Generalized Schur function determinants using Bazin-Sylvester identity

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(Joint with Meesue Yoo)

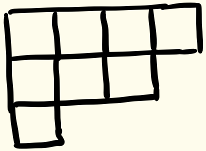
Outline

1. Review known determinant formulas for Schur function
   - Jacobi-Trudi, dual Jacobi-Trudi
   - Giambelli
   - Lascoux-Pragacz
   - Hamel-Goulden
   - Conjecture of Morales, Pak, Panova.
2. Main results

   Generalization of all of these formulas

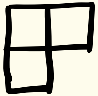
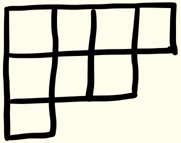
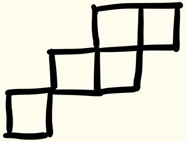
   to Macdonald's 9th variation of Schur function.

Basic definitions

- $\lambda = (\lambda_1, \dots, \lambda_\ell)$ is a **partition** of n
  if $\lambda_1 \geq \cdots \geq \lambda_\ell > 0$ and $\lambda_1 + \cdots + \lambda_\ell = n$
- **Young diagram** of $\lambda = (4,3,1)$ is 

$$\leq$$

$$\wedge \quad \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 2 \\ \hline 2 & 3 & 3 & \multicolumn{1}{|c}{} \\ \cline{1-3} 4 & \multicolumn{3}{|c}{} \\ \cline{1-1} \end{array}$$

- **semistandard Young tableau**
- $\mu =$  $\subseteq \lambda =$  **skew shape** $\lambda/\mu =$ 
- **Schur function**

$$s_{\lambda/\mu}(x) = \sum_{T \in SSYT(\lambda/\mu)} x^T, \qquad T = \begin{array}{ccc} & & \boxed{1}\boxed{3} \\ & \boxed{1}\boxed{2} & \\ \boxed{2} & & \end{array}, \qquad x^T = x_1^2 x_2^2 x_3^1$$

Jacobi-Trudi formula (1833)

For partitions λ, μ of length $\leq n$

$$s_{\lambda/\mu} = \det\left(h_{\lambda_i-\mu_j+i-j}\right)_{i,j=1}^{n},$$

$$h_k = s_{(k)} = s_{\square\square\square\cdots\square} = \sum_{i_1\leq\cdots\leq i_k} x_{i_1}\cdots x_{i_k}$$

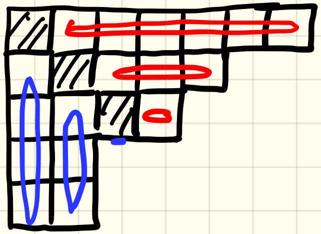
Dual Jacobi-Trudi formula

$$s_{\lambda/\mu} = \det\left(e_{\lambda_i'-\mu_j'+i-j}\right)_{i,j=1}^{n},$$

$$e_k = s_{(1^k)} = s_{\begin{smallmatrix}\square\\\square\\\vdots\\\square\end{smallmatrix}} = \sum_{i_1<\cdots<i_k} x_{i_1}\cdots x_{i_k}$$

Def) The Frobenius notation of λ is

$$\lambda = (\alpha \mid \beta) = (\alpha_1, \alpha_2, \ldots, \alpha_r \mid \beta_1, \beta_2, \ldots, \beta_r)$$



$\alpha_1 = 6$
$\alpha_2 = 3$
$\alpha_3 = 1$

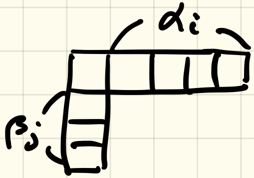
$\beta_1 = 4$, $\beta_2 = 3$, $\beta_3 = 0$

$$(7,5,4,2,2) = (6,3,1 \mid 4,3,0)$$

Giambelli formula (1903)

