Algorithms to compute topological invariants of symmetric semi algebraic sets

Geometry of Real Polynomials, Convexity and Optimization
Banff 2019

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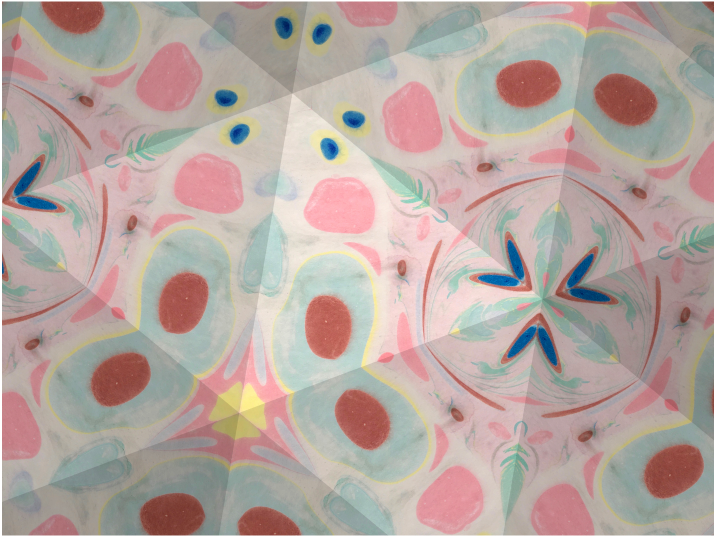
28 May, 2019

A first beautiful image

or my talk in a nutshell



Introduction

The objects we are looking at

In the sequel we will use

- a real closed field $\mathbf{R}$
- a finite set $\mathcal{P} \subset \mathbf{R}[X_1, \ldots, X_k]$ defining a real variety $\mathrm{V}_{\mathbf{R}}(\mathcal{P}) \subset \mathbf{R}^k$,
- or more generally a semi-algebraic $S \subset \mathbf{R}^k$, which can be described by $\mathcal{P}$
- $\mathcal{S}_k$ the symmetric group on k elements.

We are interested in the Betti numbers of S, i.e.,

$$b_i(S, \mathbb{F}) = \dim_{\mathbb{F}} H_i(S, \mathbb{F}),$$

and want to compute them in the case when S is symmetric.

Complexity

Geometric vs. algorithmic

Belief

The worst-case topological complexity of a class of semi-algebraic sets (measured by the Betti numbers for example) serve as a rough lower bound for the complexity of algorithms for computing topological invariants or deciding topological properties of this class of sets.

Cohomology of the quotient of symmetric sets

Theorem (Basu, R. 18)

Let $\mathcal{P} \subset \mathbf{R}[X_1, \ldots, X_k]^{S_k}_{\leq d}$ where $|\mathcal{P}| = s$ and $1 < d < s, k$. Consider a closed semi algebraic set $S \subset \mathbf{R}^k$ defined by $\mathcal{P}$. Then the following holds:

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1. *The sum of the Betti numbers of $S/\mathcal{S}_k$ is bounded by:*

$$b(S/\mathcal{S}_k, \mathbb{F}) \leq d^{O(d)} s^d k^{\lfloor d/2 \rfloor - 1}.$$

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2. *We have $b^i(S/\mathcal{S}_{\mathbf{k}}, \mathbb{F}) = 0$ for all $i \geq d$.*

Reminder : Vandermonde Varieties

Definition

Consider the continuous map

$$\Psi_d^{(k)}(\mathbf{x}) = (p_1^{(k)}(\mathbf{x}), \ldots, p_d^{(k)}(\mathbf{x})),$$

where p_i^k are the Newton powersums. Then, for $\mathbf{y} \in \mathbf{R}^{d'}$ we call

$$V_{d',\mathbf{y}} := (\Psi_{d'}^{(k)})^{-1}(\mathbf{y})$$

a Vandermonde variety.

Let $\mathcal{W}_k = \{\mathbf{x} \in \mathbf{R}^k \, : \, x_1 \leq x_2 \leq \ldots \leq x_k\}$ denote the Weyl-chamber.

