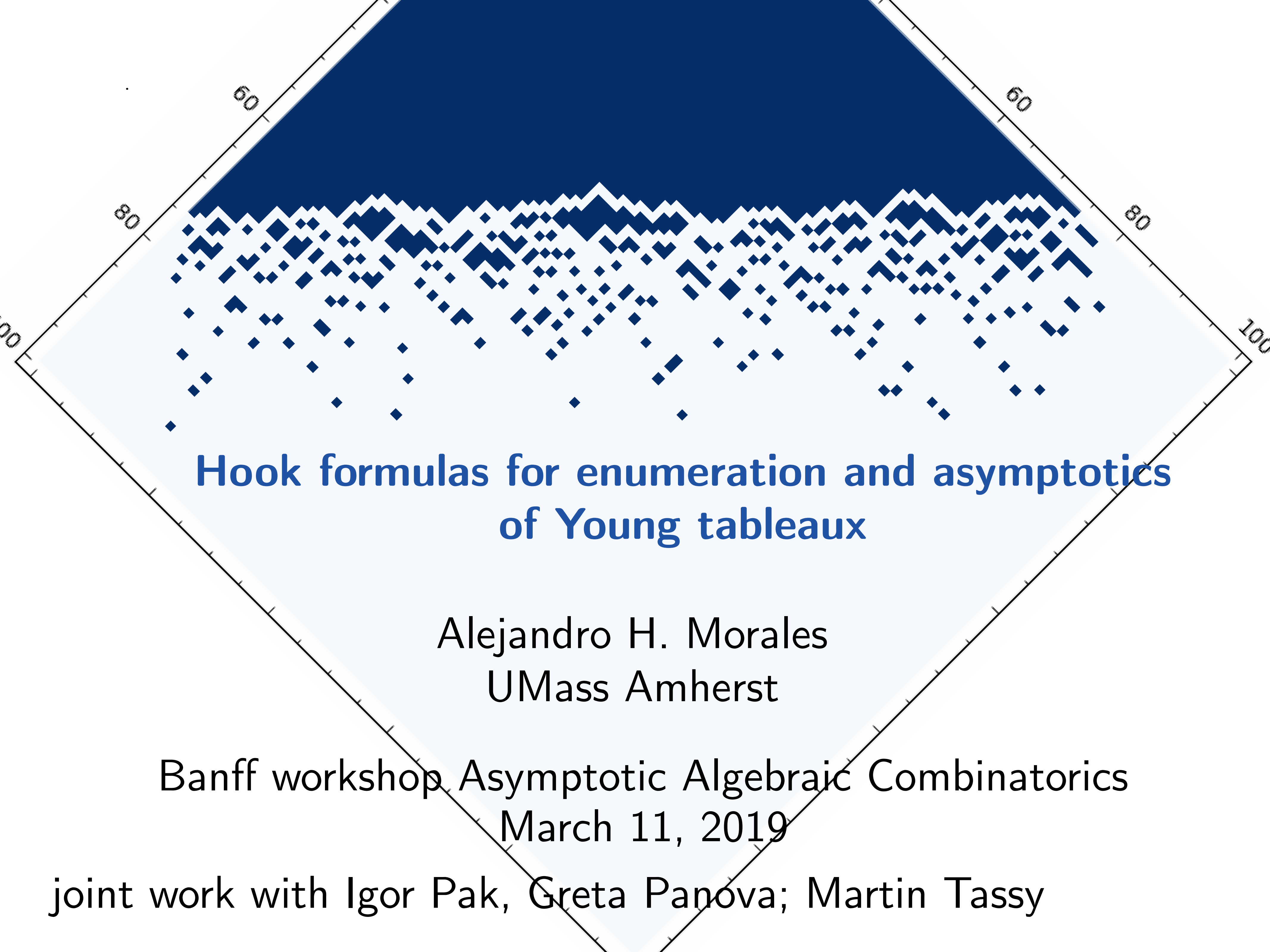


Hook formulas for enumeration and asymptotics of Young tableaux

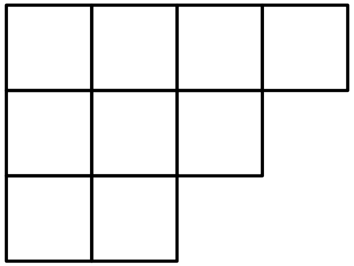
Alejandro H. Morales
UMass Amherst

Banff workshop Asymptotic Algebraic Combinatorics
March 11, 2019

joint work with Igor Pak, Greta Panova; Martin Tassy

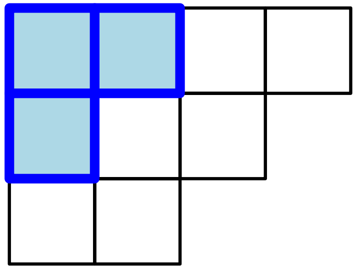
Young diagrams of (skew) partitions

λ: partition (straight) shape



$(4, 3, 2)$

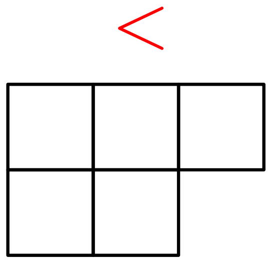
λ/μ: skew shape



$(4, 3, 2)/(2, 1)$

Linear extensions: standard Young tableaux

λ: straight shape



A **standard tableau** is a filling of the Young diagram with $1, 2, \ldots, n$ increasing in rows and columns.

