Topological Free Entropy

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χ_{top}

idea: matricial microstates with approximating noncommutative moments replaced by "norm-microstates"

Connection with "largest eigenvalue" (or norm) facts for random matrices

χ_{top} C^*-algebraic quantity
(i.e. "topological")

versus

χ v. Neumann algebraic
quantity (i.e. "measurable")

[Letters in Mathematical Physics 62
pp. 71-82, 2002]

Free Entropy (microstates) $\chi(X_1, \dots, X_n)$

$X_j = X_j^* \in (M, \tau)$, $1 \le j \le n$

v. Neumann alg. trace-state

$\Gamma(X_1, \dots, X_n : m, k, \varepsilon)$

$(A_1, \dots, A_n) \in (\mathcal{M}_k^{sa})^n$

$|\tau(X_{i_1} \cdots X_{i_p}) - k^{-1} \mathrm{Tr}_k(A_{i_1} \cdots A_{i_p})| < \varepsilon$

$1 \le p \le m$

$$\limsup_{k\to\infty}\left(k^{-2}\log \operatorname{vol}\Gamma(\ldots)+\frac{n}{2}\log k\right)$$ (4

$$\inf_{\varepsilon>0}\ \inf_{m\in\mathbb{N}}$$

$$\chi(X_1,\ldots,X_n).$$

Topological Free Entropy

$\chi_{top}(a_1, \ldots, a_n)$, $a_j = a_j^* \in A$ unital C^*-algebra

$\Gamma_{top}(a_1, \ldots, a_n; k, \varepsilon, P_1, \ldots, P_m)$ $k \in \mathbb{N}, \varepsilon > 0$

$P_1 \ldots P_m \in \mathbb{C}\langle X_1, \ldots, X_n \rangle$

$C_1, \ldots C_n \in \mathcal{M}_k^{sa}$

$$\big| \|P_j(C_1, \ldots, C_n)\| - \|P_j(a_1, \ldots, a_n)\| \big| < \varepsilon$$

$1 \le j \le m$

$$\limsup_{k\to\infty}\left(k^{-2}\log \mathrm{vol}\, \Gamma_{top}(\cdots) + \frac{n}{2}\log k\right)$$

$$\inf_{\varepsilon>0\ m\in\mathbb{N},\, P_1\cdots P_m\in\mathbb{C}\langle X_1\cdots X_n\rangle}$$

Subadditive

$$\chi_{top}(a_1,\dots a_n, b_1 \cdots b_m) \le \chi_{top}(a_1\cdots a_n) + \chi_{top}(b_1\cdots b_m)$$

Upper Semicontinuity

$$(a_1^{(p)},\dots,a_m^{(p)}) \in A^{(p)}_{sa},\ (a_1\cdots a_n)\in A$$

converge in "norm-distribution"

$$\lim_{p\to\infty} \|P(a_1^{(p)}\cdots a_m^{(p)})\| = \|P(a_1\cdots a_n)\|,\ P\in\mathbb{C}\langle X..X_n\rangle$$

$$\Rightarrow \limsup_{p\to\infty} \chi_{top}(a_1^{(p)}\cdots a_m^{(p)}) \le \chi_{top}(a_1\cdots a_n)$$

Change of Variables

$F_1, \dots, F_n$, $G_1, \dots, G_n$ noncommutative power series in n variables

$(F_1 \dots F_n)$, $(G_1 \dots G_n)$ inverses with lots of convergence properties

then

$$\chi_{top}(F_1(a_1 \dots a_n), \dots, F_n(a_1, \dots a_n)) \leq$$

$$\leq \chi_{top}(a_1, \dots a_n) + n \log \|\tilde{D} F(a_1 \dots a_n)\|$$

$\tilde{D} F \in M_n \otimes A \otimes A^{op}$ (spatial $\otimes$)

Competing quantity: <u>Free Capacity</u>

$a_j = a_j^* \in A$, $1 \le j \le n$ generate A

$$K(a_1, \dots, a_n) = \sup_{\substack{\tau \in TS(A) \\ \text{trace states}}} \chi(a_1(\tau), \dots, a_n(\tau))$$

$(a_1(\tau), \dots, a_n(\tau)) \in A(\tau) = W^*(A, \tau)$

<u>Subadditive</u>

Change of Variables

$$K(F_1(a_1,\dots,a_m),\dots,F_m(a_1,\dots,a_m)) \leq$$

$$\leq K(a_1,\dots,a_m) + \sup_{\tau\in TS(A)} \log|J|(F_1\dots F_n)(a_1(\tau),\dots a_m(\tau))$$

(Kadison-Fuglede |det| of $\widetilde{D}F$)

Case $n=1$ $K(a)$ $(A = C(\sigma(A)))$

$$\tau \in TS(A) \longleftrightarrow \mu \in \operatorname{Prob}(\sigma(a))$$

$$\chi(a(\tau)) = \theta + \iint \log|s-t|\, d\mu(s)\, d\mu(t)$$

$$e^{K(a)-\theta} = \operatorname{cap}(\sigma(a))$$ logarithmic capacity

Big Problem:

$$K(a_1, \dots, a_n) = \chi_{top}(a_1 \dots a_n)$$

(A generated by $a_1 \dots a_n$) ?

