Limit shape of shifted staircase SYT

and 132-avoiding random sorting networks

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March 11, 2019

SYT

A Standard Young Tableau (SYT) is a Ferrer's diagram of a partition $\lambda \vdash m$ with the numbers $1, 2, \ldots, m$ filled in. Example of a SYT of staircase shape $cs_n = (n, n-1, n-2, \ldots, 1)$, here $n = 4$.

1	3	5	8
2	6	9	
4	7		
10			

Shifted Standard Young Tableaux

A **shifted** Standard Young Tableau (SYT) is a diagram of a **strict** partition $\lambda \vdash m$ with the numbers $1, 2, \ldots, m$ filled in. Strict means $\lambda_1 > \lambda_2 > \ldots$.
Example of a shifted SYT of staircase shape $cs_n = (n, n-1, n-2, \ldots, 1)$, here $n = 5$.

1	2	4	5	7
	3	6	8	11
		9	10	13
			12	14
				15

Shifted SYT

We normalise the shifted SYT to a triangle with side length 1 and the height of a point is the number in that box divided by $\binom{n+1}{2}$. What does a random shifted SYT look like?

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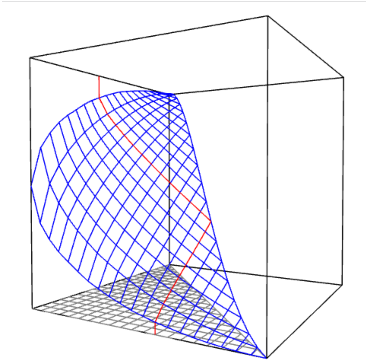

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Theorem (L-Potka-Sulzgruber)

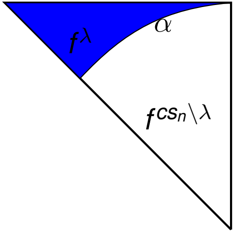
A random shifted SYT of staircase shape converges to a surface $L(x,y)$ given implicitly by

$$x+y=\frac{2}{\pi}(x-y)\arctan\left(\frac{(1-2L(x,y))(x-y)}{\sqrt{4L(x,y)(1-L(x,y))-(x-y)^2}}\right)+\frac{2}{\pi}\arctan\left(\frac{\sqrt{4L(x,y)(1-L(x,y))-(x-y)^2}}{1-2L(x,y)}\right),$$

for $0\leq x\leq y\leq 1-x\leq 1$. Use symmetry for $1-x\leq y$.

Proof sketch

Using the hook-length formula for f^λ, the number of shifted SYT, twice we can get a probability for a certain shape α of the filled in part of the shifted SYT at a given time as $\frac{f^\lambda f^{cs_n \setminus \lambda}}{f^{cs_n}}$.



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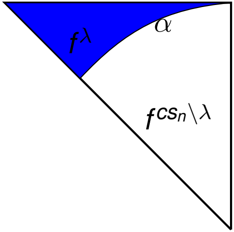

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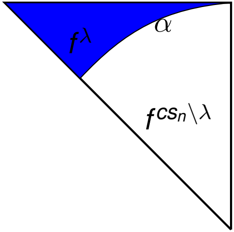

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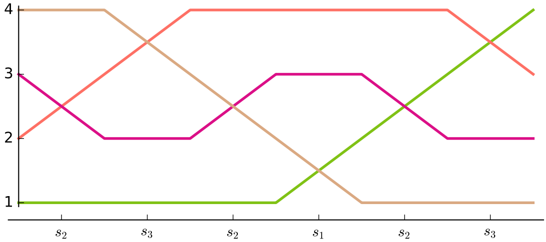
This strategy was used first by Pittel and Romik to determine the limit shape of a square SYT. In fact, we don't have to do the complicated analysis, because we get essentially the same variational problem as they solved and thus half of the same limit shape as they did.

Random Sorting Networks

A sorting network is a way to go from the identity permutation $12\ldots n$ to the reverse $n\ldots 21$ in as few steps as possible, $\binom{n}{2}$.
An example of size 4:



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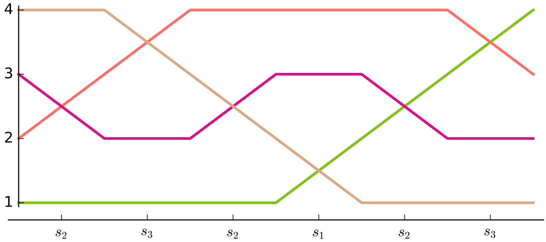


This corresponds to a reduced word for the reverse permutation w_0.