1	2	3
4	5	

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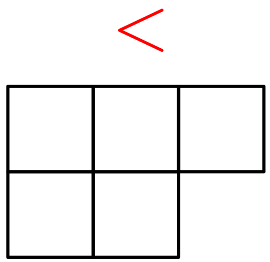

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4	5	

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3	5	

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3	4	

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2	4	

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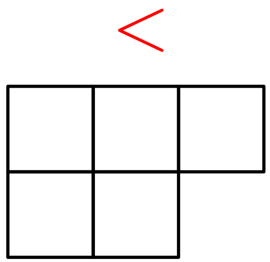

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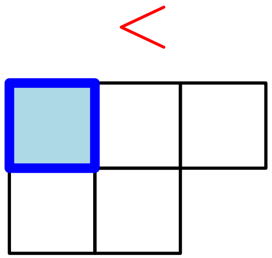
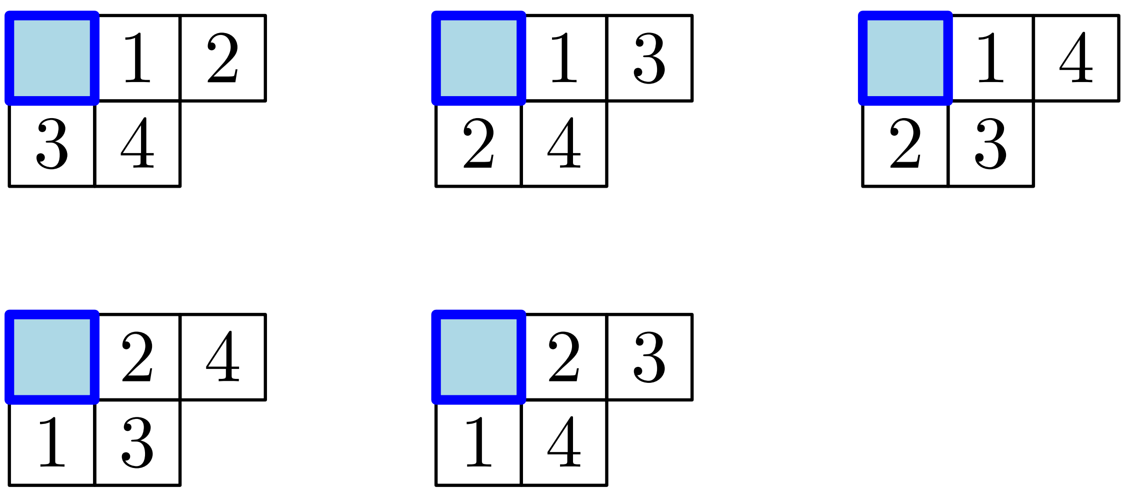
Let f^{λ} be number of such tableaux.

Standard Young tableaux skew shape

λ/μ: skew shape



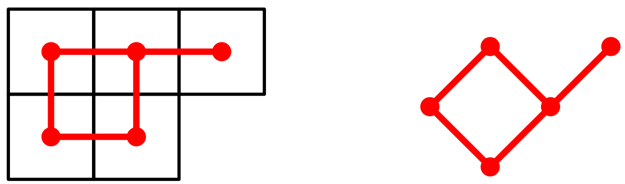


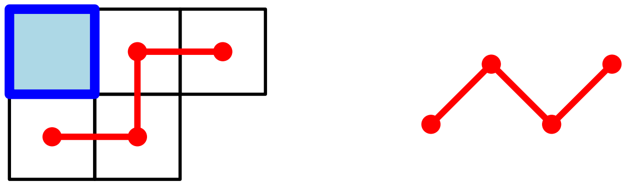
Let $f^{\lambda/\mu}$ be the number of such tableaux.

Why do we want to compute f^λ and $f^{\lambda/\mu}$

Enumeration

- f^λ, $f^{\lambda/\mu}$ count linear extensions of certain posets.





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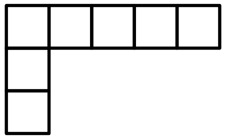

binomial coefficients

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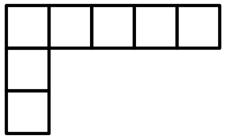

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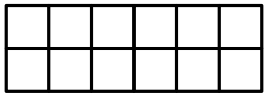

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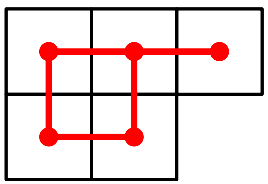
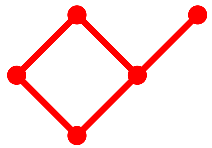
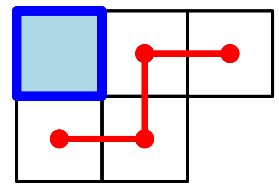


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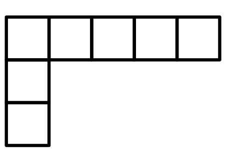

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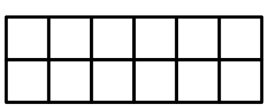

