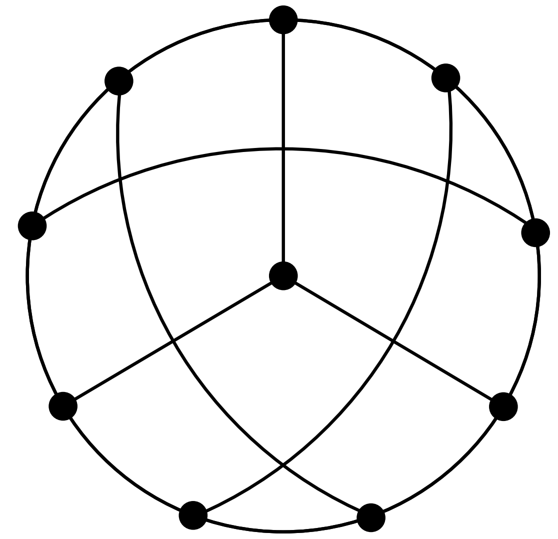
Average mixing of quantum walks

Krystal Guo

Université libre de Bruxelles

Quantum walks and information tasks, BIRS, April 25, 2019

Linear algebra and graph theory



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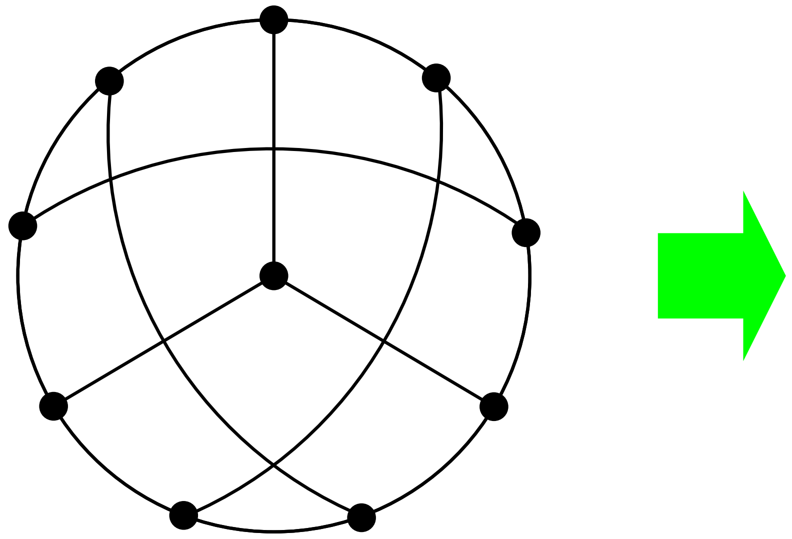
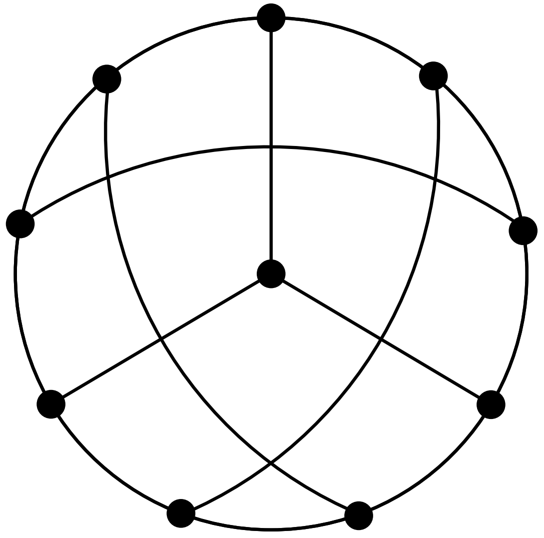


Eigenvalues of adjacency matrix:

$$3, 1, 1, 1, 1, 1, -2, -2, -2, -2$$

Linear algebra and graph theory



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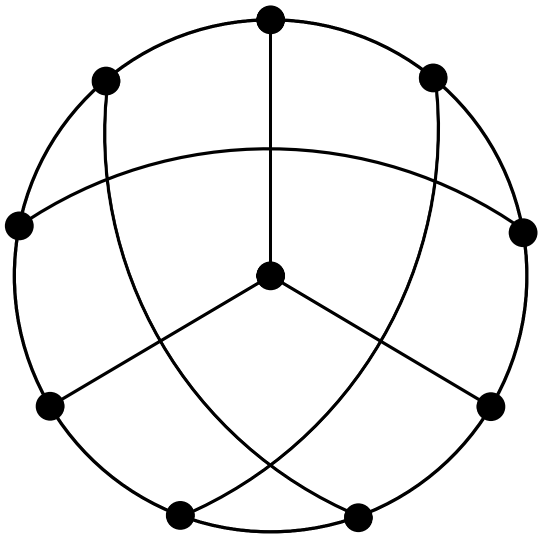

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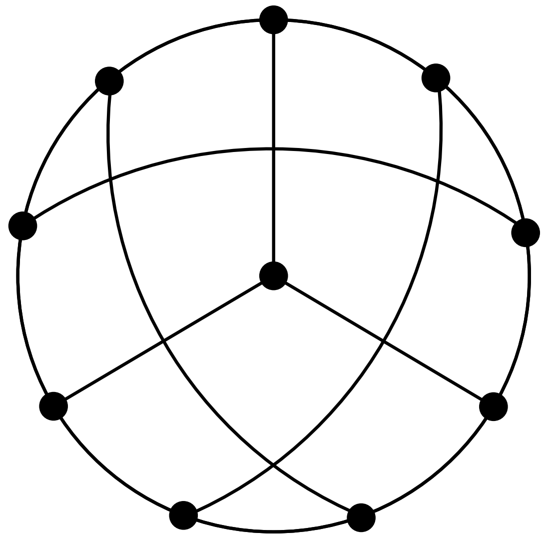
- 10 vertices and 15 edges

Linear algebra and graph theory



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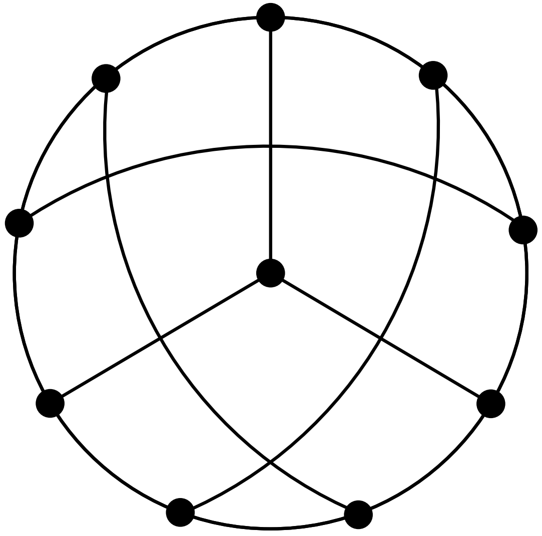