Theorem (Arnold Giventhal and Kostov)

For $1 \leq d \leq k$ any every $\mathbf{y} \in \mathbf{R}^k$ the intersection of the Weyl chamber and the Vandermonde variety

$$V_{d,\mathbf{y}} \cap \mathcal{W}_k$$

is contractable.

Ideas behind the proof

Geometry of the Weyl chamber

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- Now consider $\mathcal{W}_k^{d\star} \subset \mathcal{W}_k^d$ the union of those faces corresponding to compositions of the form $(1, \ell_1, 1, \ell_2, \ldots)$.

Observation

$\mathcal{W}_k^{d\star}$ is the union of $\binom{k-\lceil d/2\rceil-1}{\lfloor d/2\rfloor-1} = (O_d(k))^{\lfloor d/2\rfloor-1}$ faces.

Lemma

Let $S_{k,d} = S \cap \mathcal{W}_k^{d\star}$. Then

$$\mathrm{H}^*(S_{k,d}, \mathbb{F}) \cong \mathrm{H}^*(S/\mathcal{S}_k, \mathbb{F}).$$

Algorithmic consequences

Theorem (Basu. R. '18)

For every fixed $d \geq 0$, there exists an algorithm that takes as input a

$$\mathcal{P}\text{-closed formula } \Phi, \text{ where } \mathcal{P} \subset \mathbf{R}[X_1, \ldots, X_k]^{\mathcal{S}_k}_{\leq d},$$

and outputs

$$b^i(S/\mathcal{S}_k, \mathbb{F}), 0 \leq i < d,$$

where $S = \mathrm{Reali}(\Phi, \mathbf{R}^k)$ whose complexity is bounded by

$$(|\mathcal{P}|kd)^{2^{O(d)}}$$

(which is polynomial in the $|\mathcal{P}|$ and k).

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Question

Can we also get hold of the Betti numbers efficiently?

Action on a space

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- If $\mathrm{char}(\mathbb{F}) = 0$ then

$$\mathrm{H}^*(X/G, \mathbb{F}) \xrightarrow{\sim} \mathrm{H}^*_G(X, \mathbb{F}) \xrightarrow{\sim} (\mathrm{H}^*(X, \mathbb{F}))^G.$$

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- So the results on the cohomology of the quotient are in fact results on the multiplicity of the *trivial* representation of S_k in $\mathrm{H}^*(X, \mathbb{F})$.

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- Let $(\lambda_1, \ldots, \lambda_l) \vdash k$ then the so called *Young-module* is

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- For each $\lambda \vdash k$ Young's rule gives

$$M^\lambda = \bigoplus_{\mu \vdash k} K(\lambda, \mu) \mathfrak{S}^\mu,$$

where $K(\lambda, \mu)$ are the so called Kostka-numbers.

Isotypic-Dcomposition

Theorem (Basu, R. 18+)

Let $P \in \mathbf{R}[X_1, \ldots, X_k]$ *symmetric with* $\deg(P) = d$ *and define* $V = V_{\mathbf{R}}(P)$*. We consider the decomposition*

$$\mathrm{H}^*(V, \mathbb{F}) = \bigoplus_{\mu \vdash k} m_\mu \mathfrak{S}^\mu .$$

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Then:

1. $m_\mu \leq k^{O(d^2)} d^d$;

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Then:

1. $m_\mu \leq k^{O(d^2)} d^d$;
2. $m_\mu \neq 0$ *only if the Durfee-square of μ is of size d. (the number of such partitions is polynomial in k).*

Observation

Given these bounds it seems hopeful, that there is a polynomial algorithm to compute all the m_μ - and thus all the Betti numbers.

Mirrored spaces

Let (W, S) be a Coxeter system, X be a CW-complex and $\mathcal{U}$ be a CW-complex obtained by pasting together copies of X, one for each element of W.
Then $(\mathcal{U}, W, S)$ is called a *mirrored space*.