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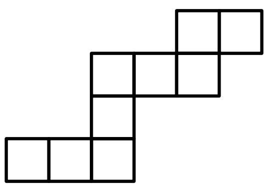

permutations with given descents

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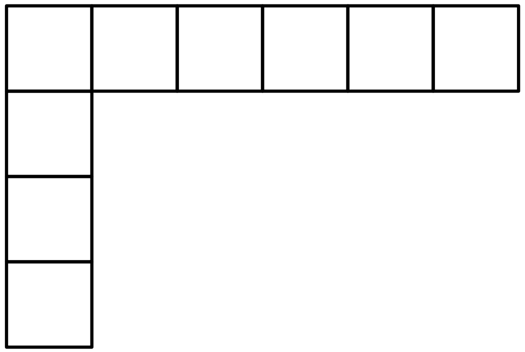
- $f^{\lambda/\mu}$ gives the dimension of irreducible representations of affine Hecke algebras.

number of SYT of straight shape

Example: hooks



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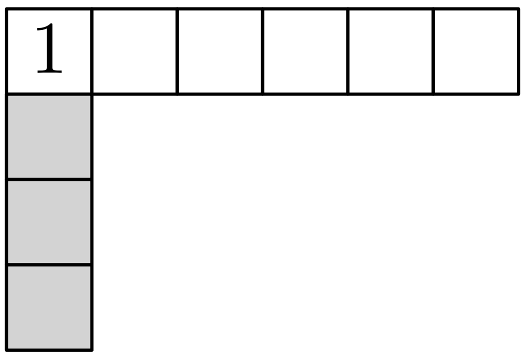

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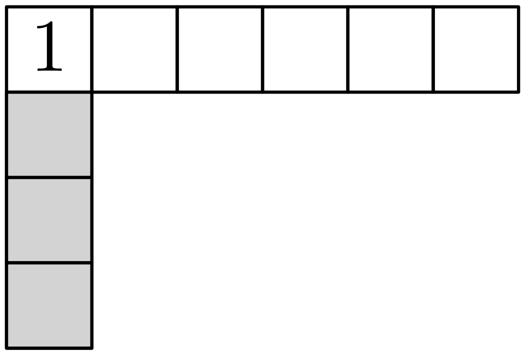

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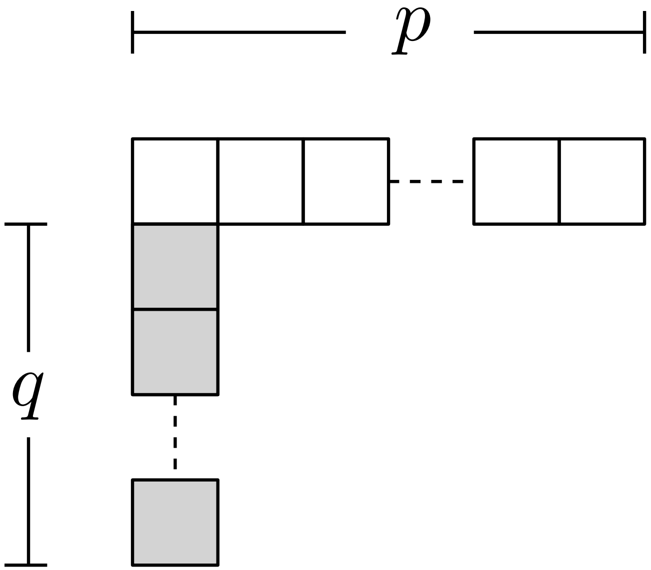
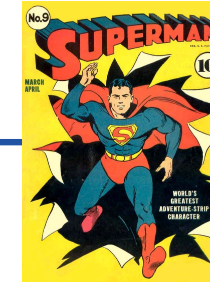


$$f^{(p,1^q)} = \binom{p+q-1}{q}$$

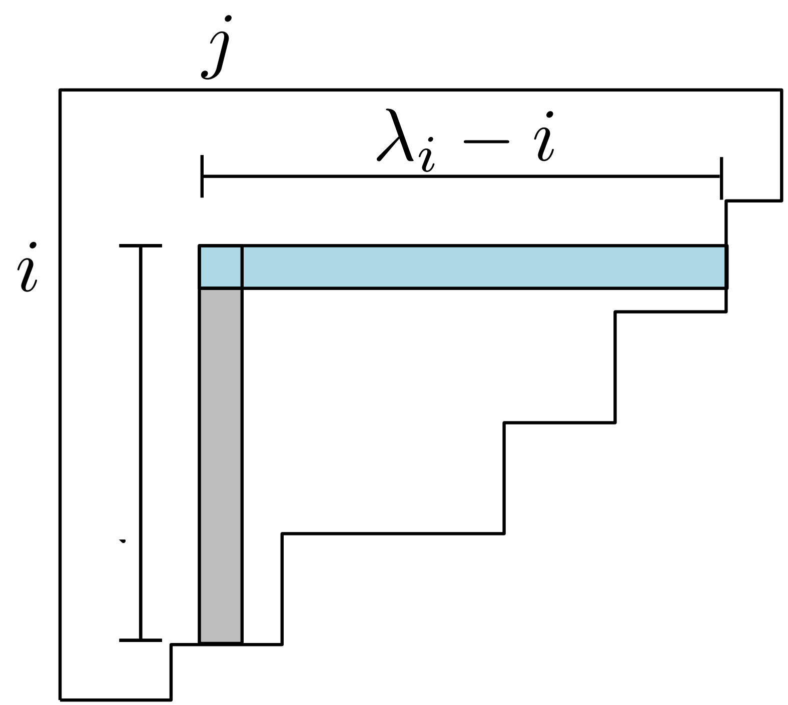
Hook-length formula



Theorem (Frame-Robinson-Thrall 1954)