A (uniformly) random sorting network was studied in a 2007 paper by Angel, Holroyd, Romik and Virag. They posed several intriguing conjectures, now proven by Dauvergne.

Random sorting networks

Theorem (Dauvergne, conjectured by AHRV)

The trajectory of a given number is almost surely a half-sine curve.

As permutation matrices

Theorem (Dauvergne, conjectured by AHRV)

At a given time t the permutation matrix is supported on an ellipse with an archimedean measure.

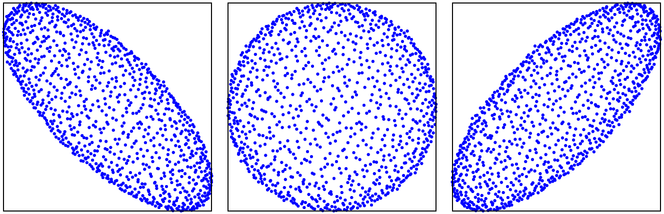
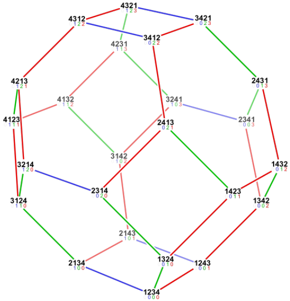


Figure: The intermediate permutation matrices $M(\sigma_t)$ of a 1000-element random sorting network at times $t = \frac{1}{4}, \frac{1}{2}$ and $\frac{3}{4}$.

On the permutahedron

Theorem (Dauvergne, conjectured by AHRV)

Let $w(n)$ be an n-element uniform sorting network. For each n there exists a great circle C_n such that $d_\infty(w(n), C_n) = o(n)$ in probability as $n \to \infty$.



Permutahedron for permutations of S_4. (Image stolen from internet.)

Notation

We perform the compositions of swaps s_{w_i} corresponding to a word $w = w_1 \ldots w_m$ from the left. As an example, consider S_4 and the reduced word 1213. Composing the swaps $s_1 s_2 s_1 s_3$ from the left yields the permutation $(3, 2, 4, 1)$. In terms of permutation matrices, we have,

$$s_1 = \begin{pmatrix} \mathbf{2} & \mathbf{1} & \mathbf{3} & \mathbf{4} \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, s_1 s_2 = \begin{pmatrix} \mathbf{2} & \mathbf{3} & \mathbf{1} & \mathbf{4} \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, s_1 s_2 s_1 s_3 = \begin{pmatrix} \mathbf{3} & \mathbf{2} & \mathbf{4} & \mathbf{1} \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

s_i corresponds to swapping the columns i and $i + 1$.

Edelman-Greene bijection

A number bumps the smallest larger or equal number. **If equal the bumped number increases by 1.** As an example, consider the reduced word $w = 321232$. Then the insertion form the following sequence

$$\begin{array}{|c|}\hline 3\\\hline\end{array} \longrightarrow \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array} \longrightarrow \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array} \longrightarrow \begin{array}{|c|c|}\hline 1 & 2\\\hline 2 & \\\cline{1-1} 3 & \\\cline{1-1}\end{array} \longrightarrow \begin{array}{|c|c|c|}\hline 1 & 2 & 3\\\hline 2 & & \\\cline{1-1} 3 & & \\\cline{1-1}\end{array} \longrightarrow \begin{array}{|c|c|c|}\hline 1 & 2 & 3\\\hline 2 & \mathbf{3} & \\\cline{1-2} 3 & & \\\cline{1-1}\end{array}$$

so that

$$P(321232) = \begin{array}{|c|c|c|}\hline 1 & 2 & 3\\\hline 2 & 3 & \\\cline{1-2} 3 & & \\\cline{1-1}\end{array} \text{ and } Q(321232) = \begin{array}{|c|c|c|}\hline 1 & 4 & 5\\\hline 2 & 6 & \\\cline{1-2} 3 & & \\\cline{1-1}\end{array}.$$

$P(w)$ is the *insertion tableau* and $Q(w)$ is the *recording tableau* which records the order the growth of $P(w)$.

Edelman-Greene

Theorem (Edelman-Greene)

This modified RSK gives a bijection between sorting networks w and SYT of staircase shape, $Q(w)$.

