Mean hitting times of open quantum walks in terms of generalized inverses

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This talk is in part motivated by some conversations with F. A. Grünbaum (Berkeley) and L. Velázquez (Zaragoza), regarding the *recurrence* problem in (closed and open) quantum settings. At some point we began discussing the *mean hitting time* problem.

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In this talk I discuss some points in the open quantum case, and something else.

1. Open quantum walks
2. The mean hitting time formula (classical and quantum)
3. Generalized inverses and another formula
4. Open question: the unitary case

Open quantum walks

Let

$$\mathcal{D}_{n;k} = \{\rho = [\rho_1 \ \cdots \ \rho_n]^T : \rho_i \in M_k(\mathbb{C}),\ \rho_i \geq 0,\ \sum_{i=1}^{n} \mathrm{Tr}(\rho_i) = 1\}$$

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$$\Phi_{ij}(\cdot) = B_{ij} \cdot B_{ij}^{\dagger}, \quad B_{ij} \in M_k(\mathbb{C}), \quad i = 1, \ldots, n$$

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and let

$$T(\rho) = \begin{bmatrix} \Phi_{11} & \cdots & \Phi_{1n} \\ \Phi_{21} & \cdots & \Phi_{2n} \\ \vdots & \ddots & \vdots \\ \Phi_{n1} & \cdots & \Phi_{nn} \end{bmatrix} \cdot \begin{bmatrix} \rho_1 \\ \rho_2 \\ \vdots \\ \rho_n \end{bmatrix} := \begin{bmatrix} \Phi_1(\rho) \\ \Phi_2(\rho) \\ \vdots \\ \Phi_n(\rho) \end{bmatrix}, \quad \rho \in \mathcal{D}_{n;k}$$

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We assume trace preservation, that is,

$$\mathrm{Tr}\Big(\sum_{i=1}^{n} \Phi_i(\rho) \Big) = \mathrm{Tr}(\rho), \quad j = 1, \ldots, n, \quad \rho \in \mathcal{D}_{n;k}$$

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We say T is an **open quantum walk (OQW)** on n vertices and internal degree of freedom k.

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We define the following quantities for an OQW starting at state ρ:

$\pi_r(\rho \to j) =$ probability of reaching vertex j for the first time in r steps.

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Let $\mathbb{P}_j$ be the projection matrix on vertex j and let $\mathbb{Q}_j = \mathbb{I} - \mathbb{P}_j$ be its complement. This projection is such that if ρ is an OQW density then

$$\mathbb{P}_j\Big(\sum_i \rho_i \otimes |i\rangle\langle i|\Big) = \rho_j \otimes |j\rangle\langle j|$$

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So

$$\pi_r(\rho \to j) = \mathrm{Tr}(\mathbb{P}_j T(\mathbb{Q}_j T)^{r-1}\rho)$$

We introduce the following matrix-valued generating functions,

$$\mathbb{G}_{ij}(z) = \sum_{n \geq 1} \mathbb{P}_i T (\mathbb{Q}_i T)^{n-1} \mathbb{P}_j z^{n-1} = \mathbb{P}_i T (I - z\mathbb{Q}_i T)^{-1} \mathbb{P}_j, \quad z \in \mathbb{D}$$

where $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

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where $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Then we can write

$$\pi(\rho_j \to i) = \mathrm{Tr}(\hat{h}_{ij}\rho_j), \qquad \hat{h}_{ij} := \begin{cases} \lim_{x\uparrow 1} \mathbb{G}_{ij}(x) & \text{if } i \neq j \\ I & \text{if } i = j \end{cases} \tag{1}$$

$$\tau(\rho_i \to i) = \mathrm{Tr}(\hat{r}_i\rho_i), \qquad \hat{r}_i := \lim_{x\uparrow 1} \frac{d}{dx} x\mathbb{G}_{ii}(x) \tag{2}$$

$$\tau(\rho_j \to i) = \mathrm{Tr}(\hat{k}_{ij}\rho_j), \qquad \hat{k}_{ij} := \lim_{x\uparrow 1} \frac{d}{dx} x\mathbb{G}_{ij}(x), \quad \text{if } i \neq j \tag{3}$$