$$f^\lambda = |\lambda|! \prod_{(i,j)\in\lambda} \frac{1}{h(i,j)},$$

$h(i,j) = \lambda_i - i + \lambda'_j - j + 1$ is the **hook-length** of (i,j)



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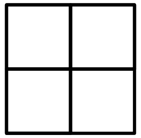

$$f^{\square\!\square} = \left| \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} \right|$$

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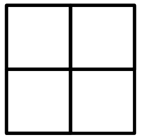

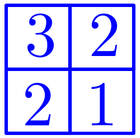

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- probabilistic proof by Greene-Nijenhuis-Wilf 79.
- bijective proof by Novelli-Pak-Stoyanovskii 97.

Asymptotics of large f^λ

From representation theory or the *RSK bijection*:

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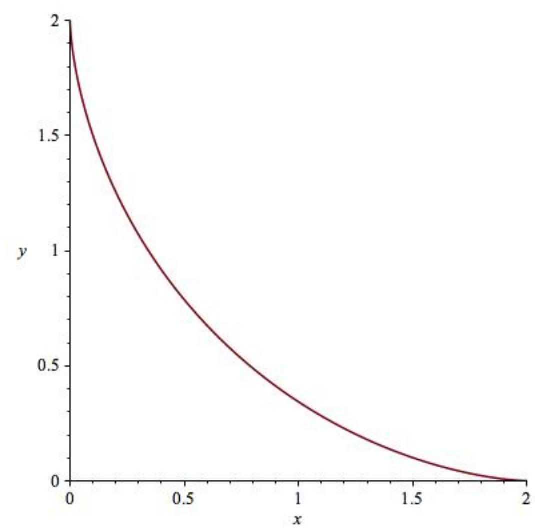

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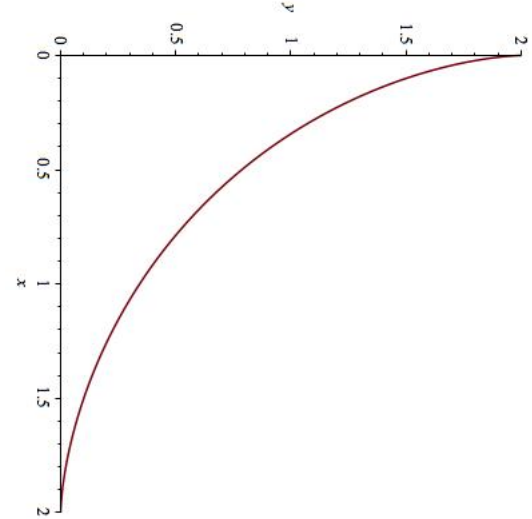
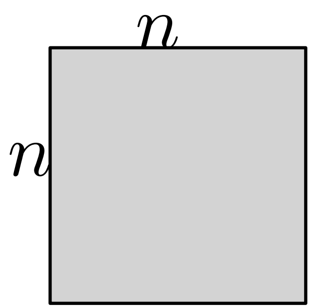
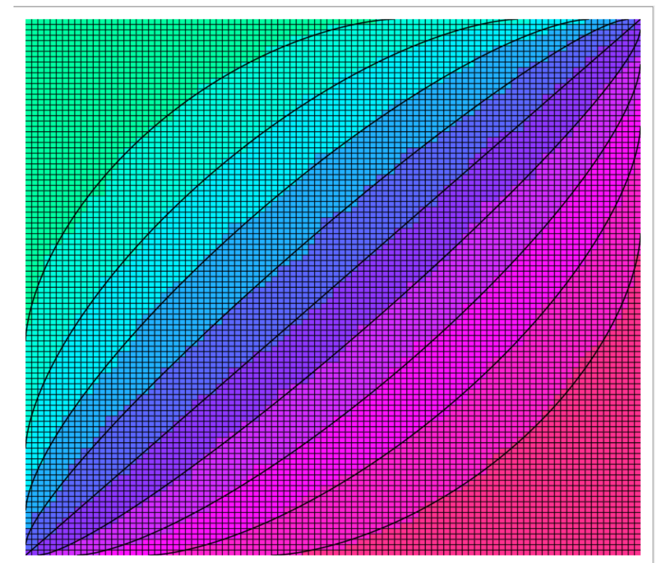


Asymptotics of f^λ for certain shapes

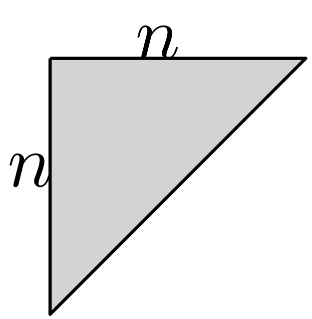
precise asymptotics and limit shapes (Pittel-Romik)

