Carleman estimates for geodesic X-ray transforms

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Joint work with Mikko Salo

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Setting

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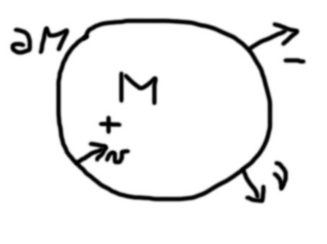

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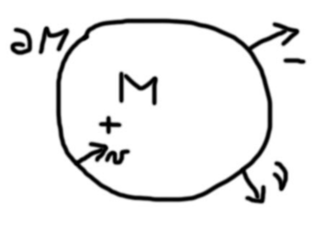

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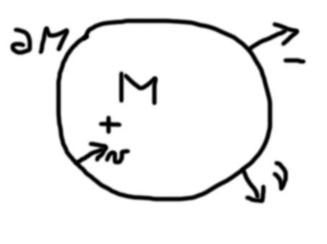

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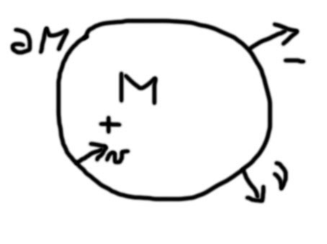
- We will assume ∂M is strictly convex (positive definite second fundamental form).



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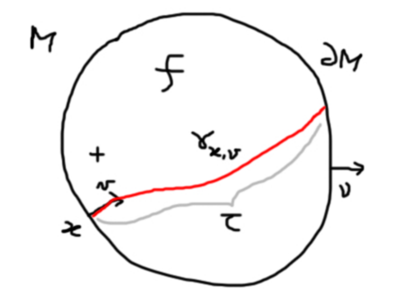
By Morse theory a non-trapping manifold with strictly convex boundary is contractible (otherwise it would contain a closed geodesic).



- Given $f \in C(M, \mathbb{R})$ define for $(x, v) \in \partial_+ SM$

$$If(x,v) := \int_0^{\tau(x,v)} f(\gamma_{x,v}(t))\, dt$$

where $\gamma_{x,v}$ is the unique geodesic determined by (x, v).

Different methodologies for determining f from If:

1. *Explicit inversion formulas* in symmetric geometries (e.g. homogeneous spaces).

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3. *Energy methods* for transport PDE (Mukhometov 1977, Sharafutdinov,..., P-Salo-Uhlmann 2013). Pestov identity.

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Let φ_t denote the geodesic flow of (M, g) and X the geodesic vector field on SM, so that X acts on smooth functions on SM by

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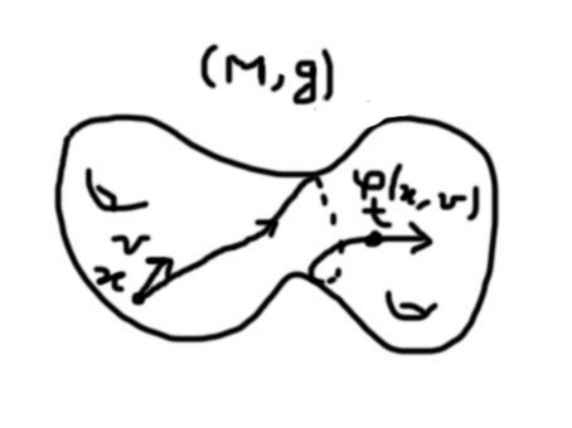

If (M, g) is a strictly convex domain in $\mathbb{R}^2$ with the usual Euclidean metric we have

$$\begin{aligned} X = v \cdot \nabla_x &= e^{i\theta}\partial + e^{-i\theta}\bar{\partial} \\ &= \cos\theta \frac{\partial}{\partial x_1} + \sin\theta \frac{\partial}{\partial x_2} \end{aligned}$$

where θ is the angle v makes with the vector $e_1 = (1, 0)$.

Vertical Fourier Analysis

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The operator $X : C^\infty(SM) \to C^\infty(SM)$ admits a nice splitting $X = X_+ + X_-$, where $X_\pm : \Omega_m \to \Omega_{m\pm1}$.

The Carleman estimate

Theorem 1 (P-Salo 2018)

Let (M, g) *be compact with sectional curvature* $\leq -\kappa$ *where* $\kappa > 0$*. Let also* $\phi_l = \log(l)$*. For any* $\tau \geq 1$ *and* $m \geq 1$*, one has*

$$\sum_{l=m}^{\infty} e^{2\tau\phi_l} \|u_l\|^2 \leq \frac{(d+4)^2}{\kappa\tau} \sum_{l=m+1}^{\infty} e^{2\tau\phi_l} \|(Xu)_l\|^2$$

whenever $u \in C^\infty(SM)$ *(with* $u|_{\partial(SM)} = 0$ *in the boundary case).*

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The case $m = 2$ is at the heart of the proof that closed negatively curved manifolds are spectrally rigid (Guillemin-Kazhdan 1980, Croke-Sharafutdinov 1998).