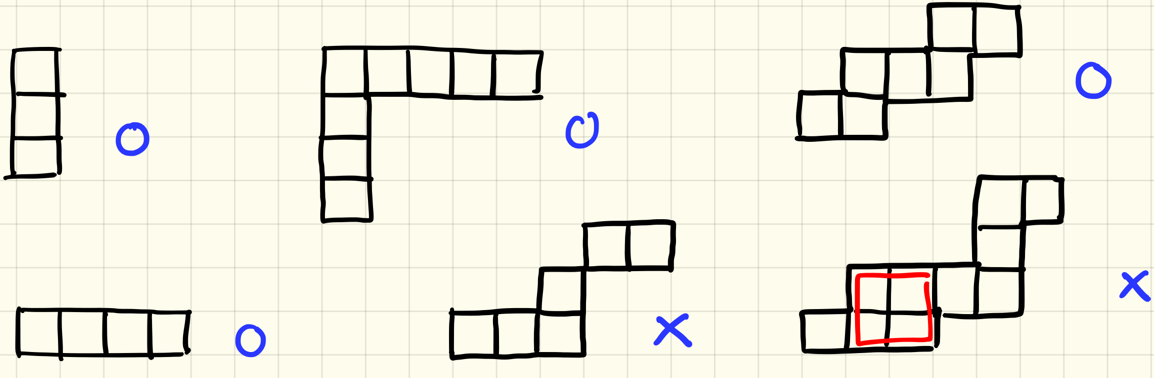
$$S_{(\alpha|\beta)} = \det\left(S_{(\alpha_i | \beta_j)} \right)_{i,j=1}^{r}$$

Note: $(\alpha_i | \beta_j)$ is a hook shape :



Def) A **border strip** is a connected skew diagram that does not contain 2×2 squares

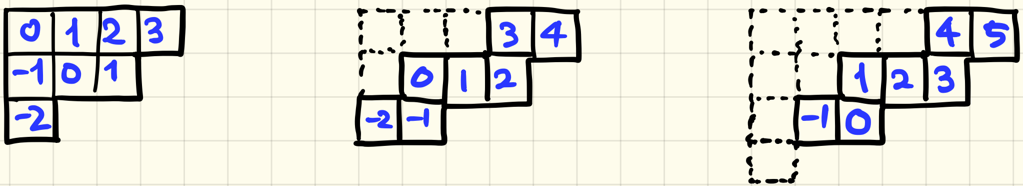
ex)



Def) The **content** of a cell $(i,j) \in \lambda/\mu$ is

$$c(i,j) = j - i.$$

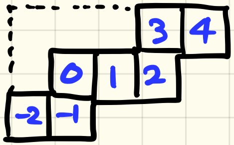
ex)

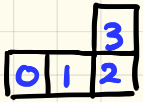


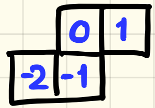
Def) For a border strip γ,

$$\gamma[a,b] := \{ x \in \gamma \ : \ a \leq c(x) \leq b \}.$$

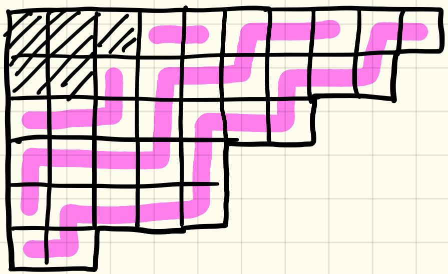
ex)

$\gamma =$ 

$\gamma[0,3] =$ 

$\gamma[-2,1] =$ 

Def) The Lascoux-Pragacz decomposition λ/μ



$\theta = (\theta_1, \theta_2, \ldots, \theta_k)$.

$\gamma = \theta_1$: the maximal outer border strip in λ/μ

Lascoux-Pragacz formula (1983)

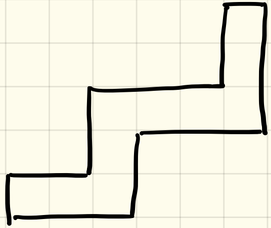
$$s_{\lambda/\mu} = \det\left(s_{\gamma[p_j, q_i]} \right)_{1 \leq i,j \leq k}$$