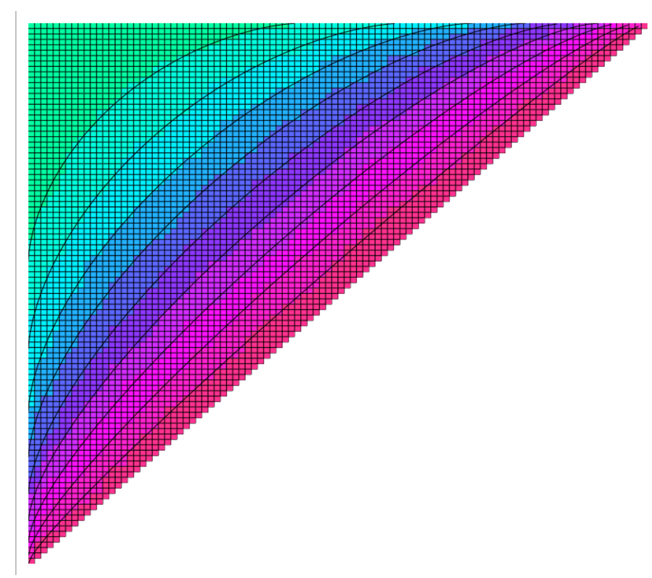
n^n





δ_n

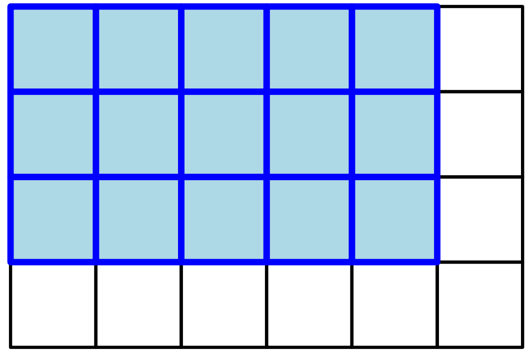
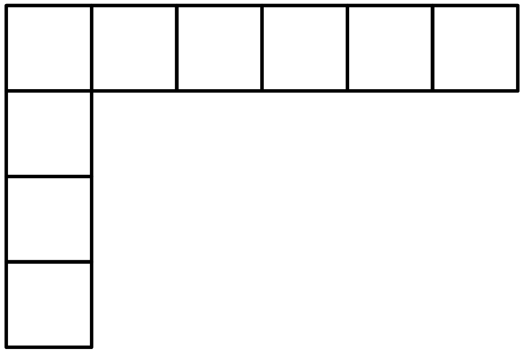




Outline

$$f^{\lambda} = \frac{|\lambda|!}{\prod_{u\in\lambda} h(u)} \qquad f^{\lambda/\mu} = ?$$

Product formula for $f^{\lambda/\mu}$?

$\#SYT$ shape  $=$ $\#SYT$ 

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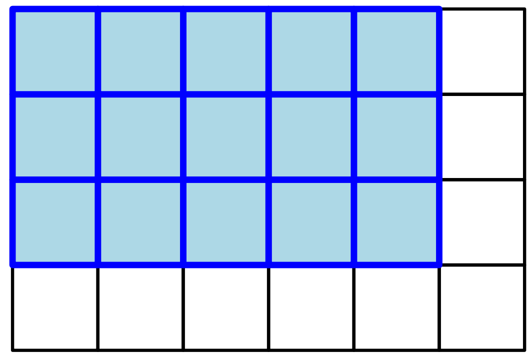
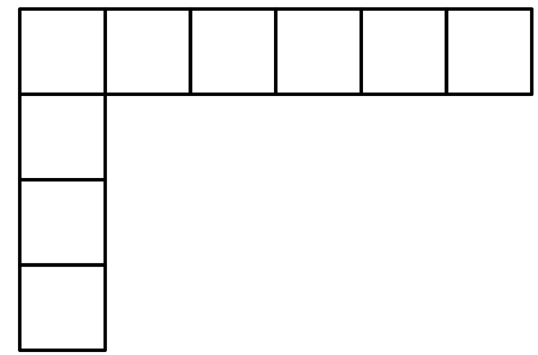
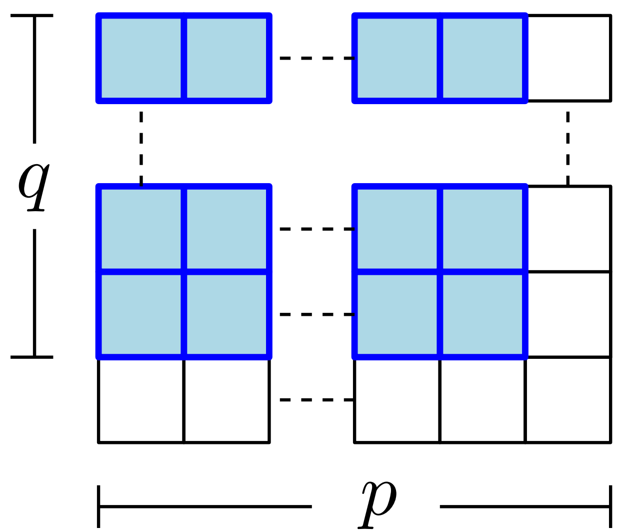
$\#SYT$ shape  $=$ $\#SYT$ 

