The X-ray transform on Anosov manifolds

A survey of recent results

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April 15th 2019

- $(\mathcal{M}, g)$ smooth closed connected n-dimensional Riemannian manifold,
- $X \in C^\infty(\mathcal{M}, T\mathcal{M})$ smooth vector field generating a transitive Anosov flow $(\varphi_t)_{t\in\mathbb{R}}$, i.e. such that there exists a continuous flow-invariant splitting

$$T\mathcal{M} = E_s \oplus E_u \oplus \mathbb{R}X,$$

with:

$$\|\mathrm{d}\varphi_t(v)\| \leq Ce^{-\lambda t}\|v\|, \ \forall v \in E_s, \forall t \geq 0,$$
$$\|\mathrm{d}\varphi_t(v)\| \leq Ce^{-\lambda |t|}\|v\|, \ \forall v \in E_u, \forall t \leq 0,$$

where the constants $C, \lambda > 0$ are uniform, $\| \cdot \| = g(\cdot, \cdot)^{1/2}$,

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- $\mathcal{G}$ set of periodic orbits.

Definition (X-ray transform)

$$I : C^0(\mathcal{M}) \to \ell^\infty(\mathcal{G}), \qquad If : \mathcal{G} \ni \gamma \mapsto \langle \delta_\gamma, f \rangle := \frac{1}{\ell(\gamma)} \int_0^{\ell(\gamma)} f(\varphi_t z) \mathrm{d}t,$$

where $z \in \gamma$, $\ell(\gamma)$ is the period of γ.

- Definition can be restricted to other regularities: C^α (Hölder), H^s (Sobolev) for $s > \frac{n}{2}$, ...
- **Question:** can we describe the kernel of I on functions with prescribed regularity?
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Theorem (Livsic '72)

Let $\alpha \in (0,1)$. Given $f \in C^\alpha(\mathcal{M})$ such that $If = 0$, there exists $u \in C^\alpha(\mathcal{M})$ such that $f = Xu$. Moreover, u is unique up to an additive constant.

Classical Livsic theorem was also proved:

- in smooth regularity i.e. $f, u \in C^\infty(\mathcal{M})$ (**de la Llave-Marco-Moriyon '86**),
- in Sobolev regularity i.e. $f, u \in H^s(\mathcal{M})$ (**Guillarmou '17**).

Other natural questions:

- What if $If \geq 0$ instead of $If = 0$? (Positive version of Livsic theorem)
- What if $If \simeq \varepsilon$ (i.e. $\|If\|_{\ell^\infty} := \sup_{\gamma \in \mathcal{G}} |If(\gamma)| \leq \varepsilon$)? (Approximate Livsic theorem)
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Theorem (Lopes-Thieullen '04, Positive Livsic theorem)

Let $\alpha \in (0,1)$. There exists $\beta \in (0,\alpha)$, $C > 0$ such that the following holds. Let $f \in C^\alpha(\mathcal{M})$ such that $If \geq 0$. Then, there exists $u, h \in C^\beta(\mathcal{M})$ such that $Xu \in C^\beta(\mathcal{M})$, $h \geq 0$ and $f = Xu + h$. (In particular, $f \geq Xu$.) Moreover, $\|h\|_{C^\beta} + \|Xu\|_{C^\beta} \leq C\|f\|_{C^\alpha}$.

Theorem (Gouëzel-L. '19, Approximate Livsic theorem)

Let $\alpha \in (0,1)$. There exists $\beta \in (0,\alpha), \nu > 0$ such that the following holds. For any $\varepsilon > 0$ small enough, given $f \in C^\alpha(\mathcal{M})$ such that $\|f\|_{C^\alpha} \leq 1$ and $\|If\|_{\ell^\infty} \leq \varepsilon$, there exists $u, h \in C^\beta(\mathcal{M})$ such that $Xu \in C^\beta(\mathcal{M})$, $\|h\|_{C^\beta} \leq \varepsilon^\nu$ and $f = Xu + h$.