$p_j = c(\text{start of } \theta_j)$

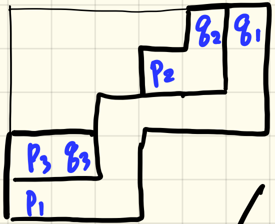
$q_i = c(\text{end of } \theta_i)$

$$s_{\lambda/\mu} = \det\left(s_{\gamma[p_j, q_i]} \right)_{i,j=1}^{r}$$

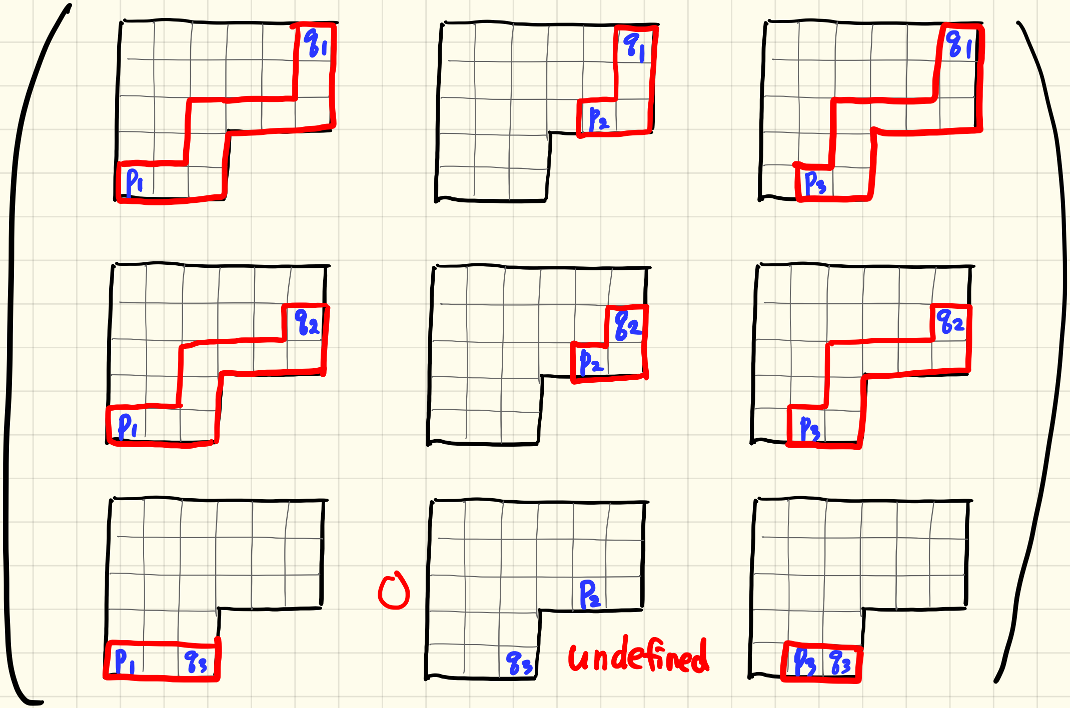
$\gamma = \theta_1 =$



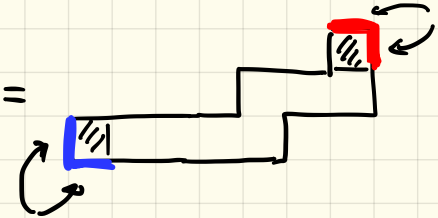
ex) $\lambda/\mu =$



$s_{\lambda/\mu} = \det$

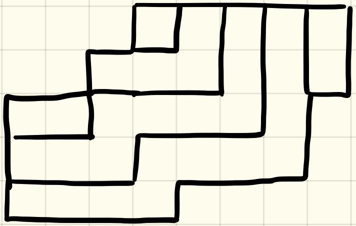


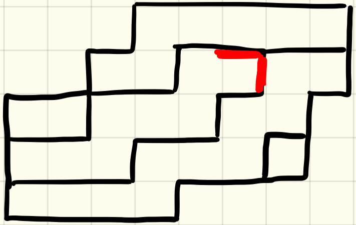
Def) An outside decomposition of λ/μ is a decomp $(\theta_1, \dots, \theta_k)$ of λ into border strips such that

$\theta_i =$ 

one of these is on the boundary of λ/μ.