Finally, define the matrices of operators

$$H = \begin{bmatrix} \hat{h}_{11} & \cdots & \hat{h}_{1n} \\ \hat{h}_{21} & \cdots & \hat{h}_{2n} \\ \vdots & \ddots & \vdots \\ \hat{h}_{n1} & \cdots & \hat{h}_{nn} \end{bmatrix}, \; K = \begin{bmatrix} \hat{k}_{11} & \cdots & \hat{k}_{1n} \\ \hat{k}_{21} & \cdots & \hat{k}_{2n} \\ \vdots & \ddots & \vdots \\ \hat{k}_{n1} & \cdots & \hat{k}_{nn} \end{bmatrix}, \; D = \begin{bmatrix} \hat{r}_1 & 0 & \cdots & 0 \\ 0 & \hat{r}_2 & 0 & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \hat{r}_n \end{bmatrix}$$

These will play a central role in the description of hitting time formulae.

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Can we avoid using the definition in order to calculate $E_i(T_j)$?

$$E_i(T_j) = \sum_t tP_i(T_j = t)$$

We can use the fundamental matrix associated with a finite ergodic (irreducible and aperiodic) Markov chain with stochastic matrix P,

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so each entry of W counts the mean number of visits to a particular vertex given some initial position, noting that we only consider pairs of vertices (i, j) for which i, j are transient (in the other situations we obtain null or infinite entries).

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We ask: is there a quantum version of such hitting time formula?

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First mean hitting time formula. Let T be an ergodic OQW acting on a finite graph with $n \geq 2$ vertices and let D denote the block diagonal matrix with block diagonal entries given by the operators $\hat{k}_{ii}$, $i = 1, \ldots, n$.

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First mean hitting time formula. Let T be an ergodic OQW acting on a finite graph with $n \geq 2$ vertices and let D denote the block diagonal matrix with block diagonal entries given by the operators $\hat{k}_{ii}$, $i = 1, \ldots, n$.

a) The mean hitting time for the walk to reach vertex i, beginning at vertex j with initial density ρ_j concentrated in vertex j is given by

$$\mathrm{Tr}(\hat{k}_{ij}\rho_j) = \mathrm{Tr}(\hat{k}_{ii}(\hat{Z}_{ii} - \hat{Z}_{ij})\rho_j)$$

a) $\mathrm{Tr}(\hat{k}_{ij}\rho_j) = \mathrm{Tr}(\hat{k}_{ii}(\hat{Z}_{ii} - \hat{Z}_{ij})\rho_j)$

b) (Random Target Lemma). If D is invertible and there is c scalar such that $\mathrm{Tr}(\hat{k}_{ii}\gamma) = c\mathrm{Tr}(\gamma)$, all i vertex, $\gamma \in M_n(\mathbb{C})$, then for every density ρ,

$$\mathrm{Tr}[(D^{-1}K)_{ij}\rho] = \mathrm{Tr}[(\hat{Z}_{ii} - \hat{Z}_{ij})\rho]$$

As a consequence,

$$t_{\odot}(\rho) := \sum_i \mathrm{Tr}[(D^{-1}K)_{ij}\rho] = \Big[\sum_i \mathrm{Tr}(\hat{Z}_{ii}\rho)\Big] - 1$$

In particular, such quantity does not depend on j.

$$\mathrm{Tr}(\hat{k}_{ij}\rho_j) = \mathrm{Tr}(\hat{k}_{ii}(\hat{Z}_{ii} - \hat{Z}_{ij})\rho_j)$$

Informally, the meaning of the theorem is: the mean hitting time from j to i is an information which can be extracted from the mean <u>return</u> time to i if we know the fundamental matrix of the walk.

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An element of the proof: study the iterates of T (which converge to Ω in the ergodic case) in terms of matrix representations.