Theorem (Gouëzel-L. '19, Finite Livsic theorem)

Let $\alpha \in (0,1)$. There exists $\beta \in (0,\alpha), \mu > 0$ such that the following holds. For any $L > 0$ large enough, given $f \in C^\alpha(\mathcal{M})$ such that $\|f\|_{C^\alpha} \leq 1$ and $If(\gamma) = 0$ for all $\gamma \in \mathcal{G}$ such that $\ell(\gamma) \leq L$, there exists $u, h \in C^\beta(\mathcal{M})$ such that $Xu \in C^\beta(\mathcal{M})$, $\|h\|_{C^\beta} \leq L^{-\mu}$ and $f = Xu + h$. This implies that $\|If\|_{\ell^\infty} \leq L^{-\mu}$.

(Second theorem is actually a corollary of the proof of the first one.)

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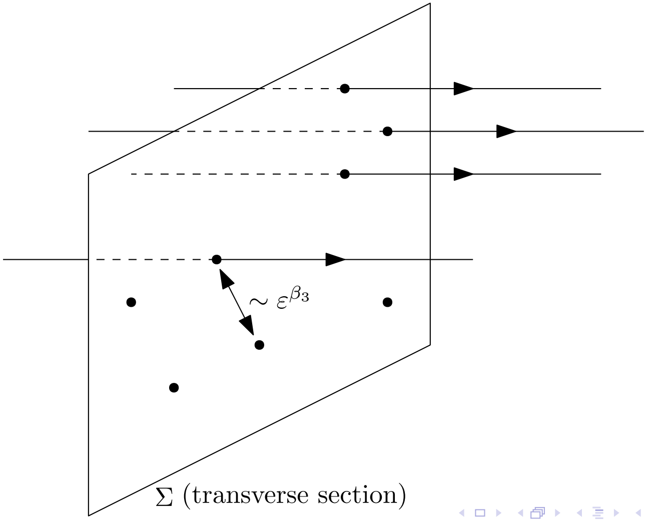
Idea for the first theorem: find a periodic orbit of length $\varepsilon^{-\beta_1}$ ($\beta_1 < 1$) that is ε^{β_2}-dense in $\mathcal{M}$ and yet ε^{β_3}-separated. Then, mimick the proof of the classical Livsic theorem.



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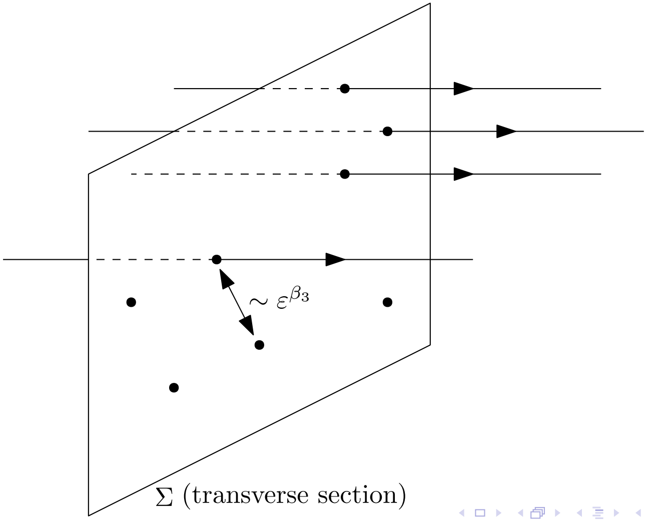

- (M, g) smooth connected closed Riemannian manifold with Anosov geodesic flow $(\varphi_t)_{t\in\mathbb{R}}$ on its unit tangent bundle $\mathcal{M} := SM$. We call (M, g) an Anosov Riemannian manifold.
- $\mathcal{C}$ is the set of free homotopy classes; there exists a unique closed geodesic in each free homotopy class $c \in \mathcal{C}$ (**Klingenberg '74**). We identify $\mathcal{G}$ and $\mathcal{C}$.
- $C^\infty(M, \otimes_S^m T^*M)$ is the vector-space of smooth symmetric m-tensors ($m \in \mathbb{N}$).