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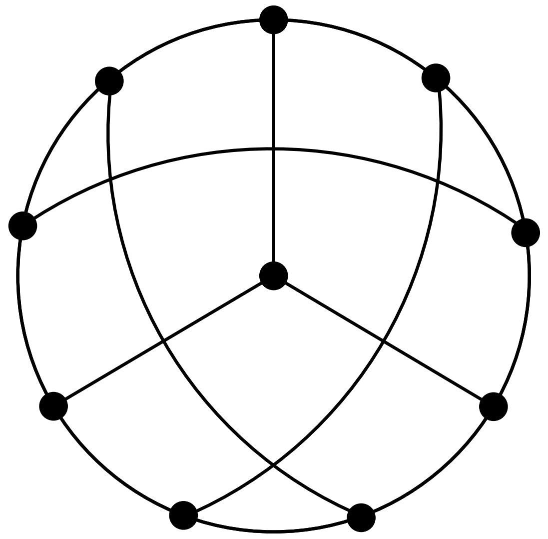

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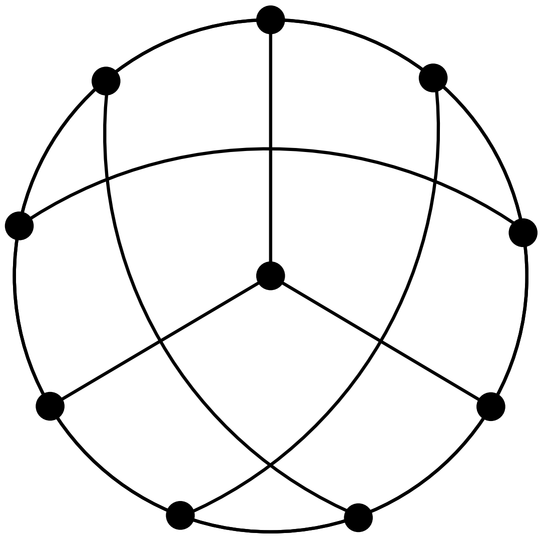
Linear algebra and graph theory



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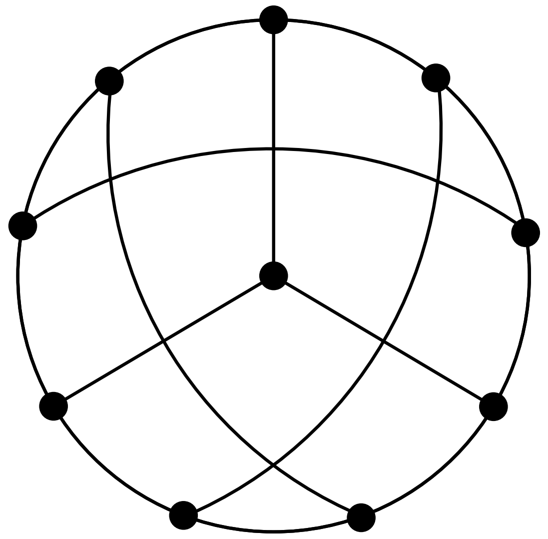

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Continous quantum walk

As in the previous talk, we will consider walks with the following transition matrix.

$$U(t) = e^{itA}$$

where A is the adjacency matrix of a graph.

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Mixing matrix

$$M(t) = U(t) \circ \overline{U(t)}$$

$e_u^T M(t) e_v$ is the probability of measuring at vertex u, having started at v, at time t.

Average mixing matrix

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$$\widehat{M} = \lim_{T \to \infty} \frac{1}{T} \int_0^T M(t)dt$$

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Theorem (Godsil 2012)

If $A(X) = \sum_r \theta_r E_r$ is the spectral decomposition of A, then

$$\widehat{M} = \sum_r E_r \circ E_r.$$

Like the eigenvalues of the adjacency matrix, the trace and rank of $\widehat{M}$ are graph invariants.

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Question: how much does the rank of $\widehat{M}$ (or the trace of $\widehat{M}$) tell us about the graph?

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Question: how much does the rank of $\widehat{M}$ (or the trace of $\widehat{M}$) tell us about the graph?

In other words, how much does the average behaviour of the quantum walk depend on the choice of the graph?

Rank of the average mixing matrix

Example: trees on 6 vertices

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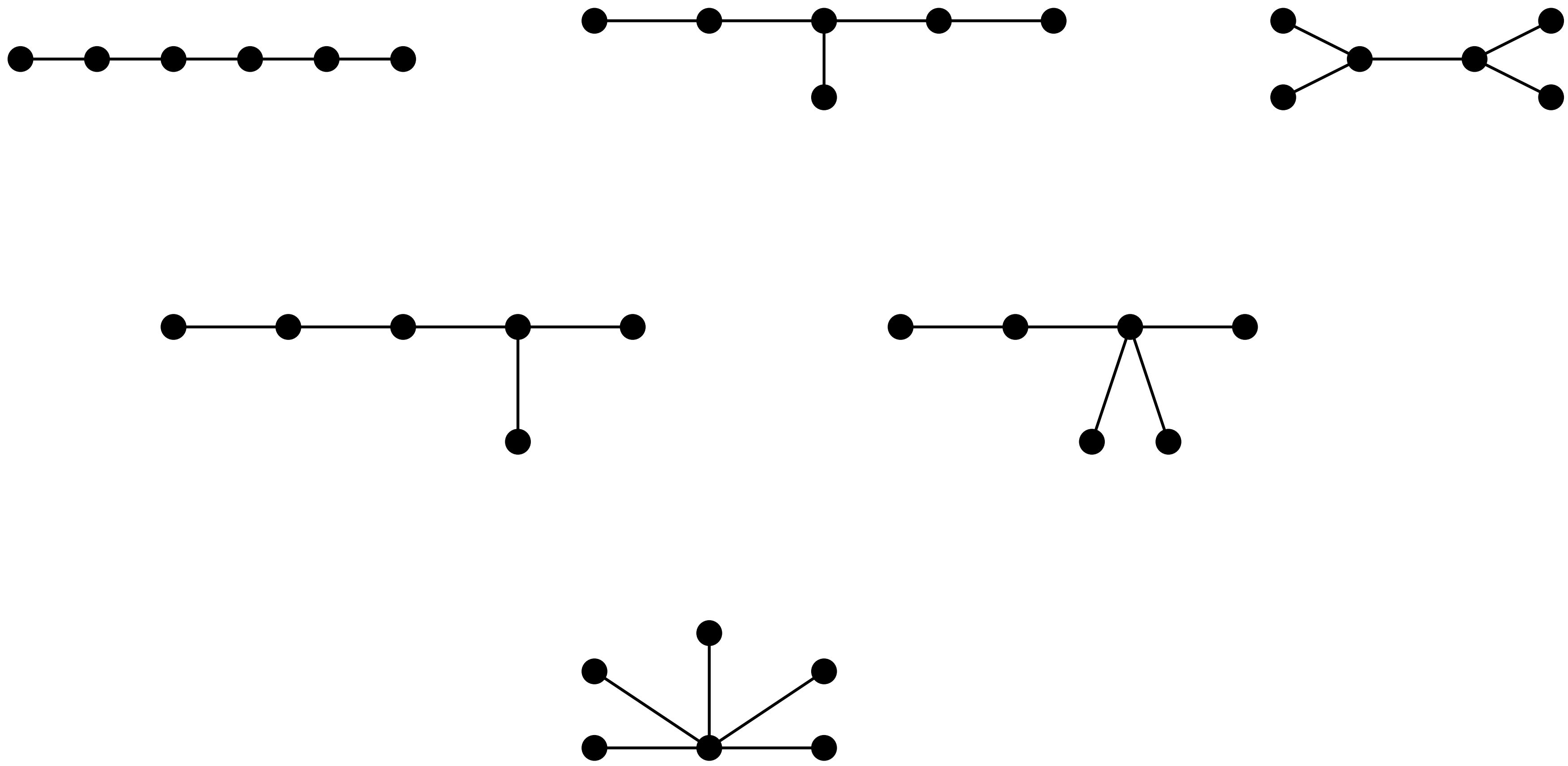