one of these is on boundary of λ/μ

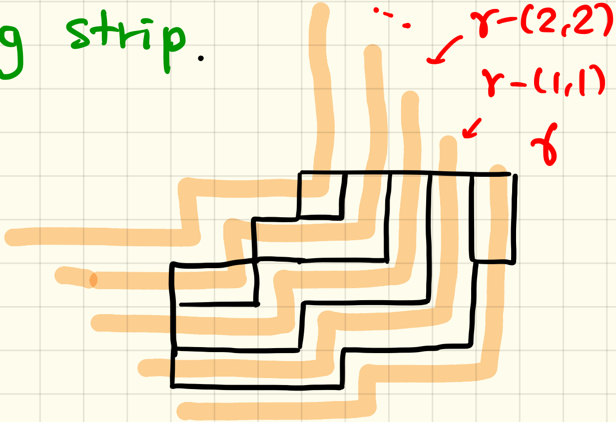
ex)  O

 X

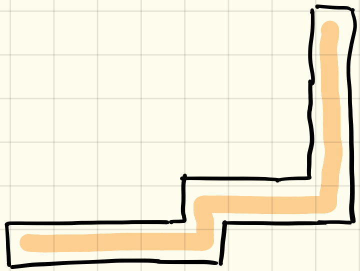
Thm (Chen, Yan, Yang, 2005)

Every outside decomposition is determined uniquely by a cutting strip.

ex)



cutting strip γ



Hamel-Goulden formula (1995)

$\Theta=(\theta_1,\dots,\theta_k)$: an outside decomp of λ/μ

γ = the cutting strip corresponding to Θ.

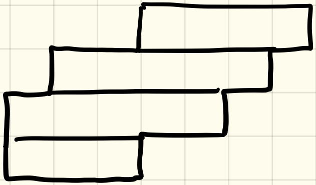
$$s_{\lambda/\mu}=\det\left(s_{\gamma[p_j,q_i]}\right)_{i,j=1}^{k}$$

$p_j = c(\text{start of } \theta_j)$

$q_i = c(\text{end of } \theta_i)$

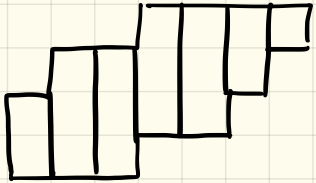
Hamel-Goulden formula contains the other formulas.

- Jacobi-Trudi



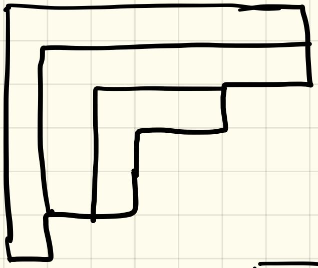
cutting strip 

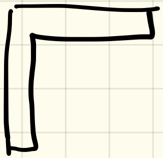
- dual Jacobi-Trudi



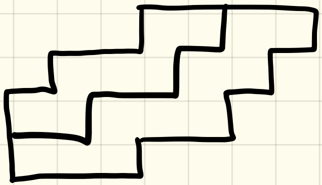
cutting strip 

- Giambelli



cutting strip 

- Lascoux-Pragacz



cutting strip = outer border

Chen, Yan, Yang showed that

JT, dual JT, Giambelli

LP, HG are all equivalent:

One formula can be obtained from another by applying elementary row operations

and

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \longrightarrow \left(\begin{array}{c|c|c} A & 0 & B \\ \hline 0 & 1 & 0 \\ \hline C & 0 & D \end{array}\right).$$

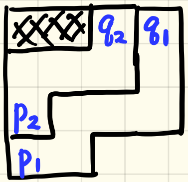
Thm (Morales, Pak, Panova, 2016)

$\Theta = (\theta_1, \dots, \theta_k)$: LP decomp of λ/μ (with some conditions)

