# Inverse problem for a semi-linear elliptic equation

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# An inverse problem

Let $(M, g)$ be a smooth Riemannian manifold with boundary and consider

$$\begin{aligned} \Delta_g u + V(x,u) &= 0, \quad \text{in } M, \\ u &= f, \quad \text{on } \partial M. \end{aligned} \tag{1}$$

There is a unique small solution to (1) for each small, smooth $f$, under suitable assumptions on $V$. We define the Dirichlet-to-Neumann map

$$\Lambda_V f = \partial_\nu u|_{\partial M}.$$

The case $V(x,z) = V_1(x)z$ corresponds to the classical:

Calderón problem. Find $V$ given $\Lambda_V$.

We will consider non-linearities of the form $V(x,z) = \sum_{k=2}^{\infty} V_k(x) z^k$.

# On previous literature in the linear case $V(x, z) = V_1(x)z$

**Standing assumption.** The dimension $n$ of $M$ satisfies $n \geq 3$.

- Linearized inverse problem in the Euclidean case: [Calderón'80] uses density of products of harmonic exponentials, that is, $e^{izx}e^{i\bar{z}x} = e^{i2\xi x}$ where $z = \xi + i\eta$, $|\xi| = |\eta|$ and $\xi \perp \eta$.
- Euclidean case: [Sylvester-Uhlmann'87] replaces harmonic exponentials by CGO solutions $e^{izx} + R$ with $R \to 0$ as $|z| \to \infty$.
- Product case $g(t, x) = dt^2 + g_0(x)$ with $g_0$ simple: [Dos Santos Ferreira-Kenig-Salo-Uhlmann'09] takes $e^{\lambda(t+ir)}a + R$ with $r$ the radial coordinate in geodesic normal coordinates.
- Product case $g(t, x) = dt^2 + g_0(x)$ with $g_0$ having invertible geodesic ray transform: [Dos Santos Ferreira-Kurylev-Lassas-Salo'16] uses a Gaussian beam construction.

We will use Gaussian beams.

# On previous literature in the non-linear case

If the inverse problem for the linear case $\Delta_g u + V_1(x)u = 0$ is solved, the non-linear case $\Delta_g u + V(x, u) = 0$ can often be reduced to the linear case via linearization.

- Euclidean case [Isakov-Sylvester'94], [Sun'10], . . . .

Multiple-fold linearization allows for solving inverse problems for non-linear wave equations in cases where the corresponding linear problem is open [Kurylev-Lassas-Uhlmann'18].

We will use multiple-fold linearizations.

Before submitting our results we became aware of an upcoming preprint of Lassas, Liimatainen, Lin and Salo, considering independently the same problem, and we agreed with them to post to arXiv at the same day[1].

[1] We didn't quite succeed, though

# Recovery of third and higher order terms

Suppose that $M \subset \mathbb{R} \times M_0$ and that, writing $x = (t, x')$,

$$g(t,x') = c(t,x')(dt^2 + g_0(x')), \quad V(t,x',z) = \sum_{k=3}^{\infty} V_k(t,x')z^k.$$

Recall that $\Lambda_V f = \partial_\nu u|_{\partial M}$ where $u$ solves

$$\begin{aligned} \Delta_g u + V(x,u) &= 0, \quad \text{in } M, \\ u &= f, \quad \text{on } \partial M. \end{aligned}$$

**Theorem** [Lassas-Liimatainen-Lin-Salo, Feizmohammadi-L.O.].
$\Lambda_V$ determines $V$ uniquely.

Both the proofs are based on showing that products of **four** harmonic functions are dense (but we give different density proofs).

# Recovery of second order terms

Suppose now that $M \subset \mathbb{R} \times M_0$ is 3-dimensional and that

$$g(t,x') = c(t,x')(dt^2 + g_0(x')), \quad V(t,x',z) = \sum_{k=2}^{\infty} V_k(t,x')z^k.$$

We choose an extension $\hat{M}_0$ of $M_0$.