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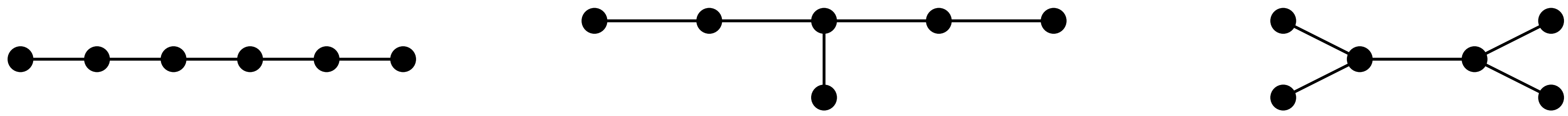
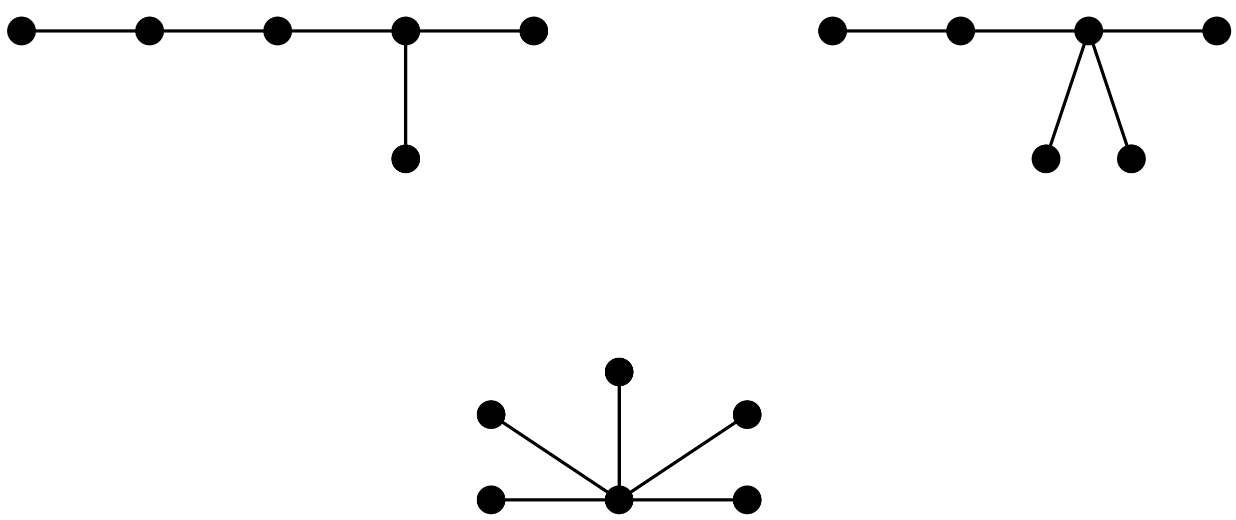


Rank of $\widehat{M} = 3$.



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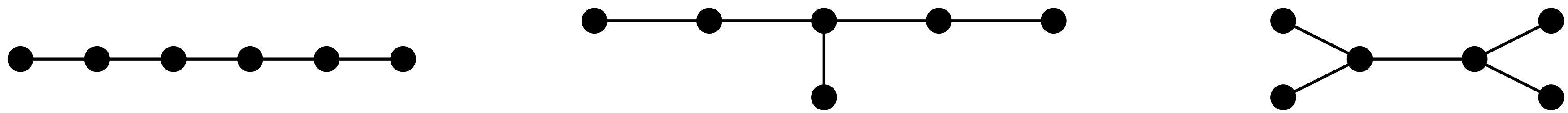

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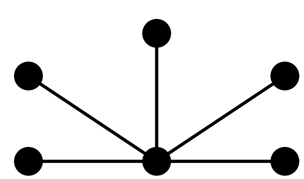

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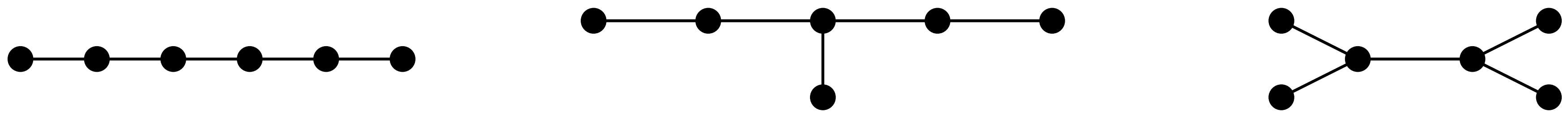

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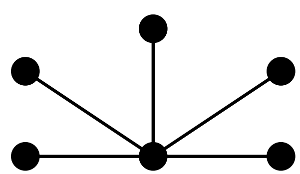
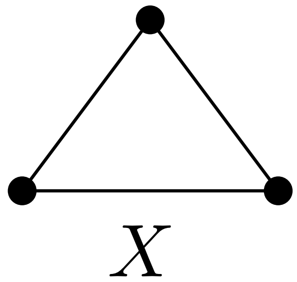


Rank of $\widehat{M} = 6$.

An algebraic interpretation of $\widehat{M}$



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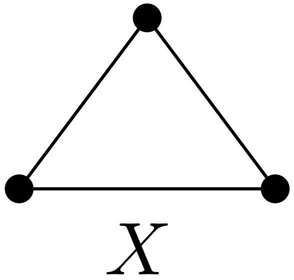

$$\widehat{M}(X) = \begin{pmatrix} 5/9 & 2/9 & 2/9 \\ 2/9 & 5/9 & 2/9 \\ 2/9 & 2/9 & 5/9 \end{pmatrix}$$

Consider the following map Ψ:

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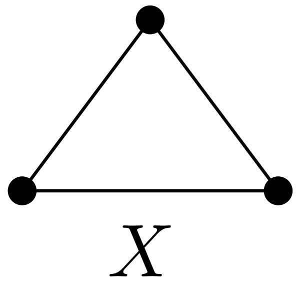

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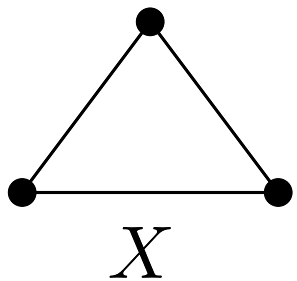

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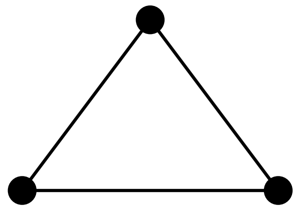

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orthgonal projection into the commutant of $A(X)$

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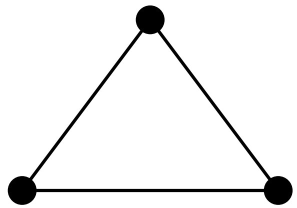