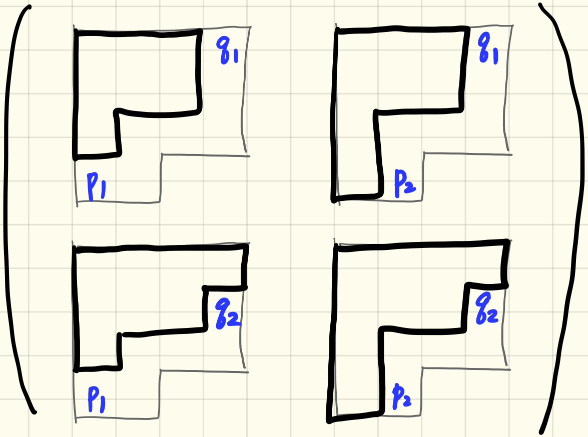
$\gamma = \theta_1$, $p_j = c(\text{start of } \theta_j)$, $q_i = c(\text{end of } \theta_i)$

$$s_\mu s_\lambda^{k-1} = \det\left(s_{\lambda \setminus \gamma[p_j, q_i]} \right)_{1 \le i,j \le k}.$$

ex) $\lambda/\mu =$



$$s_\mu s_\lambda = \det$$



Conjecture (MPP)

The same is true for factorial Schur functions.

LP vs MPP

LP: $s_{\lambda/\mu} = \det\left(s_{\gamma[p_j, q_i]} \right)_{1\le i,j\le k}$.

MPP: $s_\mu\, s_\lambda^{k-1} = \det\left(s_{\lambda \setminus \gamma[p_j, q_i]} \right)_{1\le i,j\le k}$.

<u>Goal</u> Find a common generalization of LP, MPP to Macdonald's 9th variation of Schur functions.

$h_{r,s}$ $(r, s \in \mathbb{Z},\ r \geq 1)$: independent variables

$h_{0,s} := 1$, $h_{r,s} := 0$ if $r < 0$.

$h_r := h_{r,0}$.

φ: automorphism on $\mathbb{C}[h_{r,s} : r, s \in \mathbb{Z},\ r \geq 1]$

$\varphi(h_{r,s}) := h_{r,s+1}$

Then $h_{r,s} = \varphi^s h_r$.

Macdonald's 9th variation of Schur function

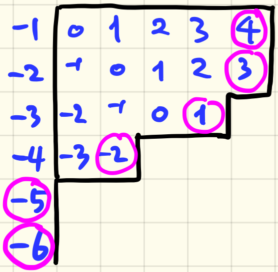
$$S_{\lambda/\mu} = \det\left(\varphi^{\mu_j - j + 1} h_{\lambda_i - \mu_j - i + j}\right)_{i,j=1}^{n}$$

- If $h_{r,s} = h_r(x)$, then $S_{\lambda/\mu}$ = usual Schur ftn
- If $h_{r,s} = h_r(x|\tau^s a)$, then $S_{\lambda/\mu}$ = factorial Schur ftn

Def) $\lambda \in \mathrm{Par}_n$ (partition with at most n parts)

$$C_n(\lambda) = \{\lambda_i - i \mid 1 \le i \le n\}$$

- λ is determined by $C_n(\lambda)$.

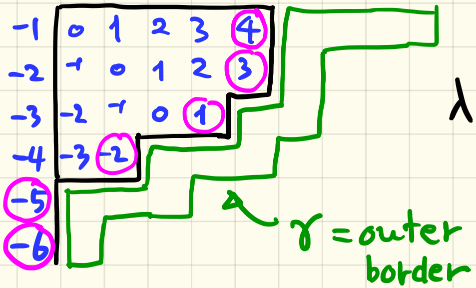


$$C_6(\lambda) = \{4, 3, 1, -2, -5, -6\}$$

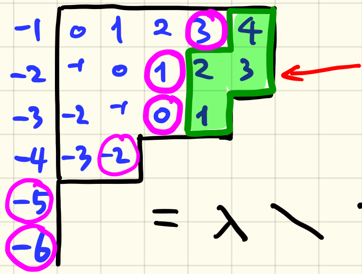
Def) For $a \in C_n(\lambda)$ and $b \geq -n$, $\lambda(a,b)$ is the unique $\mu \in \mathrm{Par}$ with $C_n(\mu) = (C_n(\lambda) - \{a\}) \cup \{b\}$.

If there is no such μ, $\lambda(a,b)$ is undefined.