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- Symmetric tensors on M can be seen as functions on the unit tangent bundle SM, polynomial in the spheric variable. Given $m \in \mathbb{N}, f \in C^0(M, \otimes_S^m T^*M)$, we define $\pi_m^* f \in C^0(SM)$ by $\pi_m^* f : (x, v) \mapsto f_x(v, ..., v)$.

Definition (Geodesic X-ray transform)

$$I_m : C^0(M, \otimes_S^m T^*M) \to \ell^\infty(\mathcal{C}),$$

$$I_m f = I \pi_m^* f : \mathcal{C} \ni c \mapsto \frac{1}{\ell(\gamma_c)} \int_0^{\ell(\gamma_c)} f_{\gamma_c(t)}(\dot{\gamma}_c(t), ..., \dot{\gamma}_c(t)) \mathrm{d}t,$$

with γ_c unique closed geodesic in c.

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- **Question:** Kernel of the X-ray transform?

- Tensor decomposition:

$$f = Dp + h,$$

with $D := \sigma \circ \nabla$ (∇ Levi-Civita connexion, σ symmetrization operator of tensors), $D^* h = 0$ where D^* is the formal adjoint of D. We call Dp the potential part and h the solenoidal part of f.

- $I_m(Dp) = 0$, that is $\{\text{potential tensors}\} \subset \ker I_m$. I_m is said to be s(olenoidal)-injective when this is an equality.

Conjecture

I_m is s-injective whenever (M, g) is an Anosov Riemannian manifold.

Known results when (M, g) Anosov; I_m is s-injective for:

- any $m \in \mathbb{N}$ on surfaces (**Paternain-Salo-Uhlmann '14, Guillarmou '17**),
- any $m \in \mathbb{N}$ in any dimension, in nonpositive curvature (**Croke-Sharafutdinov '98**),
- $m = 0, 1$ in any dimension (**Dairbekov-Sharafutdinov-11**).

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- any $m \in \mathbb{N}$ in any dimension, in nonpositive curvature (**Croke-Sharafutdinov '98**),
- $m = 0, 1$ in any dimension (**Dairbekov-Sharafutdinov-11**).

- **Question:** Once we have s-injectivity, can we obtain a stability estimate of the form

$$\|f\|_{\mathcal{H}_1} \leq C\|I_m f\|_{\mathcal{H}_2}, \quad \forall f \text{ solenoidal},$$

for some well-chosen spaces $\mathcal{H}_{1,2}$?

Theorem (Guillarmou-L. '18, Gouëzel-L. '19)

For all exponents $n/2 < s < r$, there exists $C, \nu > 0$ such that the following holds. For all solenoidal tensors f such that $\|f\|_{H^r} \leq 1$, one has:

$$\|f\|_{H^s} \leq C\|I_m f\|_{\ell^\infty}^{\nu}$$

Now, recall the Finite Livsic theorem:

Theorem (Gouëzel-L. '19, Finite Livsic theorem)

For any $L > 0$ large enough, given $f \in C^\alpha(\mathcal{M})$ such that $\|f\|_{C^\alpha} \leq 1$ and $If(\gamma) = 0$ for all $\gamma \in \mathcal{G}$ such that $\ell(\gamma) \leq L$, there exists $u, h \in C^\beta(\mathcal{M})$ such that $Xu \in C^\beta(\mathcal{M})$, $\|h\|_{C^\beta} \leq L^{-\mu}$ and $f = Xu + h$. This implies that $\|If\|_{\ell^\infty} \leq L^{-\mu}$.