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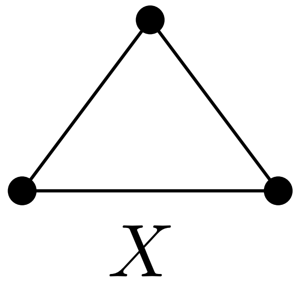

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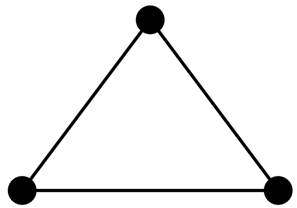

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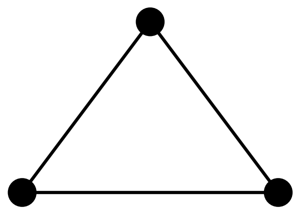

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$$\Downarrow$$

$$\begin{pmatrix} 5/9 \\ 2/9 \\ 2/9 \end{pmatrix} \Leftarrow \begin{pmatrix} 5/9 & -1/9 & -1/9 \\ -1/9 & 2/9 & 2/9 \\ -1/9 & 2/9 & 2/9 \end{pmatrix}$$

$\widehat{M}$ is the matrix of transformation of this map

Coutinho, Godsil, G. Zhan. 2018

Theorem (Continho, Godsil, G., Zhan 2018)

$$rk(\widehat{M}) = dim(Im(\Psi)).$$

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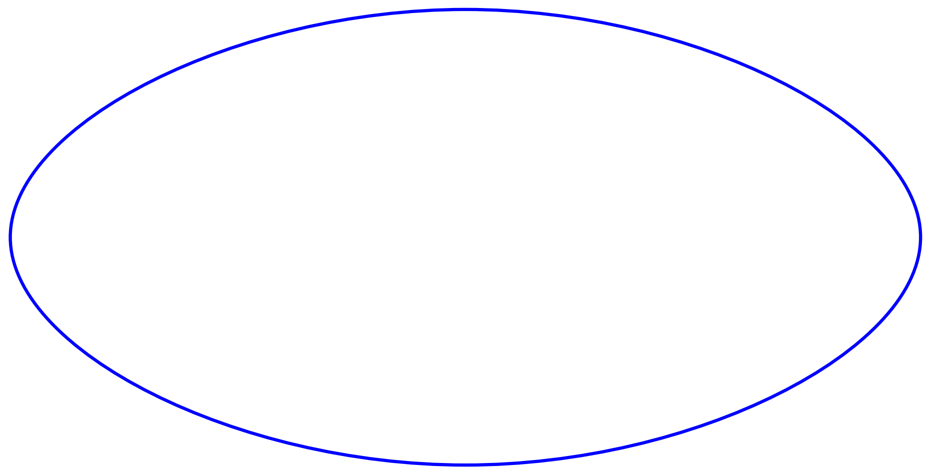

all matrices commuting with A

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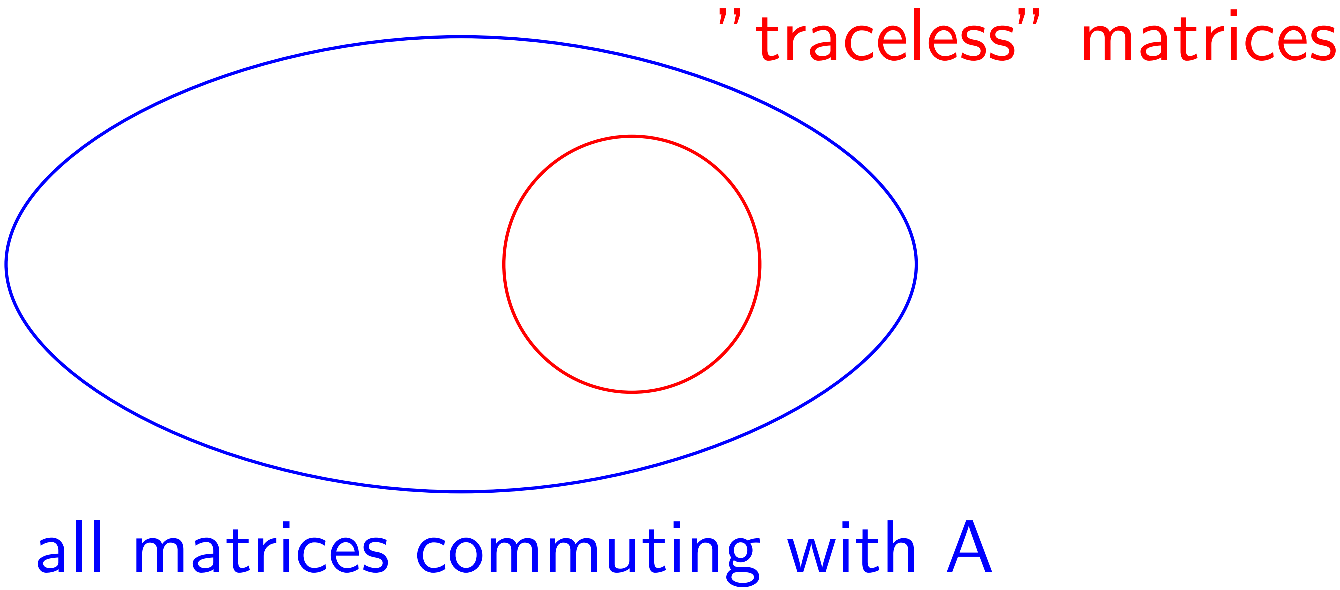

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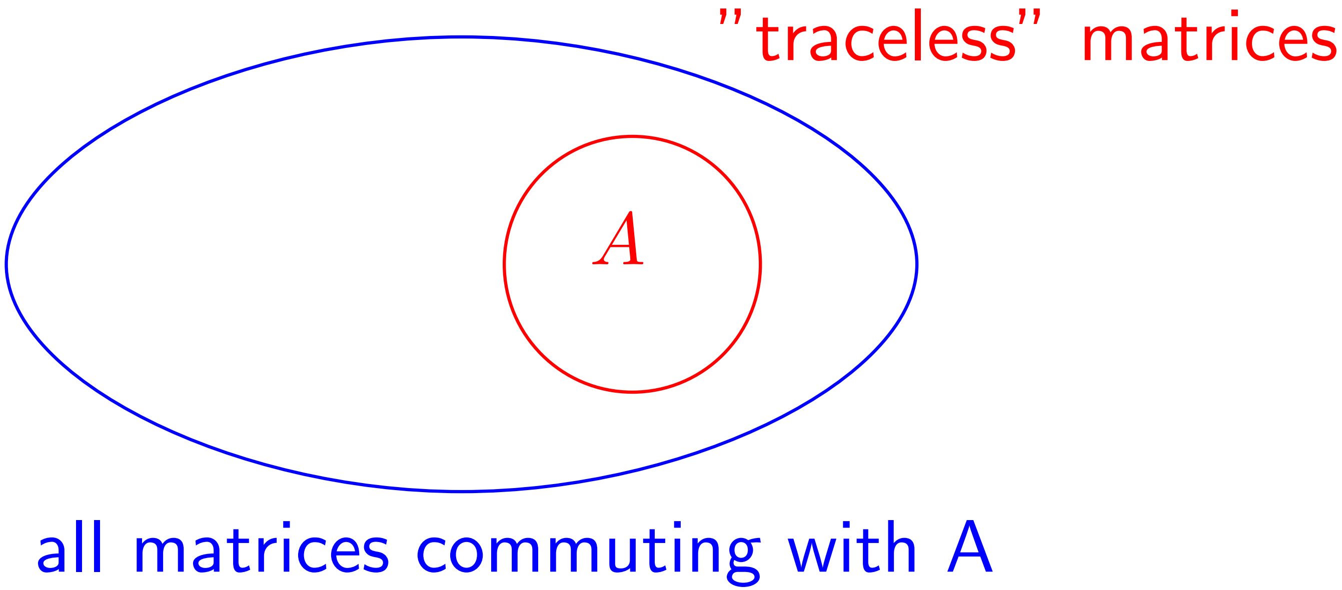