Edelman-Greene

(Rothe) **Diagram** of a permutation matrix is the non-shaded part when everything to the right and below each one has been shaded. **Frozen region** of $P(w)$ consists of cells which already have obtained minimal possible value.

Example: A reduced word w for permutation $\sigma = (5, 6, 1, 4, 2, 3)$

$D(\sigma) =$

0	0	1	0	0	0
0	0	0	0	1	0
0	0	0	0	0	1
0	0	0	1	0	0
1	0	0	0	0	0
0	1	0	0	0	0

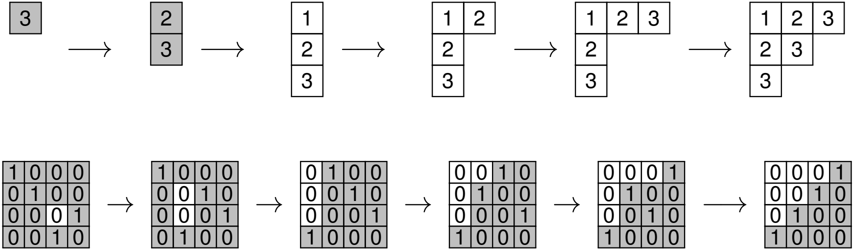
$P(w) =$

1	2	4
2	3	5
3	4	
4	5	

Figure: The diagram $D(\sigma)$ and $P = P(w)$ for any reduced word of σ.

Edelman-Greene

Example with word $w = 321232$ for S_4.



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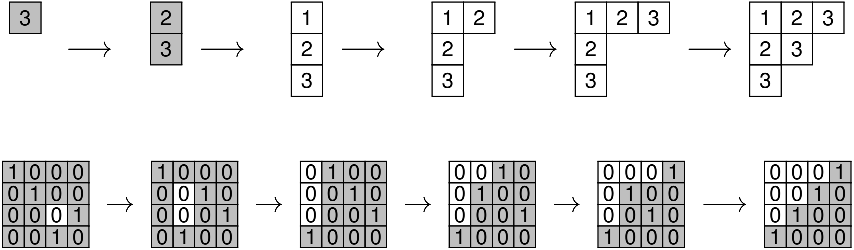

Theorem (L.-Potka)

For any permutation σ, the upper-left component of the diagram of σ has the same shape as the frozen region of $P(w)$ for any reduced word w of σ.

Frozen region

This gives a reformulation of the second theorm by Dauvergne above.

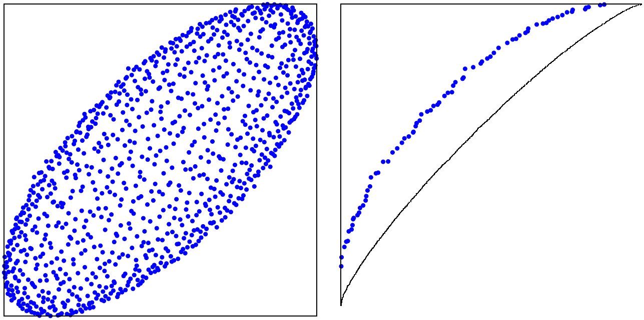

Figure: A comparison at $t = \frac{3}{4}$ of the shapes in both permutation matrices and frozen regions.

132-avoiding

A permutation σ contains the pattern 132 if the permutation matrix of 132 is a submatrix of the permutation matrix of σ. Example: 425136 contains 132 because of .25..3.

$$\begin{matrix} \mathbf{4} & \mathbf{2} & \mathbf{5} & \mathbf{1} & \mathbf{3} & \mathbf{6} \end{matrix}$$
$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \underline{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \underline{1} & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \underline{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \text{contains the pattern} \begin{matrix} \mathbf{1} & \mathbf{3} & \mathbf{2} \\ \end{matrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

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A permutation is 132-avoiding if it does not contain the pattern 132.

132-avoiding networks

We call a sorting network 132-avoiding if every intermediate permutation of the sorting network is 132-avoiding. This happens iff the insertion tableau P is frozen for every step of the insertion.

Theorem

Under the Edelman-Greene correspondence a sorting network w is 132-avoiding iff the diagonals in $Q(w)$ are increasing down to the left.

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We can shift each row to the right and obtain a so called shifted SYT, where the columns are increasing downwards.