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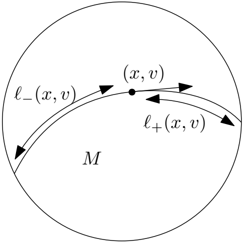
Combining the two previous theorems, we obtain the

Corollary (Gouëzel-L. '19)

For all exponents $n/2 < s < r$, there exists $\mu > 0$ such that the following holds. For any $L > 0$ large enough, given any solenoidal tensor f such that $\|f\|_{H^r} \leq 1$ and $I_m f(c) = 0$ for all $c \in \mathcal{C}$ such that $\ell(\gamma_c) \leq L$, one has: $\|f\|_{H^s} \leq L^{-\mu}$. (In particular, $L = +\infty$ is the Classical Livsic theorem.)

- The X-ray transform I has bad analytic properties (in particular, it maps to functions on a discrete set).



- **Idea (Guillarmou '17):** Mimick the case of a simple manifold with boundary (smwb). On a smwb, we can write the normal operator

$$I^*I = \int_{-\infty}^{+\infty} e^{tX} dt$$

(i.e. $I^*If(x,v) = \int_{\ell_-(x,v)}^{\ell_+(x,v)} f(\varphi_t(x,v))dt$). Then

$$I_m^* I_m = \pi_{m*} I^* I \pi_m^*$$

is a ΨDO of order -1, elliptic on solenoidal tensors.

- If $R_\pm(\lambda) := (X \pm \lambda)^{-1}$ denotes the resolvent of the generator of the geodesic flow, then $I^*I = R_+(0) - R_-(0)$.

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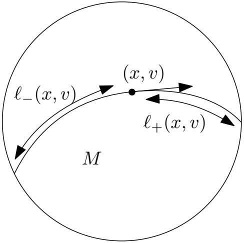

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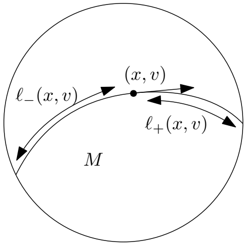

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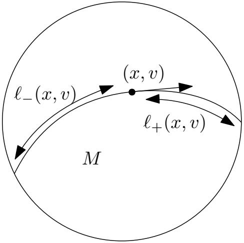

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Same construction works on a closed manifold:

- $R_\pm(\lambda) := (X \pm \lambda)^{-1}$ (initially defined for $\Re(\lambda) > 0$) can be meromorphically extended to the whole complex plane (**Faure-Sjöstrand '11**, **Dyatlov-Zworski '16**).
- $R_\pm(\lambda)$ have a pole of order 1 at 0; we denote by $R_0^\pm$ the holomorphic part of $R_\pm(\lambda)$ at $\lambda = 0$. We set:

$$\Pi := R_0^+ - R_0^- + \mathbf{1} \otimes \mathbf{1}$$

  Π is the analogue of I^*I. One has $\Pi X = 0 = X\Pi$.
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Theorem (Guillarmou '17, Guillarmou-L. '18, Gouëzel-L. '19, Properties of Π_m)

- *Π_m is a ΨDO of order -1, elliptic on solenoidal tensors,*
- *$\Pi_m D = 0 = D^* \Pi_m$,*
- *If I_m is s-injective, then Π_m is s-injective; moreover, there exists a ΨDO of order 1 such that $P\Pi_m = \pi_{\ker D^*}$, where $\pi_{\ker D^*}$ is the L^2-projection on solenoidal tensors. This implies:*

$$\|f\|_{H^s} \leq C\|\Pi_m f\|_{H^{s+1}}, \quad \forall f \text{ solenoidal}$$

- *Π is non-negative i.e. $\langle \Pi f, f\rangle_{L^2} \geq 0$ and Π_m is coercive i.e. $\langle \Pi_m f, f\rangle_{L^2} \geq C\|f\|^2_{H^{-1/2}}$ for all solenoidal f.*

- Here, $L^2 = L^2(SM, d\mu)$, where μ is the Liouville measure.
- For $f \in C^\infty(SM)$, we call $\mathrm{Var}(f) := \langle \Pi f, f\rangle_{L^2}$ the variance of f with respect to the Liouville measure. And for $f \in C^\infty(M, \otimes^m_S T^*M)$, $\mathrm{Var}_m(f) = \langle \Pi_m f, f\rangle_{L^2}$.