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$$\hat{T}^m \to \hat{\Omega} = |\pi\rangle\langle e_{I_k^n}|, \qquad |\pi\rangle, |e_{I_k^n}\rangle := \begin{bmatrix} vec(I_k) \\ vec(I_k) \\ \vdots \\ vec(I_k) \end{bmatrix} \in \mathbb{C}^{nk^2}$$

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as $m \to \infty$, where π is the unique stationary state for T and $I_k \in M_k(\mathbb{C})$ is the order k identity matrix. For instance, if $n = k = 2$, write

$$\pi = \pi_1 \otimes |1\rangle\langle 1| + \pi_2 \otimes |2\rangle\langle 2| = \begin{bmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{bmatrix} \otimes |1\rangle\langle 1| + \begin{bmatrix} \pi_{33} & \pi_{34} \\ \pi_{43} & \pi_{44} \end{bmatrix} \otimes |2\rangle\langle 2|$$

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so if we set $|\pi\rangle = \begin{bmatrix} vec(\pi_1) \\ vec(\pi_2) \end{bmatrix}$ we obtain that $\hat{\Omega} = |\pi\rangle\langle e_{I_2^2}|$

$$= \begin{bmatrix} \pi_{11} \\ \pi_{12} \\ \pi_{21} \\ \pi_{22} \\ \pi_{33} \\ \pi_{34} \\ \pi_{43} \\ \pi_{44} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \pi_{11} & 0 & 0 & \pi_{11} & \pi_{11} & 0 & 0 & \pi_{11} \\ \pi_{12} & 0 & 0 & \pi_{12} & \pi_{12} & 0 & 0 & \pi_{12} \\ \pi_{21} & 0 & 0 & \pi_{21} & \pi_{21} & 0 & 0 & \pi_{21} \\ \pi_{22} & 0 & 0 & \pi_{22} & \pi_{22} & 0 & 0 & \pi_{22} \\ \pi_{33} & 0 & 0 & \pi_{33} & \pi_{33} & 0 & 0 & \pi_{33} \\ \pi_{34} & 0 & 0 & \pi_{34} & \pi_{34} & 0 & 0 & \pi_{34} \\ \pi_{43} & 0 & 0 & \pi_{43} & \pi_{43} & 0 & 0 & \pi_{43} \\ \pi_{44} & 0 & 0 & \pi_{44} & \pi_{44} & 0 & 0 & \pi_{44} \end{bmatrix}$$

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$$k_{ij} = p_{ij} + \sum_{l \neq i} (k_{il} + 1) p_{lj} = 1 + \sum_{l \neq i} k_{il} p_{lj}, \quad r_i = 1 + \sum_{l} k_{il} p_{li}$$

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where r_i is the mean time of first return to i.

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$$\begin{bmatrix} r_1 & k_{12} & k_{13} \\ k_{21} & r_2 & k_{23} \\ k_{31} & k_{32} & r_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & k_{12} & k_{13} \\ k_{21} & 0 & k_{23} \\ k_{31} & k_{32} & 0 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix}$$

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$$\implies K = D + (K - D) = E + (K - D)P \implies E = K - (K - D)P$$

where E denotes the matrix with all entries equal to 1 and D denotes the diagonal matrix with nonzero entries equal to r_i.

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$$k_{ij}(\rho_j) = 1 + \sum_{l \neq i} k_{il}\Big(\frac{B_{lj}\rho_j B_{lj}^{\dagger}}{\mathrm{Tr}(B_{lj}\rho_j B_{lj}^{\dagger})}\Big)\mathrm{Tr}(B_{lj}\rho_j B_{lj}^{\dagger})$$

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where traces appear on the right with the purpose of making explicit the probabilistic reasoning. Then

$$k_{ij}(\rho_j) = 1 + \sum_{l \neq i} k_{il}(B_{lj}\rho_j B_{lj}^\dagger) \implies k_{ij}(\rho_j) - \sum_{l \neq i} k_{il}(B_{lj}\rho_j B_{lj}^\dagger) = 1$$

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This seems to define an open quantum version of the equation $E = K - (K - D)P$ obtained previously. In the classical case we know that $L = E$. However, in the OQW case, L (its matrix representation) does **not** have all entries equal to 1 in general.

$$L := K - (K - D)T$$

Nevertheless, we have the crucial fact that for every density ρ_j concentrated on a vertex j,

$$\mathrm{Tr}(\hat{L}_{ij}\rho_j) = 1, \quad \forall i$$

where $\hat{L}_{ij}$ is the operator corresponding to the (i, j)-th block matrix representation appearing in $\hat{L}$.