**Theorem** [Feizmohammadi-L.O.] Let $\gamma$ be a geodesic on $\hat{M}_0$, and suppose that there is such $\gamma(a) \in \hat{M}_0 \setminus M_0$ that no point along $\gamma$ is conjugate to $\gamma(a)$. Then $\Lambda_V$ determines $V(\cdot, \gamma(r), \cdot)$ for all $\gamma(r) \in M_0$.

If for each $p \in M_0$ there is a geodesic $\gamma_p$ with the above property, then our proof shows that products of **three** harmonic functions are dense.

# Gaussian beams

We consider Fermi coordinates $(r, y)$ along the geodesic $\gamma(r) = (r, 0)$. There are two families of solutions to $\Delta_g u = 0$ of the form

$$U_\lambda(t, r, y) = e^{\lambda(t+ir)+\lambda i H(r) y^2 + \dots}(Y^{-1/2}(r) + \dots),$$
$$V_\lambda(t, r, y) = e^{-\bar{\lambda}(t+ir)-\bar{\lambda} i \bar{H}(r) y^2 + \dots}(\bar{Y}^{-1/2}(r) + \dots),$$

where $\lambda = \sigma + i\tau$, $H = \dot{Y} Y^{-1}$ and $Y$ solves the Jacobi equation

$$\ddot{Y} - KY = 0 \qquad (K \text{ is the Ricci (1,1)-tensor})$$

in the (1-dimensional) orthogonal complement $\dot{\gamma}^\perp$ of $\dot{\gamma}$, together with the additional constraint:

(C) $Y(r_0) \neq 0$ and $\operatorname{Im} H(r_0) > 0$ for some $r_0$.

# Two-fold linearization

To simplify the presentation, we suppose that $V(x,z) = q(x)z^2$.
Let $f_\lambda$ be the trace of the Gaussian beam solution $U_\lambda$, let $\epsilon > 0$ be small, and let $u$ be the solution of

$$\begin{aligned} \Delta_g u + qu^2 = 0, &\qquad \text{in } M, \\ u = \epsilon f_\lambda, &\quad \text{on } \partial M. \end{aligned}$$

Then $w = -\partial_\epsilon^2 u|_{\epsilon=0}$ satisfies

$$\begin{aligned} \Delta_g w = qU_\lambda^2, &\quad \text{in } M, \\ w = 0, &\quad \text{on } \partial M. \end{aligned}$$

Observe that $\partial_\nu w|_{\partial M} = -\partial_\epsilon^2 \Lambda_V(\epsilon f_\lambda)|_{\epsilon=0}$ and therefore is known. For any $v$ satisfying $\Delta_g v = 0$ it holds that

$$\int_M qU_\lambda^2 v\, dx = \int_M (\Delta_g w) v\, dx - \int_M w \Delta_g v\, dx = \text{known bd terms.}$$

# Reduction to an integral transform

Recall the form of the Gaussian beam solutions

$$U_\lambda(t,r,y) = e^{\lambda(t+ir)+\lambda i H(r) y^2+\dots}(Y^{-1/2}(r)+\dots),$$
$$V_\lambda(t,r,y) = e^{-\bar\lambda(t+ir)-\bar\lambda i \bar H(r) y^2+\dots}(\bar Y^{-1/2}(r)+\dots),$$

where $\lambda = \sigma + i\tau$. We can recover the integral

$$\int_M q U_\lambda^2 V_{2\lambda}\, dx = \int_M q e^{4i\tau t - 4\tau r - \sigma \operatorname{Im} H y^2+\dots}(|Y|^{-1} Y^{-1/2}+\dots)\, dt\, dr\, dy.$$

Equations $H = \dot Y Y^{-1}$ and $\ddot Y - KY = 0$ imply that $(\operatorname{Im} H)^{1/2} = c|Y|^{-1}$ with a constant $c \neq 0$. Applying stationary phase for $\sigma \to \infty$ gives

$$\int_{\mathbb{R}} \hat q(-4\tau, r, 0) e^{-4\tau r} Y^{-1/2}(r)\, dr,$$

where $q$ is extended by zero outside $M$ and $\hat q$ denotes its Fourier transform with respect to $t$. This leads to inversion of an integral transform.