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So $\|f\|_{H^s} \leq \|I_m f\|^\nu_{\ell^\infty}$.

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So $\|f\|_{H^s} \leq \|I_m f\|_{\ell^\infty}^{\nu}$.

We apply the previous results in the case $m = 2$.

- (M, g) is an Anosov Riemannian manifold,
- $\mathcal{C}$ is the set of free homotopy classes; given $c \in \mathcal{C}$, there exists a unique closed geodesic $\gamma_c \in c$ (**Klingenberg '74**).

Definition (The marked length spectrum)

$$L_g : \left| \begin{array}{rl} \mathcal{C} & \to \mathbb{R}_+^* \\ c & \mapsto \ell_g(\gamma_c), \end{array} \right.$$

$\ell_g(\gamma_c)$ Riemannian length computed with respect to g.

Conjecture (Burns-Katok '85)

The marked length spectrum of a negatively-curved manifold determines the metric (up to isometries) i.e.: if g and g' have negative sectional curvature, same marked length spectrum $L_g = L_{g'}$, then there exists $\phi : M \to M$ smooth diffeomorphism such that $\phi^* g' = g$.

- The action of diffeomorphisms is a natural obstruction one cannot avoid,
- Analogue of Michel's conjecture of rigidity for simple manifolds with boundary (the boundary distance function should determine the metric up to isometries),
- Why the marked length spectrum ? The length spectrum (:= collection of lengths regardless of the homotopy) does not determine the metric (counterexamples by **Vigneras '80**)

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Known results:

- **Croke '90**, **Otal '90**: proof for negatively-curved surfaces,
- **Katok '88**: proof for g' conformal to g,
- **Besson-Courtois-Gallot '95**, **Hamenstädt '99**: proof when (M, g) is a locally symmetric space.
- Conjecture remains open in dimension > 2 for negatively-curved manifolds and in any dimension for Anosov Riemannian manifolds.

Theorem (Guillarmou-L. '18)

Let (M, g_0) be a negatively-curved manifold. Then $\exists k \in \mathbb{N}^*, \mathcal{U}$ open C^k-neighborhood of g_0 such that: if $g \in \mathcal{U}$ and $L_g = L_{g_0}$, then g is isometric to g_0.

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We pick g in a neighborhood of g_0. Ideas of the proof:

- Solenoidal reduction: there exists a diffeomorphism $\phi : M \to M$ such that $g' := \phi^* g$ is solenoidal. (Without loss of generality, we can assume g is solenoidal at the beginning.)
- Use a Taylor expansion of the marked length spectrum:

$$\frac{L_g}{L_{g_0}} = 1 + \frac{1}{2} I_2^{g_0}(g - g_0) + \mathcal{O}(\|g - g_0\|^2_{C^3}),$$

  thus, if $L_g = L_{g_0}$, $\|I_2^{g_0}(g - g_0)\|_{\ell^\infty} = \mathcal{O}(\|g - g_0\|^2_{C^3})$.
- Then, use the stability estimates on I_2

$$\|g - g_0\|_{H^s} \leq \|I_2^{g_0}(g - g_0)\|^{\nu}_{\ell^\infty}$$

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We present another proof of (a refined version) this Theorem. Let us pretend we do not know the previous proof.

Theorem (Guillarmou-Knieper-L. '19)

Let (M, g_0) be a negatively-curved manifold. Then $\exists k \in \mathbb{N}^*, \mathcal{U}$ open $\mathcal{C}^k$-neighborhood of g_0 such that: if $g \in \mathcal{U}$ and

$$\lim_{j \to +\infty} \frac{L_g(c_j)}{L_{g_0}(c_j)} \to 1,$$

for all sequences of closed geodesics $(\gamma_{c_j})_{c \in \mathbb{N}}$ such that $L_{g_0}(c_j) \to \infty$, then g is isometric to g_0.