$\lambda =$



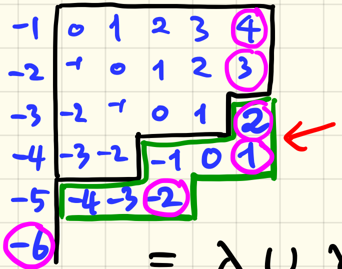
$\lambda(4,0) =$



removed

$= \lambda \setminus \gamma[1,4]$

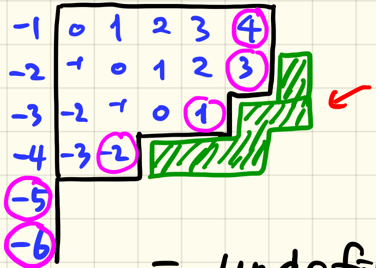
$\lambda(-5,2) =$



added

$= \lambda \cup \gamma[-4,2]$

$\lambda(3,-2)$



cannot be added

$=$ undefined.

$$\lambda(a,b) = \begin{cases} \lambda \setminus \gamma[b+1,a] & \text{if } a \geq b \\ \lambda \cup \gamma[a+1,b] & \text{if } a < b \\ \text{undefined} & \text{if the result is not a partition.} \end{cases}$$

Thm (K., Yoo)

$\lambda, \mu, \nu \in \mathrm{Par}_n$. $\quad \Theta = (\theta_1, \dots, \theta_k)$: LP decomp of λ/μ

$p_i = c(\text{start of } \theta_i)$, $q_i = c(\text{end of } \theta_i)$, $q_1 > q_2 > \cdots > q_k$

① $$s_{\nu/\lambda}^{k-1} s_{\nu/\mu} = \det\left((-1)^{\chi(p_j > q_i)} s_{\nu/\lambda(q_i, p_j - 1)} \right)_{i,j=1}^{k}$$

② $$s_{\lambda/\nu}^{k-1} s_{\mu/\nu} = \det\left((-1)^{\chi(p_j > q_i)} s_{\lambda(q_i, p_j - 1)/\nu} \right)_{i,j=1}^{k}$$

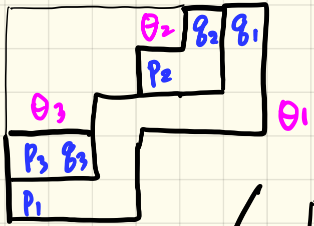
- If $\nu = \lambda$ in ①, we obtain LP theorem.
- If $\nu = \phi$ in ②, we obtain MPP conjecture.

Cor (MPP conj)

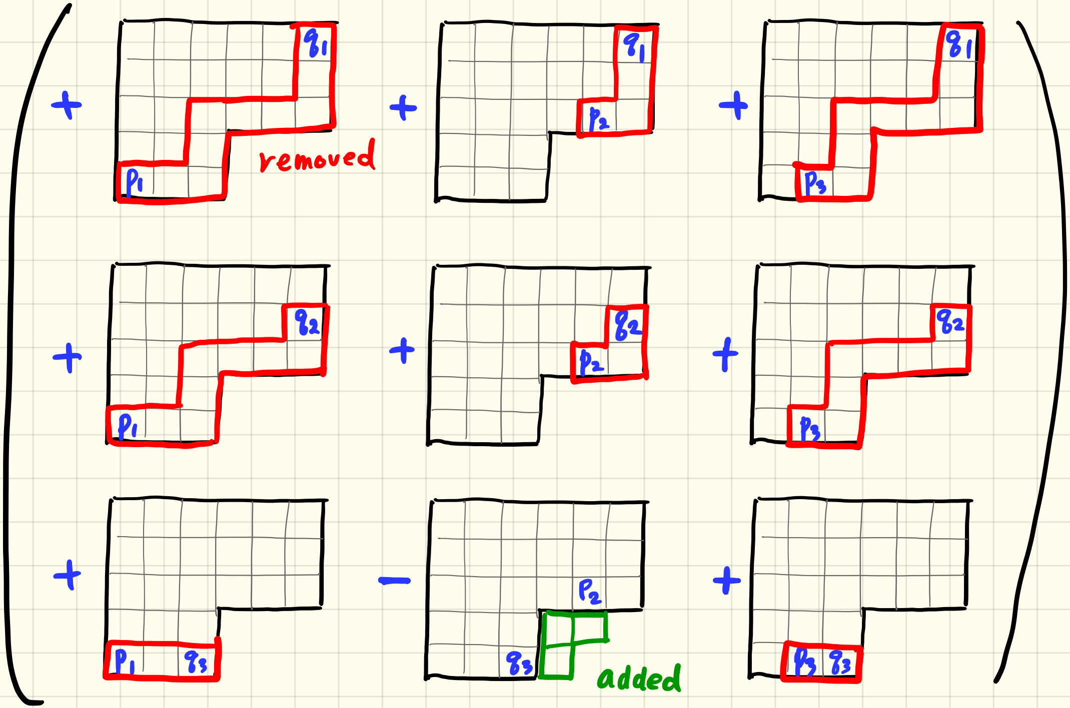
$$s_{\lambda}^{k-1} s_{\mu} = \det\left((-1)^{\chi(p_j > q_i)} s_{\lambda(q_i, p_j - 1)} \right)_{i,j=1}^{k}$$