$(6,6,6,6)/(5,5,5)$ $= \binom{8}{3}$

$\#SYT$ shape 

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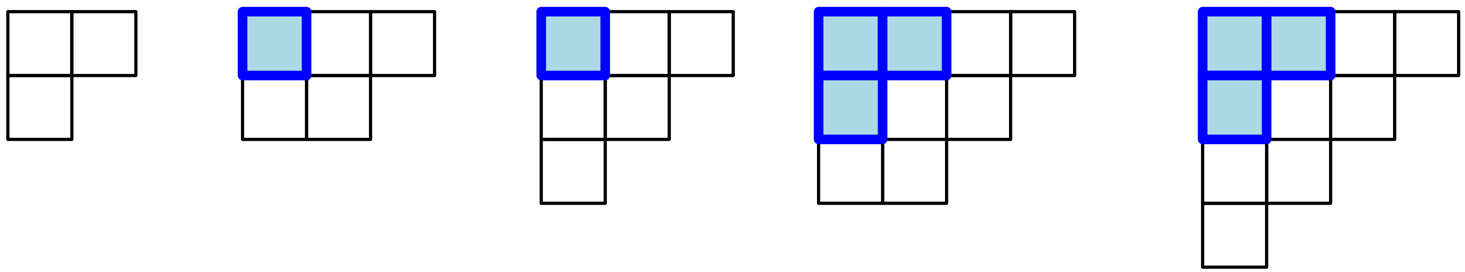
No product formula for $f^{\lambda/\mu}$

Example

λ/μ :



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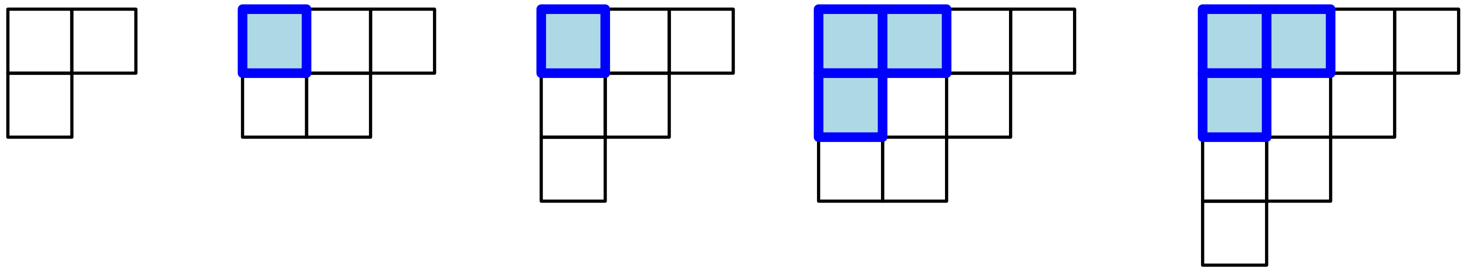

$f^{\lambda/\mu}$: 2 5 16 61 272 $\ldots$

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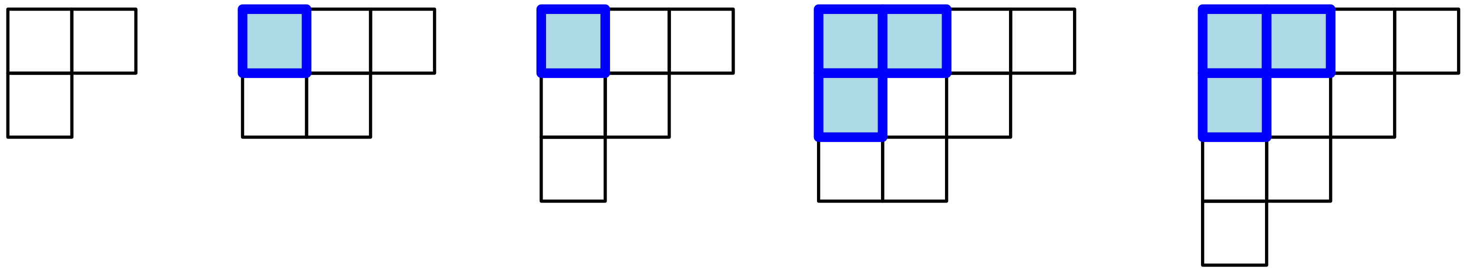

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Euler numbers E_n

$$E_{2n+1} = f^{\delta_{n+2}/\delta_n}$$

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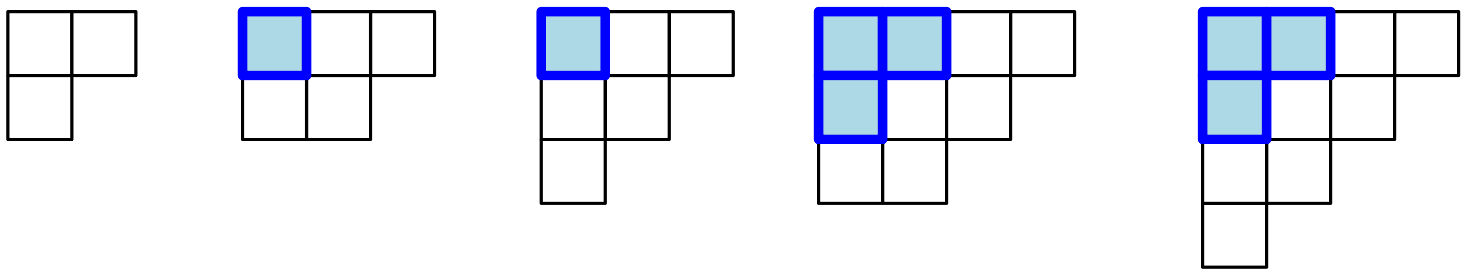