For simplicity, we denote this assumption by $L_g / L_{g_0} \to 1$.

- The geodesic flows $(\varphi_t^{g_0})_{t\in\mathbb{R}}$ and $(\varphi_t^{g})_{t\in\mathbb{R}}$ are orbit-conjugate, that is there exists a homeomorphism $\psi_g : SM \to SM$ (differentiable in the flow direction) such that $\mathrm{d}\psi_g(X_{g_0}) = a_g X_g$,
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Definition (Geodesic stretch)

The geodesic stretch of g with respect to the Liouville measure $\mathrm{d}\mu_{g_0}$ is

$$\mathcal{I}_{\mathrm{d}\mu_{g_0}}(g_0, g) := \int_{SM} a_g \, \mathrm{d}\mu_{g_0}$$

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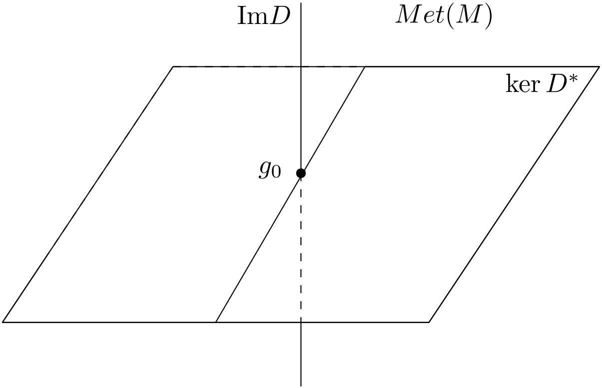
Let us do some "geometry" in $Met(M)$, the space of metrics on M.

- $\mathrm{Met}(M) \subset C^\infty(M, \otimes_S^2 T^*M)$, $\mathcal{O}(g_0) := \{\phi^* g_0 \mid \phi \in \mathrm{Diff}_0(M)\}$,
- $T_{g_0}\mathrm{Met}(M) \simeq C^\infty(M, \otimes_S^2 T^*M) \simeq \ker D_{g_0}^* \oplus \mathrm{Im}\ D_{g_0} \simeq \ker D_{g_0}^* \oplus T_{g_0}\mathcal{O}(g_0)$



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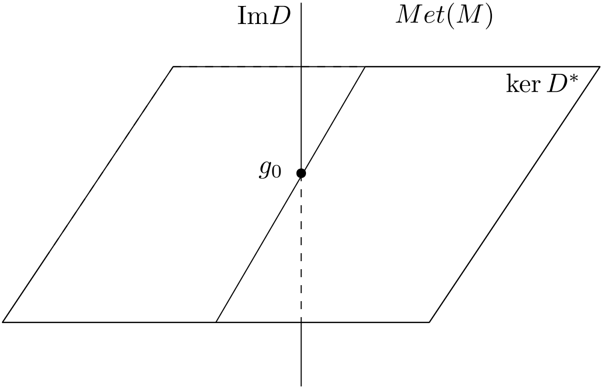
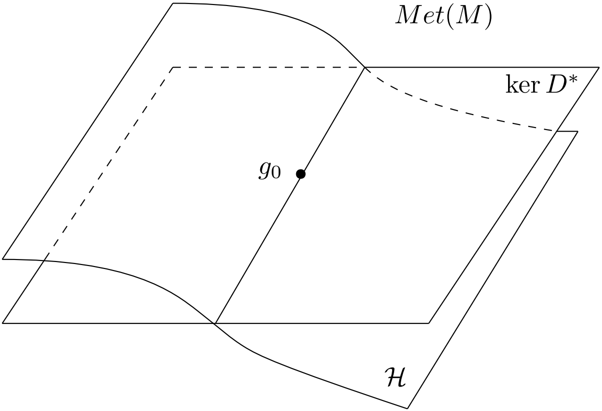
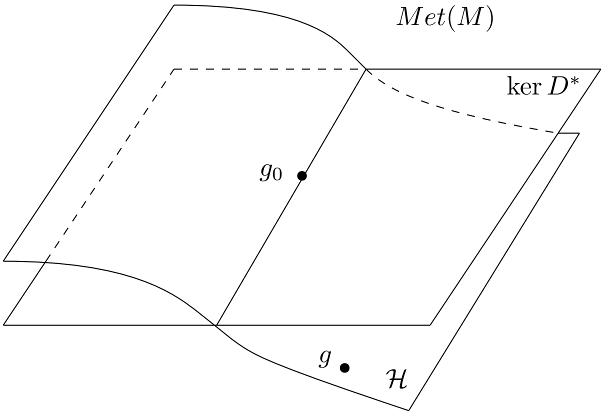