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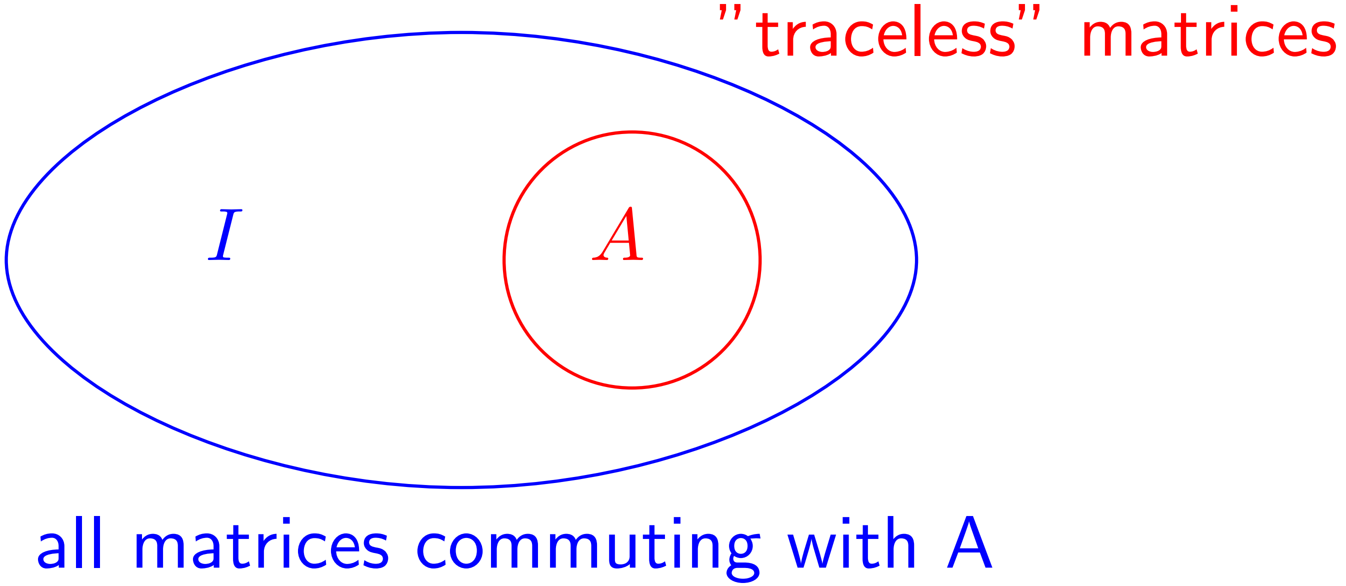

Corollary

If X is a graph with simple eigenvalues on n vertices, then $rk(\widehat{M}) < n - 1$.

Theorem (Continho, Godsil, G., Zhan 2018)

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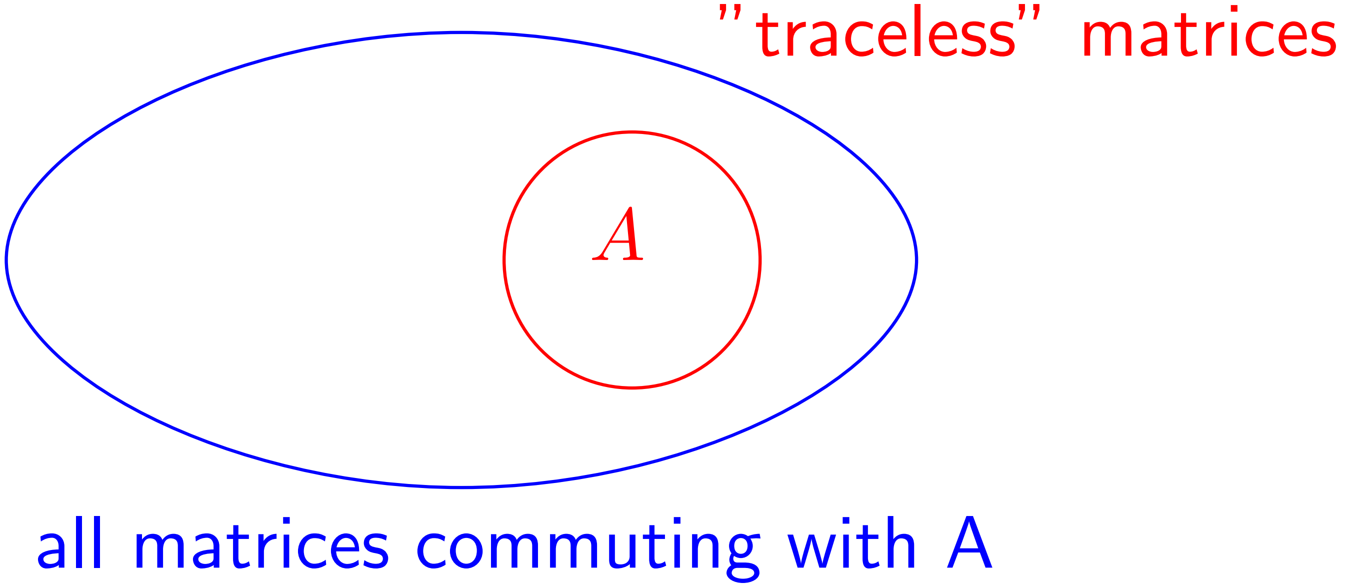
Bipartite