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Recall

$$1 + E_1 x + E_2 \frac{x^2}{2!} + E_3 \frac{x^3}{3!} + E_4 \frac{x^4}{4!} + \ldots = \sec(x) + \tan(x).$$

Alternating formulas for $f^{\lambda/\mu}$

Jacobi-Trudi formula

$$f^{\lambda/\mu} = |\lambda/\mu|! \cdot \det \left[\frac{1}{(\lambda_i - \mu_j - i + j)!} \right]_{i,j=1}^{\ell(\lambda)}.$$

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Example

$$f^{(3,2)/(1)} = 4! \cdot \det \begin{bmatrix} \frac{1}{2!} & \frac{1}{4!} \\ \frac{1}{1!} & \frac{1}{2!} \end{bmatrix}$$

$$= 4! \cdot \left(\frac{1}{4} - \frac{1}{24} \right) = 5.$$

Positive formulas for $f^{\lambda/\mu}$

Littlewood-Richardson rule (1934, 1976)

$$f^{\lambda/\mu} = \sum_{\nu} c^{\lambda}_{\mu,\nu} f^{\nu},$$

where $c^{\lambda}_{\mu,\nu}$ are the **Littlewood-Richardson coefficients**.

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- number of terms of formula is $\sum_{\nu} c^{\lambda}_{\mu,\nu}$

Positive formulas for $f^{\lambda/\mu}$

Okounkov-Olshanski 1998

$$f^{\lambda/\mu} = \frac{|\lambda/\mu|!}{\prod_{u\in\lambda} h(u)} \sum_{T} \prod_{(i,j)\in\mu} (\lambda_{d+1-T(i,j)} + j - i),$$

sum is over SSYT of shape μ entries $\leq d := \ell(\lambda)$.

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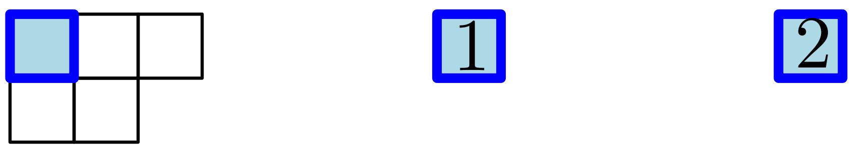
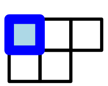


f 

$$= \frac{4!}{2\cdot 3\cdot 4} \cdot ((2+0) + (3+0))$$

$$= 5.$$

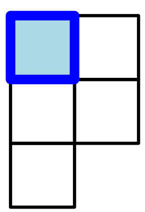
Positive formulas for $f^{\lambda/\mu}$

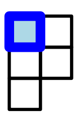
Okounkov-Olshanski 1998

$$f^{\lambda/\mu} = \frac{|\lambda/\mu|!}{\prod_{u\in\lambda} h(u)} \sum_{T} \prod_{(i,j)\in\mu} (\lambda_{d+1-T(i,j)} + j - i),$$

sum is over SSYT of shape μ entries $\leq d := \ell(\lambda)$.

Example



f  $= \frac{4!}{2\cdot 3\cdot 4} \cdot ((1+0)+(2+0)+(2+0))$

$= 5.$

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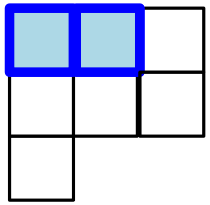
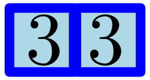
- a priori some terms might vanish or be negative





$$(1-0)(1-1)$$

Naruse's "hook-length" formula for $f^{\lambda/\mu}$

Theorem (Naruse 2014)

$$f^{\lambda/\mu} = |\lambda/\mu|! \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{(i,j) \in \lambda \setminus D} \frac{1}{h(i,j)},$$

where $\mathcal{E}(\lambda/\mu)$ is the set of **excited diagrams** of λ/μ.