$$s_\lambda^{k-1} s_\mu = \det\left((-1)^{\chi(p_j > q_i)} s_{\lambda(q_i,\, p_j - 1)} \right)_{i,j=1}^{k}$$

ex) $\lambda/\mu =$



$s_\lambda^2 s_\mu = \det$



$\theta = (\theta_1, \dots, \theta_k)$: LP decomp of λ/μ

$$\pm = (-1)^{\chi(p_j > q_i)}$$

① $s_{\nu/\lambda}^{k-1} s_{\nu/\mu} = \det\left(\pm s_{\nu/\lambda(q_i, p_j - 1)}\right)$

② $s_{\lambda/\nu}^{k-1} s_{\mu/\nu} = \det\left(\pm s_{\lambda(q_i, p_j-1)/\nu}\right)$

$\nu = \lambda$ in ① $\Rightarrow$

$$s_{\lambda/\mu} = \det\left(\pm s_{\lambda/\lambda(q_i, p_j - 1)}\right)_{i,j=1}^{k} \qquad \text{(LP)}$$

$$= \det\left(s_{\gamma[p_j, q_i]}\right)$$

$$\lambda(q_i, p_j - 1) = \begin{cases} \lambda \setminus \gamma[p_j, q_i] & \text{if } p_j \le q_i + 1 \\ \lambda \cup \gamma[q_i + 1, p_j - 1] & \text{if } p_j \ge q_i + 2 \end{cases}$$

$\nu = \phi$ in ② $\Rightarrow$

$$s_\lambda^{k-1} s_\mu = \det\left(\pm s_{\lambda(q_i, p_j - 1)}\right)_{i,j=1}^{k} \qquad \text{(MPP)}$$

$\rightsquigarrow$ Divide both sides by s_λ^k

$$s_\mu / s_\lambda = \det\left(\pm s_{\lambda(q_i, p_j-1)} / s_\lambda\right)_{i,j=1}^{k}.$$

LP vs MPP

$$s_{\lambda/\mu} = \det\left(\pm\, s_{\lambda/\lambda(q_i,p_j-1)}\right)_{i,j=1}^{k} \qquad \text{LP}$$

$$s_\mu / s_\lambda = \det\left(\pm\, s_{\lambda(q_i,p_j-1)}/s_\lambda\right)_{i,j=1}^{k}. \qquad \text{MPP}$$

Let $f_\mu^\lambda = s_{\lambda/\mu}$, $g_\mu^\lambda = s_\mu/s_\lambda$.

Then

$$f_\mu^\lambda = \det\left(\pm\, f_{\lambda(q_i,p_j-1)}^{\lambda}\right) \qquad \text{LP}$$

$$g_\mu^\lambda = \det\left(\pm\, g_{\lambda(q_i,p_j-1)}^{\lambda}\right) \qquad \text{MPP}$$

$M = (M_{ij})_{i \in \mathbb{Z},\ 1 \le j \le n}$.

$a = (a_1, \dots, a_n) \in \mathbb{Z}^n$

$[a] := \det (M_{a_i, j})_{1 \le i,j \le n}$.