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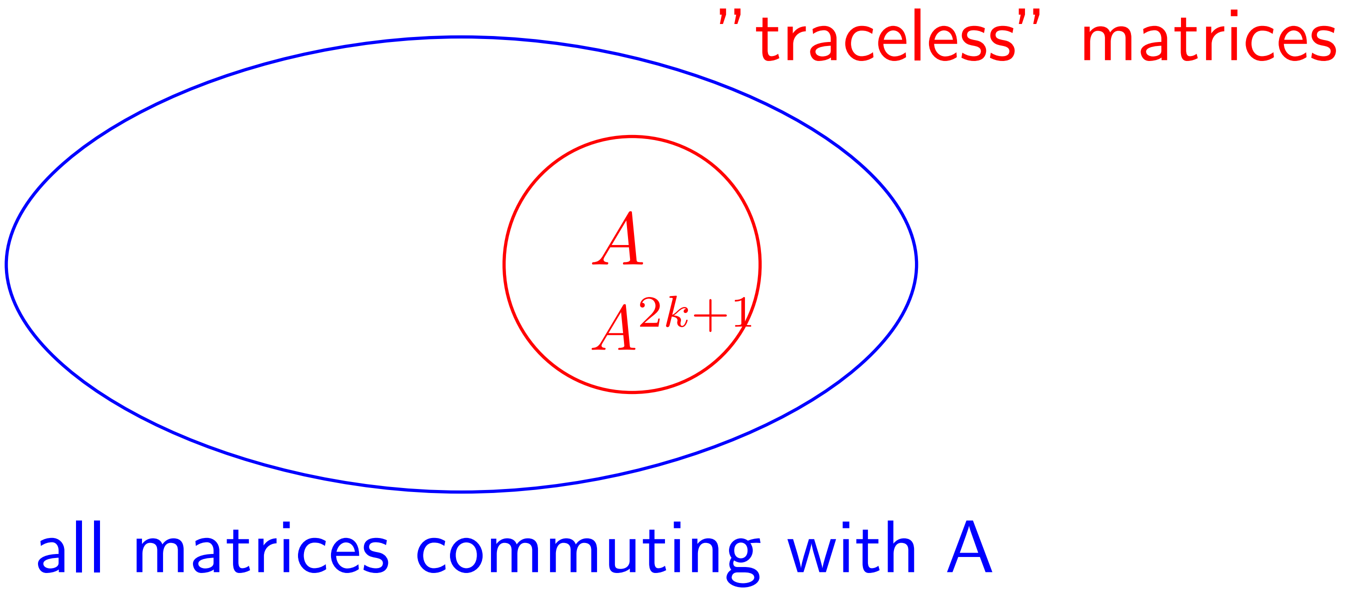

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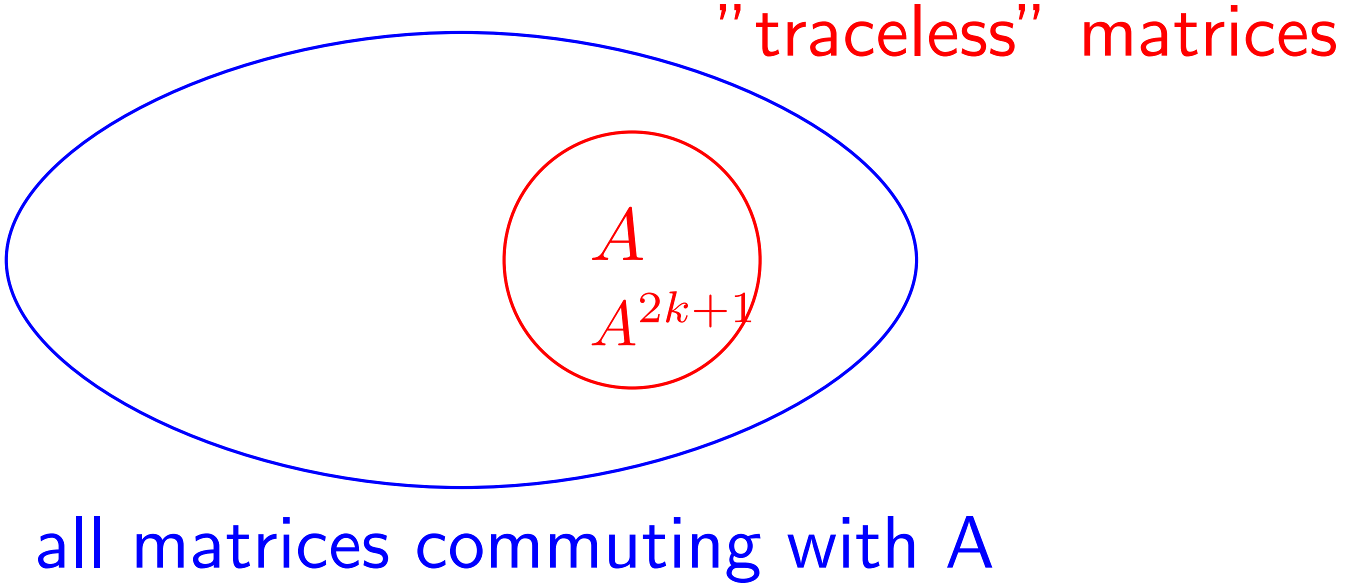

Corollary

If X is a bipartite graph with simple eigenvalues on n vertices, then $rk(\widehat{M}) \leq \lceil \frac{n}{2} \rceil$.

How large can the rank be?

Theorem (Tao and Vu, 2017)

As n goes to infinity, the proportion of graphs on n vertices which have simple eigenvalues goes to 1.

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Roughly speaking, this implies that the average mixing matrices of most graphs will not have full rank.

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There are examples of graphs where $\widehat{M}$ has full rank, including the star graph and the complete graphs.

How small can the rank be?

$\widehat{M}$ has rank 0: null graph

$\widehat{M}$ has rank 1: K_1 or K_2

$\widehat{M}$ has rank 2: ????

It is possible that there is an infinite family of graphs with $\widehat{M}$ having rank 2 and simple eigenvalues.

Theorem (Godsil, G., Sinkovic 2018)

If T is a tree with simple eigenvalues with at least 4 vertices and T is not P_4, then the rank of $\widehat{M}(T)$ is at least 3.

Trees on n vertices:

n	2	3	4	5	6	7	8	9	10	11	12

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min rk of $\widehat{M}$	1	2	2	3	3	4	4	5	4	5	5

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n	13	14	15	16	17	18	19	20			

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n	13	14	15	16	17	18	19	20
min rk of $\widehat{M}$	6	6	7	7	8	7	8	8

Open problem

Is there a non-constant, increasing function $f(n)$ which lower bounds the minimum rank of $\widehat{M}$ amongst trees on n vertices?

Trees with simple eigenvalues

A tree on n vertices with n distinct eigenvalues has rank of $\widehat{M}$ at most $\lceil n/2 \rceil$.

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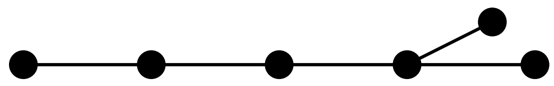
Theorem (Godsil, G., Sinkovic)

> For every positive real number c, there exists a tree T with simple eigenvalues such that
>
> $$\lceil n/2 \rceil - rk(\widehat{M}(T)) > c.$$

A graph operation



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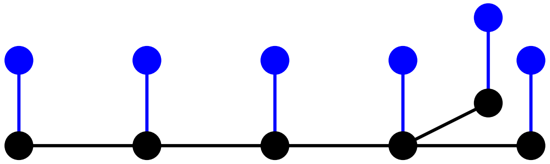

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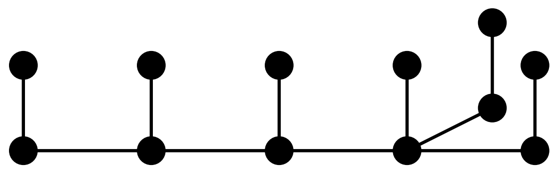