- We introduce a codimension 1 submanifold $\mathcal{H}$ of $Met(M)$ (defined by an implicit equation $F(g) = 0$ for some $F : Met(M) \to \mathbb{R}$) passing through g_0 such that $\{g \mid L_g/L_{g_0} \to 1\} \subset \mathcal{H}$. Moreover, $\mathcal{H}$ is transverse to $\ker D^*$.



- We pick g such that $L_g/L_{g_0} \to 1$ (thus $g \in \mathcal{H}$). We can first do a solenoidal reduction. Here $g' = \phi^* g$.



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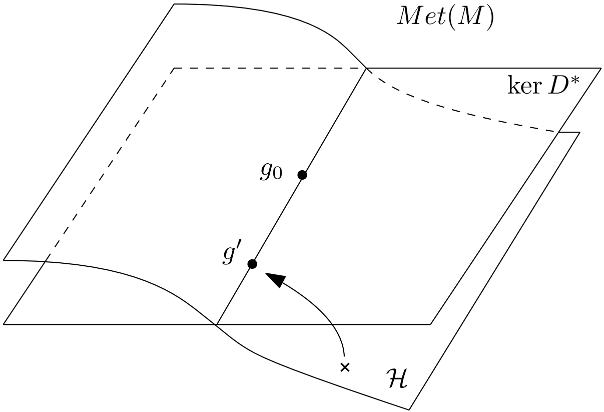
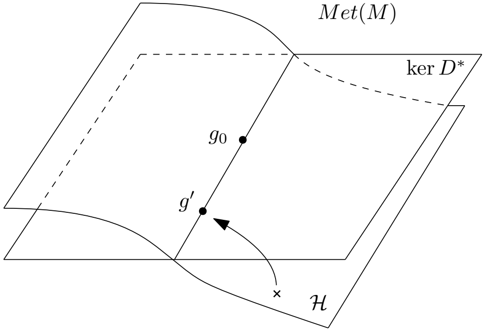


The geodesic stretch functional $\Psi : g \mapsto \mathcal{I}_{\mathrm{d}\mu_{g_0}}(g_0, g)$ has the following properties on $\mathcal{H} \cap \ker D^*$:

- $\Psi(g_0) = 1$
- $\mathrm{d}\Psi_{g_0} = 0$
- $\mathrm{d}^2\Psi_{g_0}(h, h) = \mathrm{Var}_2(h) = \langle \Pi_2 h, h \rangle \geq C \|h\|^2_{H^{-1/2}}$



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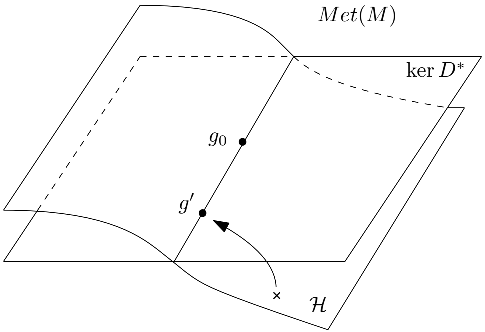