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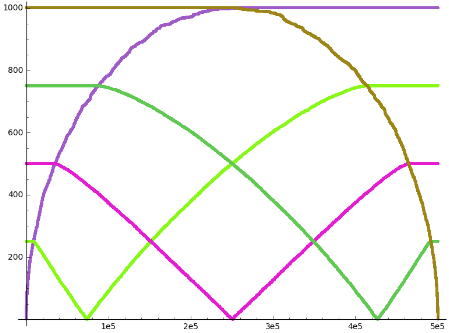
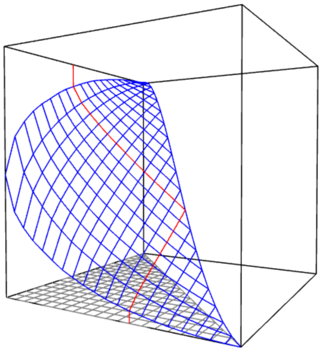
Gives a bijection between 132-avoiding sorting networks and shifted staircase $SYT(cs_n)$. Previously studied by Fishel-Nelson 2015 and Schilling-Thiery-White-Williams 2016.

Trajectories

From the limit shape we can deduce the form of the trajectories.





To the left are trajectories of the elements 1, 250, 500, 750 and 1000 in a 132-avoiding random sorting network with 1000 elements.

132-avoiding networks

And the shape of the permutation matrix during the sorting.

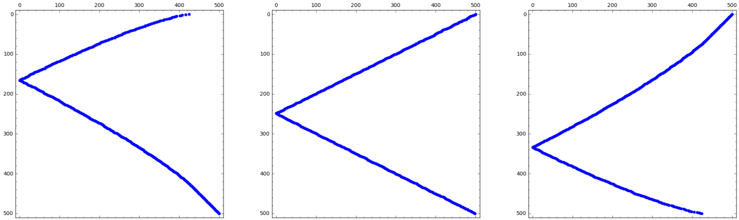


Figure: The intermediate permutation matrices of a 132-avoiding random sorting network with 500 elements at times $t = \frac{1}{4}, \frac{1}{2}$ and $\frac{3}{4}$.

Path on Permutahedron?

By the work of Angel-Holroyd-Romik-Virag we know that a 132-avoinding sorting network will not be close to a great circle on the permutahderon. What is it?

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Theorem (L-Potka-Sulzgruber)

The expected number of horisontal adjacencies for any given column in a staircase SYT is 1.

Adjacencies

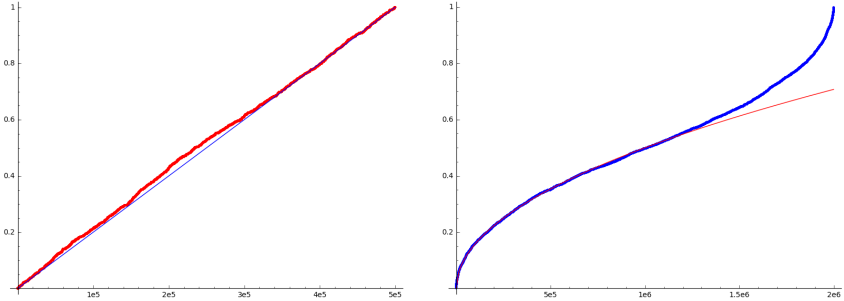


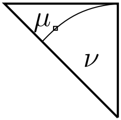
Figure: The numbers of adjacencies in an initial segment of a random sorting network of size $n = 1000$, (left) seems to grow linearly. The number of adjacencies in an initial segment of a random 132-avoiding sorting network of size $n = 2000$ (right) seems to grow like a square root for the first half.

We conjecture this to be true in general.

Shifted SYT

Let f^λ be the number of SYT of shape λ. And let cs_n be the shape of the shifted staircase SYT.

$$\sum_{\mu \cup \square \cup \nu = cs_n} f^\mu \cdot f^\nu = \binom{n+1}{2} f^{cs_n},$$



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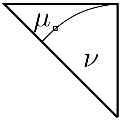

$$\sum_{\mu \cup \square\square \cup \nu = cs_n} f^\mu \cdot f^\nu = (n-1) f^{cs_n},$$

L-P-S

$$\sum_{\mu \cup \begin{smallmatrix}\square & \square \\ & \square\end{smallmatrix} \cup \nu = cs_n} f^\mu \cdot f^\nu = f^{cs_n},$$

Schilling et.al.

where the sums are taken over all disjoint unions such that μ and $\mu \cup \square$ etc are (shifted) Young diagrams.