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This will be essential to our discussion on generalized inverses as well.

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A **generalized inverse**, or g-inverse, of a matrix A is any matrix A^{-} such that

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Usually one can obtain several generalized inverses, but by imposing additional conditions one may have uniqueness.

Theorem. Let T be an irreducible OQW on a finite graph with stationary density π. Let $|t\rangle, |u\rangle \in \mathbb{C}^n$ be such that $\langle e_I|t\rangle \neq 0$ and $\langle u|\pi\rangle \neq 0$. Then $I - T + |t\rangle\langle u|$ is invertible and its inverse is a g-inverse of $I - T$.

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Corollary. Under the above conditions any g-inverse G_0 of $I - T$ can be written in one of the following forms:

$$a)\ G_0 = (I - T + |t\rangle\langle u|)^{-1} + H\frac{|t\rangle\langle e_I|}{\langle e_I|t\rangle} + \frac{|\pi\rangle\langle u|}{\langle u|\pi\rangle}H - \frac{|\pi\rangle\langle u|H|t\rangle\langle e_I|}{\langle u|\pi\rangle\langle e_I|t\rangle}$$

$$b)\ G_0 = (I - T + |t\rangle\langle u|)^{-1} + \frac{|\pi\rangle\langle u|}{\langle u|\pi\rangle}F + G\frac{|t\rangle\langle e_I|}{\langle e_I|t\rangle}$$

$$c)\ G_0 = (I - T + |t\rangle\langle u|)^{-1} + |\pi\rangle\langle f| + |g\rangle\langle e_I|$$

where f, g are arbitrary vectors, F, G, H are arbitrary matrices.

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Corollary. Let T be an irreducible OQW on a finite graph with stationary density π and let $\Omega = |\pi\rangle\langle e_I|$. Then

$$Z = (I - T + \Omega)^{-1}$$

is a generalized inverse of $I - T$

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Corollary. Let T be an irreducible OQW on a finite graph with stationary density π and let $\Omega = |\pi\rangle\langle e_I|$. Then

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is a generalized inverse of $I - T$ (fundamental matrix of T).

$$G_0 = (I - T + |t\rangle\langle u|)^{-1} + H\frac{|t\rangle\langle e_I|}{\langle e_I|t\rangle} + \frac{|\pi\rangle\langle u|}{\langle u|\pi\rangle}H - \frac{|\pi\rangle\langle u|H|t\rangle\langle e_I|}{\langle u|\pi\rangle\langle e_I|t\rangle}$$

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Going back to the question of obtaining hitting time formulae, what can we do with an arbitrary generalized inverse?

$$G_0 = (I - T + |t\rangle\langle u|)^{-1} + H\frac{|t\rangle\langle e_I|}{\langle e_I|t\rangle} + \frac{|\pi\rangle\langle u|}{\langle u|\pi\rangle}H - \frac{|\pi\rangle\langle u|H|t\rangle\langle e_I|}{\langle u|\pi\rangle\langle e_I|t\rangle}$$

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Going back to the question of obtaining hitting time formulae, what can we do with an arbitrary generalized inverse? Inspired by a very interesting result [J. J. Hunter, Lin. Alg. Appl. 45:157-198 (1982)], we are able to prove the following:

Hunter's formula for OQWs.

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a) The mean hitting time for the walk to reach vertex i, beginning at vertex j with initial density ρ_j is given by

$$\mathrm{Tr}(\hat{k}_{ij}\rho_j) = \mathrm{Tr}\Big(\Big[D\Big(\Omega G - (\Omega G)_d E + I - G + G_d E\Big)\Big]_{ij}\rho_j\Big)$$

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b) By setting $G = (I - T + |u\rangle\langle e_I|)^{-1} + |f\rangle\langle e_I|$, with $|f\rangle$ arbitrary and $|u\rangle$ such that $\langle u|\pi\rangle \neq 0$, we have that for every vertex i and initial density ρ_j on vertex j,

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Informally, the meaning of the theorem is: the mean time of first visit from j to i is an information which can be extracted from the mean <u>return</u> time to vertices if we have knowledge of any generalized inverse of $I - T$.