[assuming A has sufficient trace-states]

Fact: A generated by $a_1 \dots a_n$

$$\chi_{top}(a_1 \dots a_n) \leq K(a_1 \dots a_n)$$

Some examples of equality

1°. Semicircular case

$A = C^*(S_1, \ldots, S_n)$

$S_1, \ldots, S_n$ semicircular system

$\chi_{top}(S_1, \ldots, S_n) = K(S_1, \ldots, S_n) =$

$= \chi(S_1, \ldots, S_n)$

(A has unique trace-state)

essential ingredient: Haagerup-Thorbjoernsen Th.

$$\|P(T_1(k),\dots,T_n(k))\| \xrightarrow[k\to\infty]{a.s.} \|P(S_1,\dots,S_n)\|$$

hermitian Gaussian
k×k i.i.d. RM

[generalizations of H–T by Capitaine, Donati-Martin, Male, Collins, Bordenave should be relevant for going beyond the semicircular case].

2°. Universal n-tuple of Hermitian Contractions

$A = C([-1,1]) * \dots * C([-1,1])$ full free product

T_j identical function in j-th copy of $C([-1,1])$

$$(T_1,\dots,T_n)$$

$$\chi_{top}(T_1,\dots,T_n) = K(T_1,\dots,T_n)$$

maximum in definition of K is attained for $\tau = \tau(1) * \dots * \tau(1)$

$\tau(1)$ classical equilibrium measure on $[-1,1]$

$$\pi^{-1}(1-x^2)^{-1/2}dx$$

<u>Free Equilibrium Trace-state for generator $a_1,\dots,a_n$ of A</u> (assume $TS(A) \neq \emptyset$)

$\tau \in TS(A)$ so that $K(a_1,\dots a_n) = \chi(a_1(\tau)\dots a_n(\tau))$

always exists such a factor-state (extreme pt) not necessarily unique.

Remarks:

a) in paper dealt with

$\chi_{top}(a_1, \dots a_n : b_1 \dots b_m)$

(in the presence of $b_1 \dots b_m$)

simplified exposition in talk

b) Semimicrostates $\Gamma_{top\,1/2}(\dots)$

instead of $\big| \|P(a_1, \dots a_n)\| - \|P(C_1 \dots C_n)\| \big| < \varepsilon$

require only

$\|P(C_1, \dots, C_n)\| \leq \|P(a_1, \dots a_n)\| + \varepsilon$

$\chi_{top\,1/2} \geq \chi_{top}$ in general

If "norm-analogue of Connes-conjecture" holds for $a_1, \dots a_n$ *) i.e.

$$\Gamma_{top}(a_1, \dots a_n : k_0, \varepsilon, P_1 \cdots P_r) \neq \emptyset$$

(given $\varepsilon, P_1 \cdots P_r$ can find k_0 so that - -)

then

$$\chi_{top\,1/2}(a_1 \dots a_n) = \chi_{top}(a_1 \dots a_n)$$

*) $a_1 \dots a_n$ generate an MF-algebra in the Blackadar-Kirchberg sense.

Topological Free Entropy Dimension

free entropy dimension $\delta(X_1, \ldots X_n)$
initially defined (V) as a "Minkowski content" quantity derived from $\chi(X_1 \cdots X_n)$.
Later Kenley Jung found a direct equivalent definition based on ε-nets, which is better suited for adaptation to the topological context.

$\mathcal{X}$ metric space, $N_\varepsilon(\mathcal{X})$ minimum # of points in an ε-net of $\mathcal{X}$.

$$D_\varepsilon(a_1, \ldots, a_n) = \limsup_{k \to \infty} k^{-2} \log N_\varepsilon(\Gamma_{top}(a_1 \ldots a_n : k, \varepsilon, P_1 \ldots P_m))$$

N_ε w.r.t. uniform norm

$$\inf_m \inf_{P_1 \cdots P_m}$$

$$\delta_{top}(a_1 \ldots a_n) = \limsup_{\varepsilon \to 0} \frac{D_\varepsilon(a_1 \ldots a_n)}{|\log \varepsilon|}$$

Competing quantity: Free Entropy Dimension Capacity

$$K\delta(a_1,\dots a_n) = \sup_{\tau\in TS(A)} \delta_0(a_1(\tau),\dots,a_n(\tau))$$

$a_1\dots a_n$ hermitian generator of A

δ_0 the "modified free entropy dimension"

technical ....

Examples:

$\chi_{top}(a_1,\dots a_n) > -\infty \Longrightarrow \delta_{top}(a_1,\dots,a_n) = n$

$\delta_{top}(S_1\dots S_n) = \delta_{top}(T_1\dots T_n) =$

$= K\delta(S_1\dots S_n) = K\delta(T_1\dots T_n) = n$

Big Questions:

1°. $\kappa\delta$ versus δ_{top}
equality?

2°. is $\delta_{top}(a_1, \dots a_n)$ independent
of the choice of the hermitian
generator $a_1 \cdots a_n$ of A?

Several results on evaluating $\delta_{top}(a_1, \ldots a_n)$ and on question 2° (δ_{top} for generators the same, question) obtained by Don Hadwin and Junhao Shen etc. Will show a few results, which don't require additional technical definitions.

$a_1, \dots, a_n$ generator of finite-dimensional C^*-algebra A

$$\delta_{top}(a_1, \dots, a_n) = 1 - \frac{1}{\dim_{\mathbb{C}} A}$$

(arXiv: 0708.0164)

$A_i = C(K_i)$, $K_i \subset \mathbb{R}$ compact

X_i identical function on K_i as element of $A = A_1 * \cdots * A_n$ full free product

$$\delta_{top}(X_1, \dots X_n) = n - \sum_{i=1}^{n} \frac{1}{\operatorname{card} K_i}$$

(arXiv: 0802.0281)

In $A \oplus B$ results on

$\delta_{top}(a_1 \oplus b_1, \ldots a_n \oplus b_n)$.

(arXiv: 0708.01687

More general assuming nuclearity conditions . –

(see for instance Li–Hadwin–Li–Shen J. Operator Theory 71 (2014) no. 1 pp. 15 – 44)