Theorem (Davis)

For each $t \in T$ we define X_t is the intersection of X with the wall corresponding to t and for $T \subset S$ we set $X^T := \bigcup_{t \in T} X_t$. Then:

$$H_*(\mathcal{U}) \cong \bigoplus_{T \subset S} \mathrm{H}_*(X, X^T) \otimes_{\mathbb{Q}} \Psi^k_{S,S-T}.$$

where each $\Psi^k_{S,S-T}$ is a representation defined by Solomon.

Generalizing Arnold's work

- In case $W = \mathcal{S}_k$, the set of Coxeter generators S will be the set of transpositions $S = \{s_1, \ldots, s_{k-1}\}$, $s_i = (i, i+1), 1 \leq i \leq k-1$.
- One has for each $T \subseteq S$ the representation $\Psi^k_{S,S-T}$ may be understood as analogs of Specht-modules, but defined in terms of *MacMahon's tableau* rather than Young's tableau. Unlike the Specht-modules, they need not be irreducible!

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Theorem (Basu, R. '19+)

Let $d, k \in \mathbb{N}$, $3 < d \leq k$, $\mathbf{y} = (y_1, \ldots, y_d) \in \mathbf{R}^d$, and let $V^{(k)}_{d,\mathbf{y}}$ denote the Vandermonde variety defined by $p^{(k)}_1 = y_1, \ldots, p^{(k)}_d = y_d$, where $p^{(k)}_j = \sum_{i=1}^k X_i^j$. Then, for all $\lambda \vdash k$:

(a)

$$\mathrm{mult}_{\mathbb{S}^\lambda}(\mathrm{H}^i(V^{(k)}_{d,\mathbf{y}})) = 0, \ \textit{for } i \leq \mathrm{length}(\lambda) - 2d + 1,$$

(b)

$$\mathrm{mult}_{\mathbb{S}^\lambda}(\mathrm{H}^i(V^{(k)}_{d,\mathbf{y}})) = 0, \ \textit{for } i \geq k - \mathrm{length}({}^t\lambda) + 1,$$

Polynomial algorithm

Theorem (Basu, R. '19+)

For every fixed $d \geq 0$ and every fixed $\ell \geq 0$, there exists an algorithm that takes as input a $\mathcal{P}$-closed formula Φ, where $\mathcal{P} \subset \mathbf{R}[X_1, \ldots, X_k]^{\mathcal{S}_k}_{\leq d}$, and outputs $b^{\ell}(S, \mathbb{F}), 0 \leq i$, where $S = \mathrm{Reali}(\Phi, \mathbf{R}^k)$ whose complexity is bounded by by a quantity which is polynomial in the $|\mathcal{P}|$ and k.

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We may use Davis' formula to decompose the task of computing $b_i(S) = \dim_{\mathbb{Q}} \mathrm{H}^i(S)$ into two parts:

1. computing the dimensions of $\mathrm{H}^i(S_k, S_k^T)$;
2. computing the isotypic decompositions of the modules $\Psi_T^{(k)}$ for various subsets $T \subset Coxeter(k)$.

In oder to compute $b_i(S)$ for $i \leq \ell$, we only need to consider $T \subset S$ with $|T| < \ell + 2d - 1$.

Want to know more?

1. Bounding the equivariant Betti numbers of symmetric semi-algebraic sets *Adv. Math.* 305, pp. 803-855 (2017).
2. Efficient algorithms for computing the Euler-Poincaré characteristic of symmetric semi-algebraic sets. *Contem. Math*. 697, pp. 53-81 (2017).
3. On the equivariant Betti numbers of symmetric definable sets: vanishing, bounds and algorithms. *Selecta Math.* 24(4), pp 3241–3281 (2018).
4. On the isotypic decomposition of cohomology modules of symmetric semi-algebraic sets: polynomial bounds on multiplicities. to appear in *Int. Math. Res. Notices*.
5. Vandermonde varieties, mirrored spaces, and the cohomology of symmetric semi-algebraic sets. arXiv:1812.10994

Some beauty at the end..