# Jacobi transform

We denote by $\mathbb{Y}_\gamma$ the set of Jacobi fields on a geodesic $\gamma$ that are normal (i.e. orthogonal to $\dot\gamma$) and satisfy

(C) $Y(r_0) \neq 0$ and $\operatorname{Im} \dot Y Y^{-1}(r_0) > 0$ for some $r_0$.

We define an integral transform, that we call the Jacobi transform,

$$\mathcal{J}_\gamma f(Y) = \int_{\mathbb{R}} f(r)\, Y^{-1/2}(r) dr, \quad Y \in \mathbb{Y}_\gamma,$$

where $f \in C(\mathbb{R})$ is supported on $\gamma^{-1}(M_0)$.

**Proposition** [Feizmohammadi-L.O.] Let $\gamma$ be a geodesic on $\hat M_0$, and suppose that there is such $\gamma(a) \in \hat M_0 \setminus M_0$ that no point along $\gamma$ is conjugate to $\gamma(a)$. Then the Jacobi transform $\mathcal{J}_\gamma$ is injective.

# Inversion of the Jacobi transform

We suppose that $\mathcal{J}_\gamma f = 0$ and show that $f = 0$.

Consider the normal Jacobi fields $Y_1$ and $Y_2$ satisfying

$$\begin{cases} Y_1(a) = 0, \\ \dot{Y}_1(a) = 1, \end{cases} \qquad \begin{cases} Y_2(a) = 1, \\ \dot{Y}_2(a) = 0. \end{cases}$$

We set $Y = Y_1 - i\epsilon Y_2$, $\epsilon > 0$, and see that $Y$ satisfies the condition (C).

By the non-conjugacy assumption, $Y_1 > 0$ on $\operatorname{supp}(f)$, and

$$0 = \int_{\mathbb{R}} f(r)\, Y^{-1/2}(r) dr = \int_{\mathbb{R}} \tilde{f}(r)\, (1 - i\epsilon X(r))^{-1/2} dr,$$

where $\tilde{f} = f\, Y_1^{-1/2}$ and $X = Y_2 Y_1^{-1}$. By expanding in Taylor series in $\epsilon$,

$$\int_{\mathbb{R}} \tilde{f}(r)\, X^k(r) dr = 0, \quad k = 0, 1, 2, \ldots.$$

# Inversion of the Jacobi transform continues

Recall that $X = Y_2 Y_1^{-1}$. It remains to show that

$$\int_{\mathbb{R}} \tilde{f}(r)\, X^k(r) dr = 0,\ \ k = 0, 1, 2, \ldots, \quad \Rightarrow \quad \tilde{f} = 0.$$

Supposing that we can change variables $s = X(r)$, we have

$$\int_{\mathbb{R}} h(s)\, s^k ds = 0, \quad k = 0, 1, 2, \ldots,$$

where $h(s) = \tilde{f}(r)\dot{X}(r)$. This again implies that $h = 0$.

To justify the change of variables, we show that $X$ is strictly decreasing:

$$\dot{X} = W Y_1^{-2}, \quad W = \dot{Y}_2 Y_1 - Y_2 \dot{Y}_1,$$

where the Wronskian satisfies $W(r) = W(a) = -1$.

# Open questions

- Recovery of the second order term in higher dimensions and under weaker geometric assumptions.
- Recovery of the first order term under the assumption that $V(x, z)$ is genuinely non-linear.

Related to the second question, we considered recently the following problem: Let $g$ be a globally hyperbolic Lorentzian metric on $\mathbb{R} \times \Omega$, let $T > 0$ and let $B \subset \Omega$ be open and bounded. Define $\Lambda_q f = u|_{(0,T)\times B}$ for small $f \in C_0^\infty((0, T) \times B)$ where $u$ solves

$$\begin{cases} \Box_g u + qu + u^3 = f, & \text{in } \mathbb{R} \times \Omega, \\ u|_{t<0} = 0. \end{cases}$$

**Theorem** [Feizmohammadi-L.O.]. $\Lambda_q$ determines $q$ on the causal diamond generated by $(0, T) \times B$.