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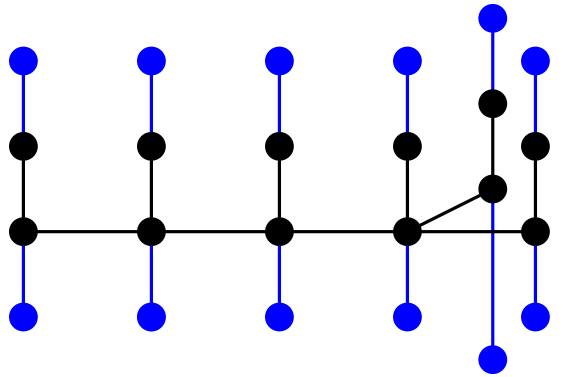

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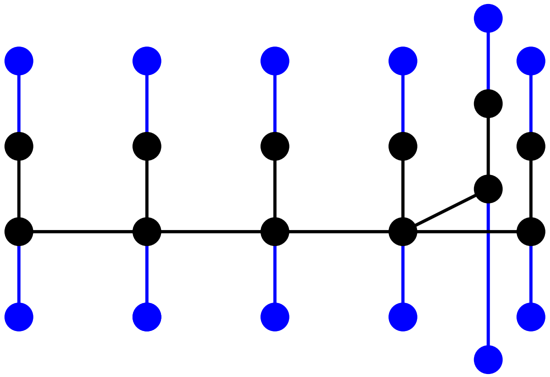

Theorem (Godsil, G., Sinkovic 2018)

graph with simple eigenvalues on n vertices with $\widehat{M}$ having rank r ⇒ graph with simple eigenvalues on $2n$ vertices with $\widehat{M}$ having rank $2r$.

Trace of the average mixing matrix

Maximum trace

For a graph X, we will consider the quantum walks using the adjacency matrix and the Laplacian matrix, with average mixing matrices $\widehat{M}_A$ and $\widehat{M}_L$, resp.

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Graphs attaining the maximum trace

n	3	4	5	6	7	8
$\widehat{M}_A$	K_3	K_4	K_5	K_6	K_7	K_8
$\widehat{M}_L$	K_3	K_4	K_5	K_6	K_7	K_8

Maximum trace

For a graph X, we will consider the quantum walks using the adjacency matrix and the Laplacian matrix, with average mixing matrices $\widehat{M}_A$ and $\widehat{M}_L$, resp.

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If X is a connected graph on n vertices, then

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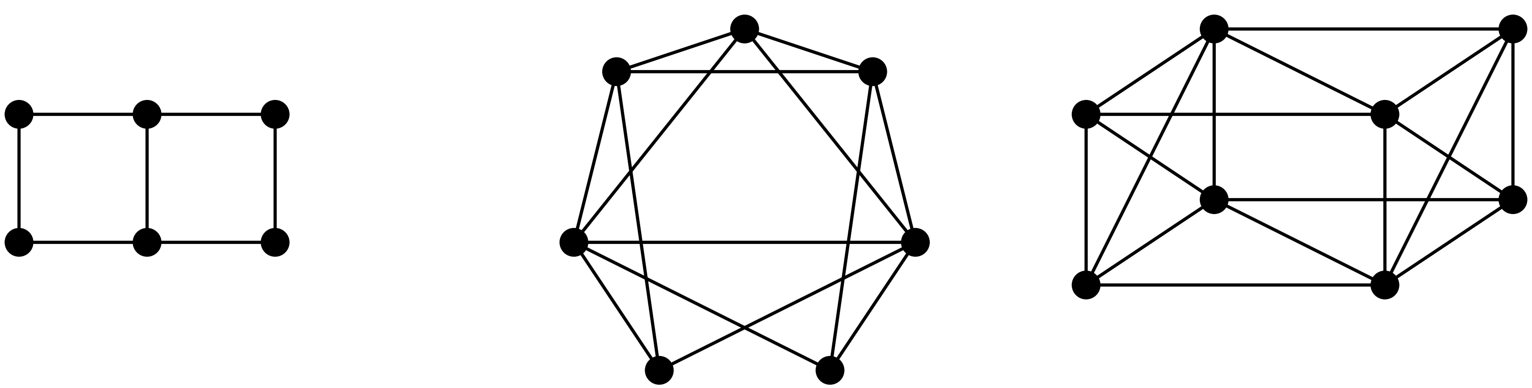
For $\widehat{M}_A$, we have P_3, P_4 and P_5 and



Diagonal entries of $\widehat{M}$

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Question: when does this matrix have a constant diagonal?

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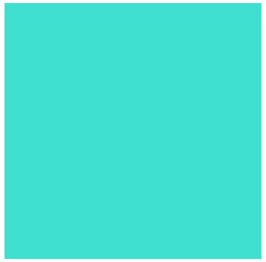
$A^k =$ 

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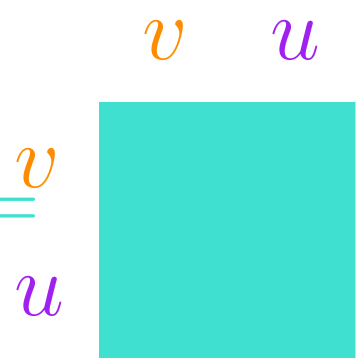

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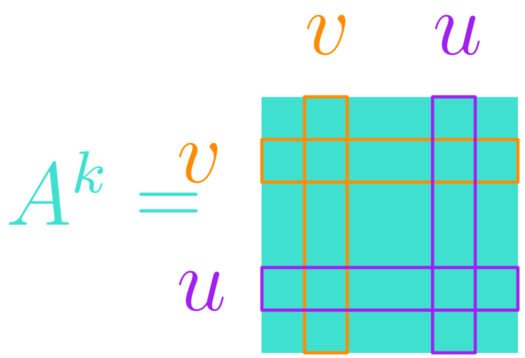

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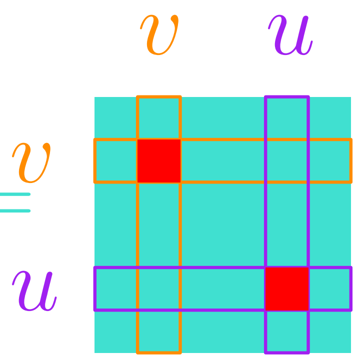
$A^k =$ 

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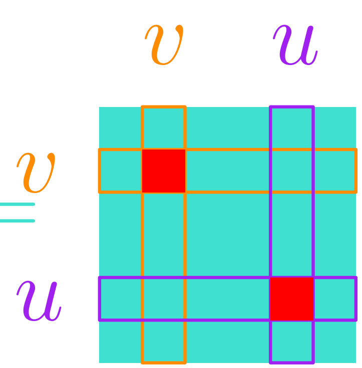

$\Leftrightarrow \quad A = \sum_\theta \theta E_\theta$ then $\forall \theta, (E_\theta)_{u,u} = (E_\theta)_{v,v}$.

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If X is walk-regular, then $\widehat{M}$ has constant diagonal.

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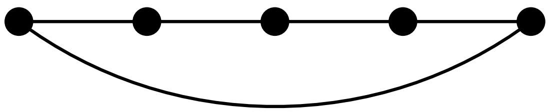
Surprisingly, X does not have to be walk-regular for this to happen.