Bazin-Sylvester identity (1851)

For any $a \in \mathbb{Z}^k$, $b \in \mathbb{Z}^k$, $c \in \mathbb{Z}^{n-k}$,

$$\boxed{\begin{aligned} &[a \sqcup c]^{k-1} [b \sqcup c] \\ &= (-1)^{\binom{k}{2}} \det\left(b_j \sqcup (a \setminus a_i) \sqcup c \right)_{1 \le i,j \le k} \end{aligned}}$$

Proof of Main Theorem.

$$S_{\nu/\lambda}^{k-1} S_{\nu/\mu} = \det\left(\pm S_{\nu/\lambda(q_i, p_j - 1)}\right)$$

Pf) Apply Bazin-Sylvester identity to

$$M = \left(\varphi^{i+1} h_{\nu_j - j - i}\right)_{i \in \mathbb{Z},\ 1 \le j \le n}$$

with

$$a = (q_1, \dots, q_k) = C_n(\lambda) \setminus C_n(\mu)$$

$$b = (p_1 - 1, \dots, p_k - 1) = C_n(\mu) \setminus C_n(\lambda)$$

$$c = (c_1 > c_2 > \cdots > c_{n-k}) = C_n(\lambda) \setminus a = C_n(\mu) \setminus b.$$

Main Theorem

$\theta = (\theta_1, \dots, \theta_k)$: LP-decomp of λ/μ

① $$s^{k-1}_{\nu/\lambda} s_{\nu/\mu} = \det\left((-1)^{\chi(p_j > q_i)} s_{\nu/\lambda(q_i, p_j - 1)} \right)_{i,j=1}^{k}$$

② $$s^{k-1}_{\lambda/\nu} s_{\mu/\nu} = \det\left((-1)^{\chi(p_j > q_i)} s_{\lambda(q_i, p_j - 1)/\nu} \right)_{i,j=1}^{k}$$

①' $$s^{k-1}_{\nu/\mu} s_{\nu/\lambda} = \det\left((-1)^{\chi(p_j > q_i)} s_{\nu/\mu(p_j - 1, q_i)} \right)_{i,j=1}^{k}$$

②' $$s^{k-1}_{\mu/\nu} s_{\lambda/\nu} = \det\left((-1)^{\chi(p_j > q_i)} s_{\mu(p_j - 1, q_i)/\nu} \right)_{i,j=1}^{k}$$

Let $\alpha = (\alpha_1, \dots, \alpha_r)$, $\beta = (\beta_1, \dots, \beta_r)$

$\gamma = (\gamma_1, \dots, \gamma_s)$, $\delta = (\delta_1, \dots, \delta_s)$

with $\alpha_1 > \cdots > \alpha_r > \gamma_1 > \cdots > \gamma_s \geq 0$

$\beta_1 > \cdots > \beta_r > \delta_1 > \cdots > \delta_s \geq 0$. $(r, s \geq 0)$.

<u>Cor</u> (Generalized Giambelli)

$$S_{(\gamma|\delta)}^{r-1} S_{(\alpha\gamma|\beta\delta)} = \det\left(S_{(\alpha_i, \gamma | \beta_j, \delta)} \right)_{1 \leq i,j \leq r}.$$

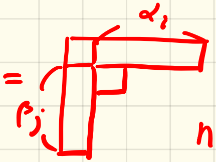
Note : If $\gamma = \delta = \phi$ then Giambelli.

If $\gamma = \delta = (0)$,

<u>Cor</u> (Jin, 2018) Let $\mu = (\alpha_1, \dots, \alpha_r, 0 \mid \beta_1, \dots, \beta_r, 0)$, $\alpha_1 > \cdots > \alpha_r \geq 1$, $\beta_1 > \cdots > \beta_r \geq 1$.

Then

$$s_1^{r-1} s_\mu = \det\left(s_{(\alpha_i, 0 | \beta_j, 0)} \right)_{i,j=1}^{r}$$

$(\alpha_i, 0 \mid \beta_j, 0) =$ 

near hook

Further Study

- Okada generalized Bazin-Sylvester identity.

  Generalize the results using Okada's identity.

- Schur's P-functions have Pfaffian expressions.

  There is a Pfaffian version of Bazin-Sylvester.

  Extend the results to P-functions.