Example. Let

$$B_{11} = B_{22} = \begin{bmatrix} a & \sqrt{1-a^2} \\ 0 & 0 \end{bmatrix},\ B_{12} = B_{21} = \begin{bmatrix} 0 & 0 \\ -\sqrt{1-a^2} & a \end{bmatrix},\ 0 < a < 1$$

and for $b := \sqrt{1-a^2}$, define the OQW on 2 vertices

$$\hat{T} := \begin{bmatrix} [B_{11}] & [B_{12}] \\ [B_{21}] & [B_{22}] \end{bmatrix} = \begin{bmatrix} a^2 & ab & ab & b^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b^2 & -ab & -ab & a^2 \\ 0 & 0 & 0 & 0 & a^2 & ab & ab & b^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ b^2 & -ab & -ab & a^2 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B_{11} = B_{22} = \begin{bmatrix} a & \sqrt{1-a^2} \\ 0 & 0 \end{bmatrix}, \; B_{12} = B_{21} = \begin{bmatrix} 0 & 0 \\ -\sqrt{1-a^2} & a \end{bmatrix}$$

$$0 < a < 1, \quad b = \sqrt{1-a^2}$$

It is a simple matter to show that T is ergodic and unital. Also

$$\hat{\Omega} = |\pi\rangle\langle e_I| = \begin{bmatrix} \Omega_{11} & \Omega_{11} \\ \Omega_{11} & \Omega_{11} \end{bmatrix}, \quad \Omega_{11} = \frac{1}{4}\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\hat{h}_{11} = \hat{h}_{22} = \begin{bmatrix} a^2 & ab & ab & b^2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ b^2 & -ab & -ab & a^2 \end{bmatrix}, \quad \hat{h}_{12} = \hat{h}_{21} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\hat{k}_{11} = \hat{k}_{22} = \begin{bmatrix} a^2 & ab & ab & b^2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 3b^2 & -3ab & -3ab & 3a^2 \end{bmatrix}, \quad \hat{k}_{12} = \hat{k}_{21} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{b^2} & \frac{a}{b} & \frac{a}{b} & 2 \end{bmatrix}$$

From this we obtain that for every density and vertex the hitting probability equals 1, as expected, since this OQW is irreducible.

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Also,

$$\mathrm{Tr}(\hat{k}_{11}\rho) = \mathrm{Tr}(\hat{k}_{22}\rho) = (3-2a^2)\rho_{11} + (1+2a^2)\rho_{22} - 4abRe(\rho_{12})$$

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$$\mathrm{Tr}(\hat{k}_{12}\rho) = \mathrm{Tr}(\hat{k}_{21}\rho) = \frac{1}{b^2}\rho_{11} + 2\rho_{22} + \frac{2a}{b}Re(\rho_{12})$$

The block matrix representation of the fundamental matrix is the order 8 matrix

$$\hat{Z} = (\hat{I} - \hat{T} + \hat{\Omega})^{-1} = \begin{bmatrix} \hat{Z}_{11} & \hat{Z}_{12} \\ \hat{Z}_{21} & \hat{Z}_{22} \end{bmatrix}$$

$$= \begin{bmatrix}
\frac{5}{8b^2} & \frac{3a}{4b} & \frac{3a}{4b} & -\frac{4a^2-3}{8b^2} & -\frac{4a^2-1}{8b^2} & -\frac{a}{4b} & -\frac{a}{4b} & -\frac{1}{8b^2} \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{8b^2} & -\frac{a}{4b} & -\frac{a}{4b} & -\frac{4a^2-5}{8b^2} & -\frac{4a^2-3}{8b^2} & -\frac{a}{4b} & -\frac{a}{4b} & \frac{1}{8b^2} \\
-\frac{4a^2-1}{8b^2} & -\frac{a}{4b} & -\frac{a}{4b} & -\frac{1}{8b^2} & \frac{5}{8b^2} & \frac{3a}{4b} & \frac{3a}{4b} & -\frac{4a^2-3}{8b^2} \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
-\frac{4a^2-3}{8b^2} & -\frac{a}{4b} & -\frac{a}{4b} & \frac{1}{8b^2} & -\frac{1}{8b^2} & -\frac{a}{4b} & -\frac{a}{4b} & -\frac{4a^2-5}{8b^2}
\end{bmatrix}$$