Recall the graph operation from before



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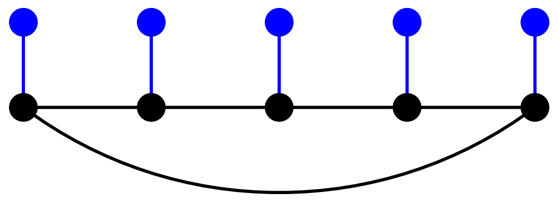

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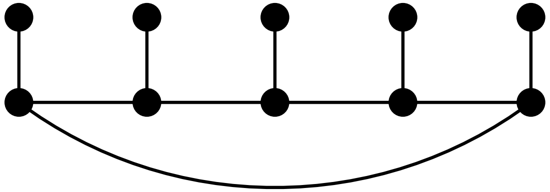
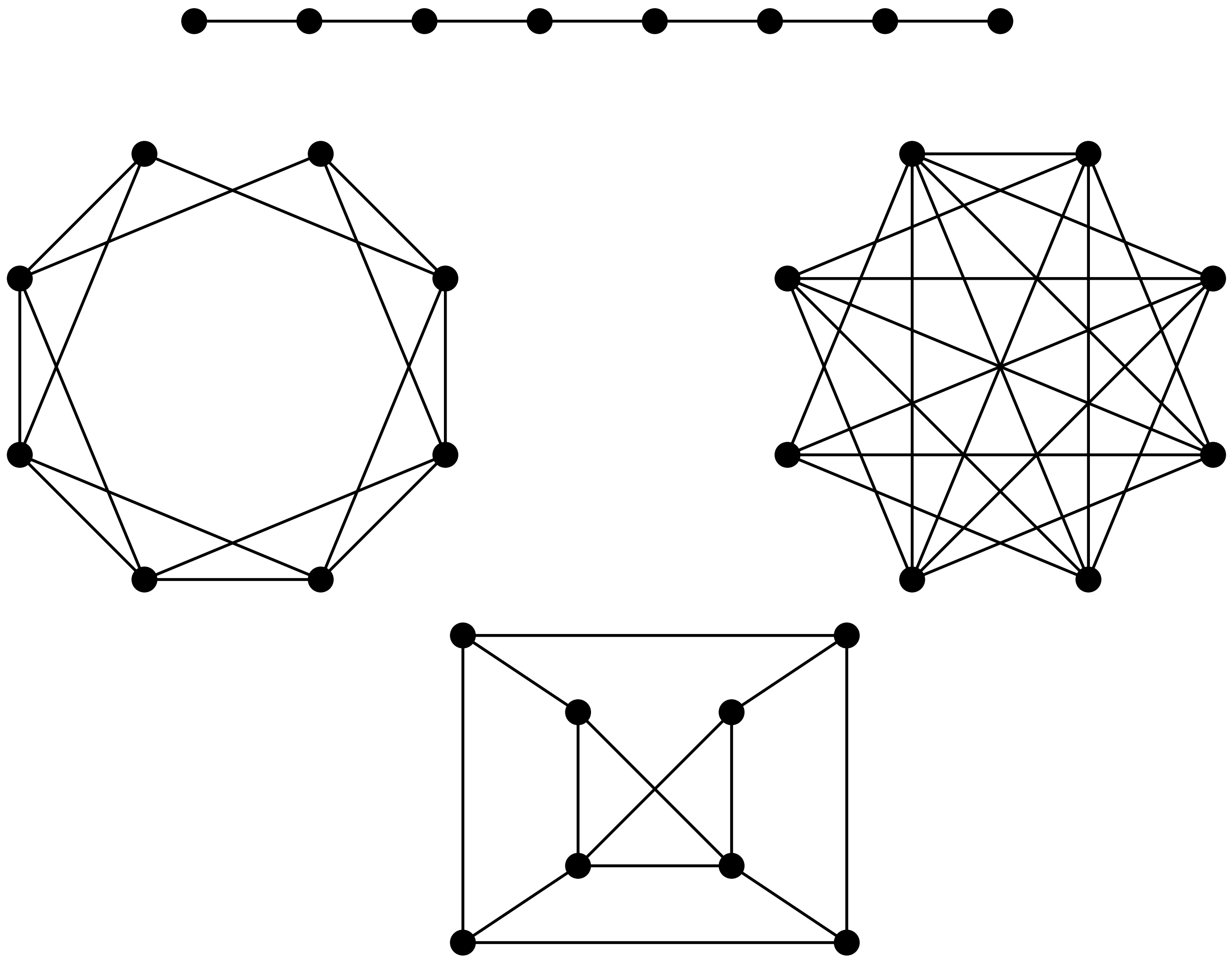


Theorem (Godsil, G., Sobchuk 2019+)

walk-regular graph ⇒ graph with $\widehat{M}$ having constant diagonal

What do these graphs have in common?



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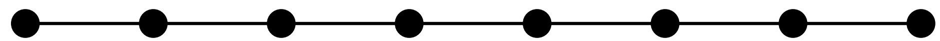

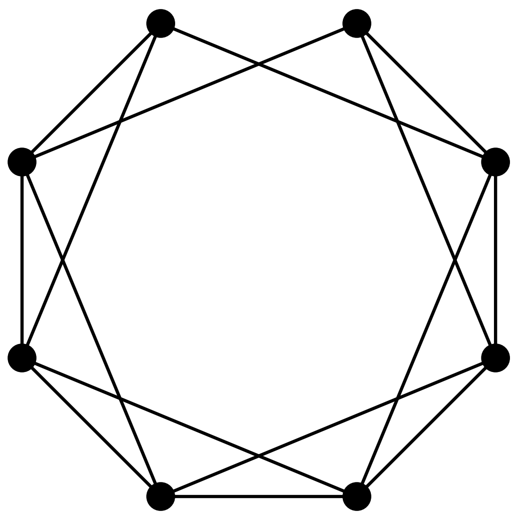

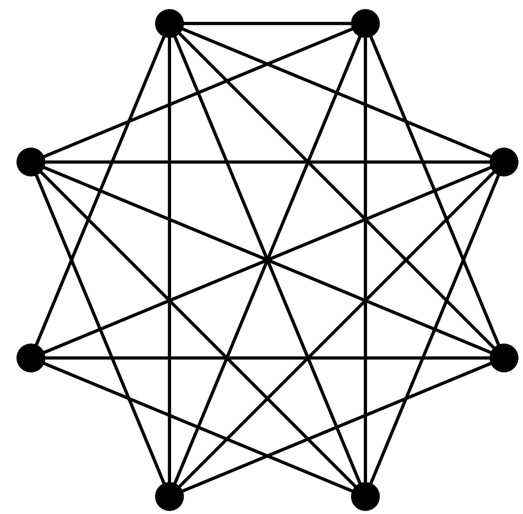

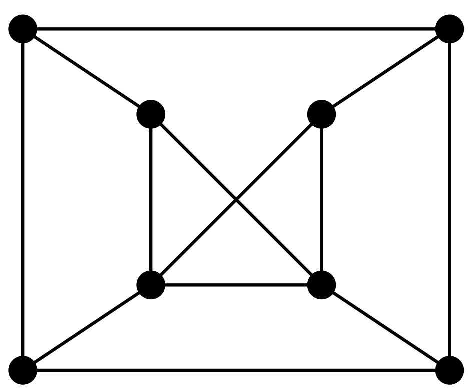

Graphs on 8 vertices attaining the minimum trace with respect to $\widehat{M}_L$.

Thanks!