}{2}}}{8985128009} & 0 & 0 & 0 \\
0 & -\frac{65024127\, 2^{\frac{5}{2}}}{8985128009} & \frac{79870808}{691163693} & -\frac{6223\, 2^{\frac{3}{2}}}{185757} & 0 & 0 \\
0 & 0 & -\frac{6223\, 2^{\frac{3}{2}}}{185757} & \frac{2373852460}{8985128009} & -\frac{1253356875\, \sqrt{2}}{8985128009} & 0 \\
0 & 0 & 0 & -\frac{1253356875\, \sqrt{2}}{8985128009} & \frac{440517770}{816829819} & -\frac{468840967}{869528517\, \sqrt{2}} \\
0 & 0 & 0 & 0 & -\frac{468840967}{869528517\, \sqrt{2}} & 1 \\
0 & 0 & 0 & 0 & 0 & -\frac{49545027}{26349349\, 2^{\frac{3}{2}}} \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

A different topic, just a sketch of work in progress.

The arcsine law for the Hadamard walk

Paul Levy for Brownian motion and Chung-Feller for the coin tossing game

Joint work with **Luis Velazquez and Jon Wilkening.**

How to make sense of "occupation times in a given subspace" for a discrete time quantum walk?

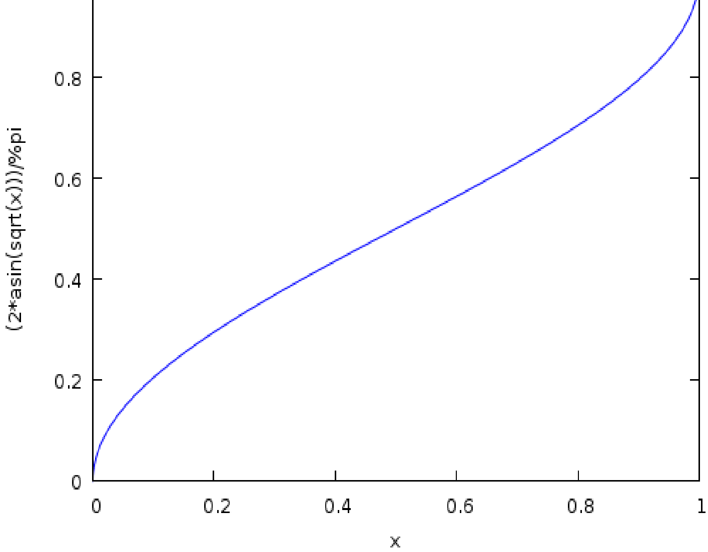
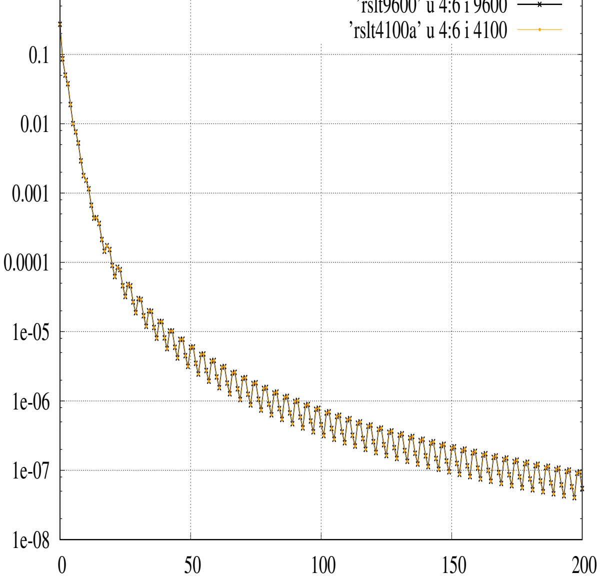


Figure: a plot of the cummulative distribution of the "proportion of time spent on the positive half line"

If you start a classical coin at the origin and run it for time n, the probability that it spends 0 time on the positive side behaves like $1/\sqrt{n}$.

In the quantum case things are very different.

0	$\frac{3(\pi-2)}{4\pi}$
1	$\frac{3(\pi-2)}{4\pi^2}$
— —	— —
5	$\frac{(\pi^4+40\pi^2+48)}{64\pi^6}(\pi-2)$
— —	— —
8	$\frac{(\pi^6+11\pi^4+32\pi^2+24)}{32\pi^9}(\pi-2)$



'rslt9600' u 4:6 i 9600
'rslt4100a' u 4:6 i 4100
0.1
0.01
0.001
0.0001
1e-05
1e-06
1e-07
1e-08
0
50
100
150
200

The arcsine law of Paul Levy is replaced by some extreme version of it.

Its "density" has a pair of deltas of strength 1/2 at each end point.

Ballistic behaviour.