With such Z we have, by the MHTF,

$$\hat{k}_{11}(\hat{Z}_{11} - \hat{Z}_{12})$$

$$= \begin{bmatrix} a^2 & ab & ab & b^2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 3b^2 & -3ab & -3ab & 3a^2 \end{bmatrix} \begin{bmatrix} \frac{1+a^2}{2b^2} & \frac{a}{b} & \frac{a}{b} & \frac{1}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{1}{2} & 0 & 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{3a^2-1}{2b^2} & \frac{a}{b} & \frac{a}{b} & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{3}{2} & 0 & 0 & \frac{3}{2} \end{bmatrix}$$

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Therefore for $\rho = (\rho_{ij})_{i,j=1,2}$ density on vertex 2,

$$vec^{-1}[\hat{k}_{11}(\hat{Z}_{11} - \hat{Z}_{12})vec(\rho)] = \begin{bmatrix} \frac{3a^2-1}{2b^2}\rho_{11} + \frac{\rho_{22}}{2} + \frac{2a}{b}Re(\rho_{12}) & 0 \\ 0 & \frac{3}{2} \end{bmatrix}$$

from which we obtain $\mathrm{Tr}(\hat{k}_{12}\rho)$, as expected.

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from which we obtain $\mathrm{Tr}(\hat{k}_{12}\rho)$, as expected. Hunter's formula is also verified, by choosing any g-inverse.

Open question

	Classical	Open Quantum
MHTF I	$E_i T_j = \frac{Z_{jj} - Z_{ij}}{\pi_j}$	$\mathrm{Tr}[\hat{k}_{ij}\rho_j] = \mathrm{Tr}[\hat{k}_{ii}(\hat{Z}_{ii} - \hat{Z}_{ij})\rho_j]$
MHTF I*	$\pi_j E_i T_j = Z_{jj} - Z_{ij}$	$\mathrm{Tr}[(D^{-1}K)_{ji}\rho_i] = \mathrm{Tr}[(\hat{Z}_{ii} - \hat{Z}_{ij})\rho_i]$
MHTF II	$E_\pi T_j = \frac{Z_{jj}}{\pi_j}$	$\mathrm{Tr}[K_{j\pi}] = \mathrm{Tr}[(DZ)_{jj} F_{j\pi}]$
Hunter	$E_i T_j = [D(I - G + G_d E)]_{ij}$	$\mathrm{Tr}[\hat{k}_{ij}\rho_j] = \mathrm{Tr}([D(I - G + G_d E)]_{ij}\rho_j)$

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Question: is there a mean hitting time formula for the unitary case?

$$\mathrm{Tr}[\hat{k}_{ij}\rho_j] = \mathrm{Tr}[\hat{k}_{ii}(\hat{Z}_{ii} - \hat{Z}_{ij})\rho_j]$$

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We may break the problem into two parts:

1. **Representations.** What is an appropriate matrix representation for the unitary case? The block matrix representation works just fine for the case of CP maps describing the statistics of quantum trajectories (individual path counting). But: should the unitary problem also be examined via a matrix approach, or should something else be employed?

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A main motivation for pursuing this set of problems comes from

F. A. Grünbaum, L. Velázquez, A. H. Werner, R. F. Werner.

Recurrence for Discrete Time Unitary Evolutions. Comm. Math.

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and

F. A. Grünbaum, L. Velázquez. A generalization of Schur functions: applications to Nevanlinna functions, orthogonal polynomials, random walks and unitary and open quantum walks. Adv. Math. 326 (2018) 352-464.

Thank you!