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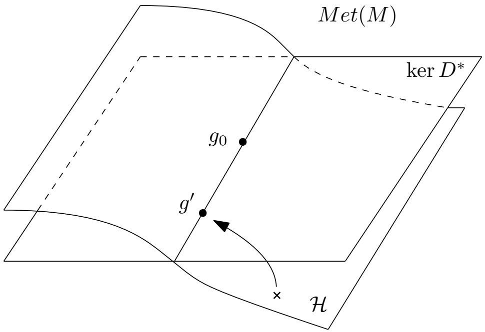

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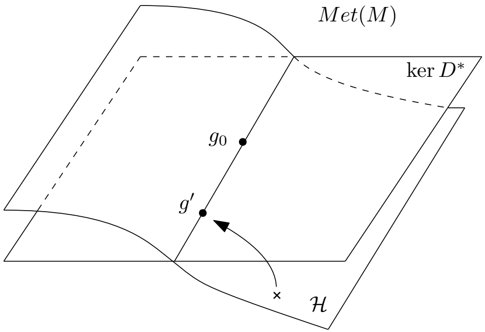

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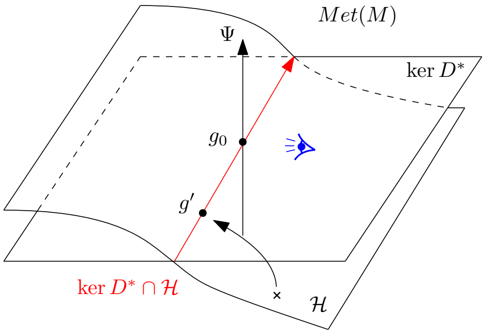

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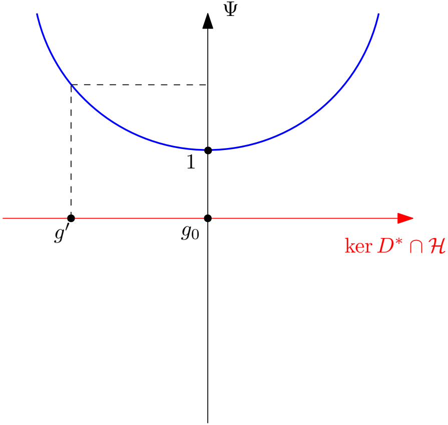
But since $L_{g'}/L_{g_0} \to 1$, $\Psi(g') = 1$. We then easily obtain that $g' = g_0$ (as long as g was chosen close enough to g_0 at the beginning). This concludes the proof.



The proof also yields a stability estimate of the form:

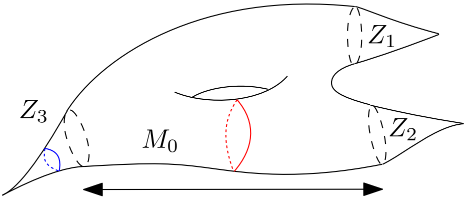
Corollary

There exists $k \in \mathbb{N}$ such that for any metric $g \in \mathcal{H}$ in a neighborhood of g_0, there exists a diffeomorphism $\phi : M \to M$ such that:

$$\|\phi^* g - g_0\|_{C^k} \lesssim |1 - \mathcal{I}_{\mathrm{d}\mu_{g_0}}(g_0, g)|$$

Perspectives:

- Further study of the geodesic stretch functional.
- Other applications of the stability estimates on I_m, for $m \neq 0, 2$?
- Study of the Marked Length Spectrum on non-compact manifolds with hyperbolic cusps (paper in preparation with Yannick Guedes Bonthonneau).



Thank you for your attention !

References:

- *Classical and microlocal analysis of the X-ray transform on Anosov manifolds*, with Sébastien Gouëzel, in preparation,
- *The marked length spectrum of Anosov manifolds*, with Colin Guillarmou, preprint (`https://arxiv.org/abs/1806.04218`),
- *Geodesic stretch and marked length spectrum rigidity*, with Colin Guillarmou and Gerhard Knieper, in preparation.