Attenuated X-ray

Let $\Phi : M \to \mathbb{C}^{n\times n}$ be given (matrix attenuation). For $f \in C(M, \mathbb{C}^n)$ and $(x, v) \in \partial_+ SM$ define

$$I_\Phi(f)(x, v) := \int_0^{\tau(x,v)} U(t) f(\gamma_{x,v}(t))\, dt,$$

where U solves

$$\dot{U} - U\Phi(\gamma_{x,v}(t)) = 0, \quad U(0) = \mathrm{Id}.$$

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When $n = 1$, and (M, g) a domain in $\mathbb{R}^2$, this reduces to the classical attenuated X-ray transform.

Injectivity

Theorem 2 (P-Salo 2018)

Assume (M, g) *is a compact simply connected manifold with strictly convex boundary and of negative sectional curvature. Let* $\Phi \in C^\infty(M, \mathbb{C}^{n \times n})$ *be given. If* $I_\Phi(f) = 0$*, then* $f = 0$*.*

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For $d = 2$, Theorem 2 has no competitor. Open for simple surfaces and $n \geq 2$ ($n = 1$ is due to Salo-Uhlmann 2011).

How to use the Carleman estimate

If $I_\Phi(f) = 0$, then it is standard that there is $u \in C^\infty(SM, \mathbb{C}^n)$ such that

$$Xu + \Phi u = f \in \Omega_0, \quad u|_{\partial SM} = 0.$$

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Since $(\Phi u)_l = \Phi u_l$, we see using the PDE that

$$\|(Xu)_l\| \leq C\|u_l\|, \qquad l \geq 1,$$

where $C = \|\Phi\|_{L^\infty(M)}$.

We input this information into the estimate

$$\sum_{l=1}^{\infty} e^{2\tau\phi_l} \|u_l\|^2 \leq \frac{(d+4)^2}{\kappa\tau} \sum_{l=2}^{\infty} e^{2\tau\phi_l} \|(Xu)_l\|^2$$

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Since $Xu_0 \in \Omega_1$, using the PDE again we obtain $f = 0$ as desired.

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- If Φ is skew-hermitian, U (and hence C_Φ) takes values in the unitary group $U(n)$.

Problem. Does C_Φ determine Φ?

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The map $\Phi \mapsto C_\Phi$ is sometimes called the non-abelian X-ray transform of Φ. This map is non-linear!

For $n = 1$, we can write

$$C_\Phi(x, v) = \exp\left(\int_0^{\tau(x,v)} \Phi(\gamma_{x,v}(t))\, dt\right)$$

and knowing C_Φ is the same as knowing the standard X-ray transform of the function Φ:

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Early work on this problem by Vertgeim (1992), R. Novikov (2002) and G. Eskin (2004).

Relation between linear and non-linear

Pseudo-linearization identity (cf. Stefanov-Uhlmann 1998 for lens rigidity) :

$$C_\Phi^{-1} C_\Psi = Id + I_{\Theta(\Phi,\Psi)}(\Psi - \Phi),$$

where $I_{\Theta(\Phi,\Psi)}$ is an attenuated X-ray transform with matrix attenuation $\Theta(\Phi,\Psi)$, an endomorphism on $\mathbb{C}^{n\times n}$ with pointwise action

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Hence the non-linear problem is solved once we solved the linear one!

Ideas for the Carleman estimate

If $u \in C^\infty(SM)$, there is a splitting (induced by Sasaki metric)

$$\nabla_{SM} u = \underbrace{(Xu)X + \overset{\mathtt{h}}{\nabla} u}_{\text{x}-\text{derivatives}} + \underbrace{\overset{\mathtt{v}}{\nabla} u}_{\text{v}-\text{derivatives}} .$$

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The basic energy identity for $P := \overset{\mathtt{v}}{\nabla} X$ (the Pestov identity) reads

$$\|Pu\|^2 = ((-X^2 - R)\overset{\mathtt{v}}{\nabla} u, \overset{\mathtt{v}}{\nabla} u) + (d-1)\|Xu\|^2$$

where R is the Riemann curvature tensor of (M, g).

- Spherical harmonics expansion in $v \in S_x M$

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- Pestov energy identity for Pu localizes in frequency.
- Multiply frequency localized estimates by suitable weights.
- Add up the weighted estimates, use negative curvature of absorb errors.