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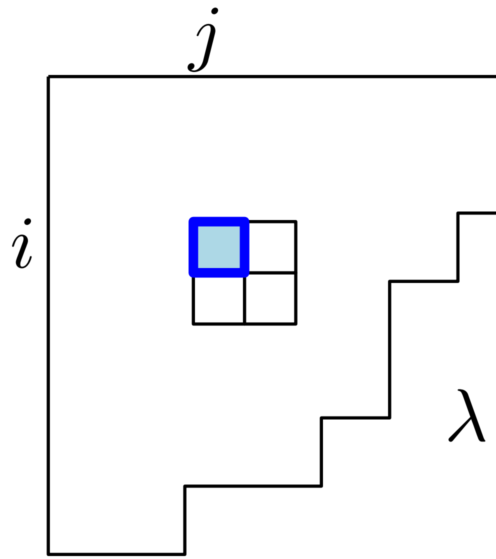

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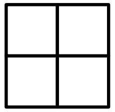

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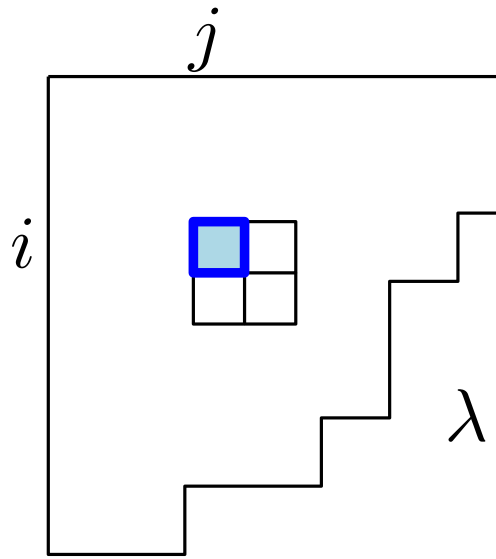

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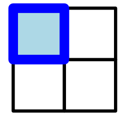

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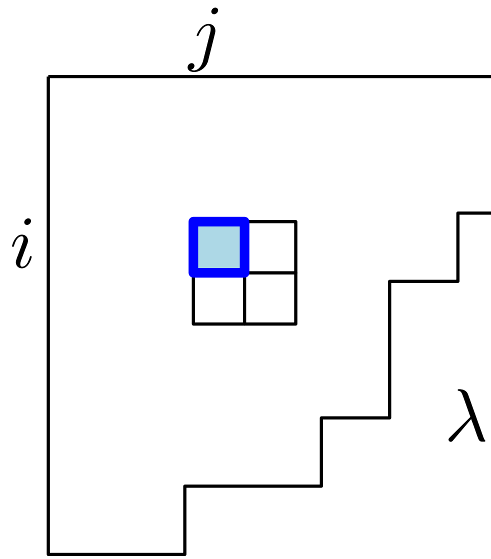

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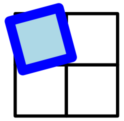

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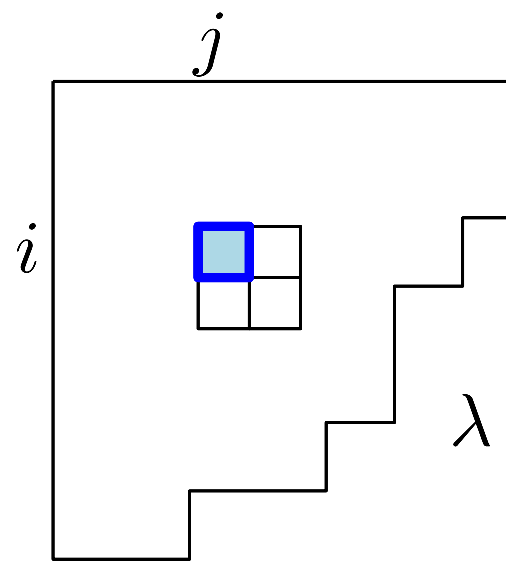

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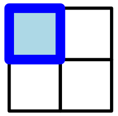

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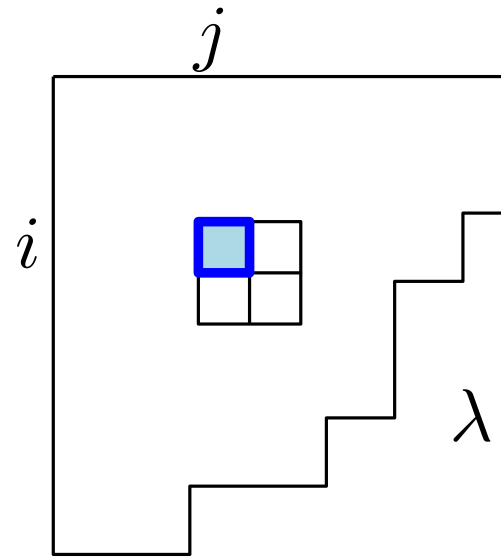

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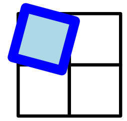

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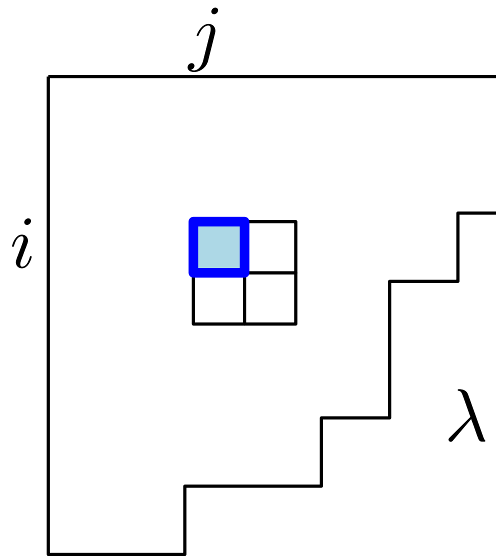

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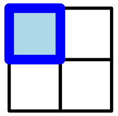

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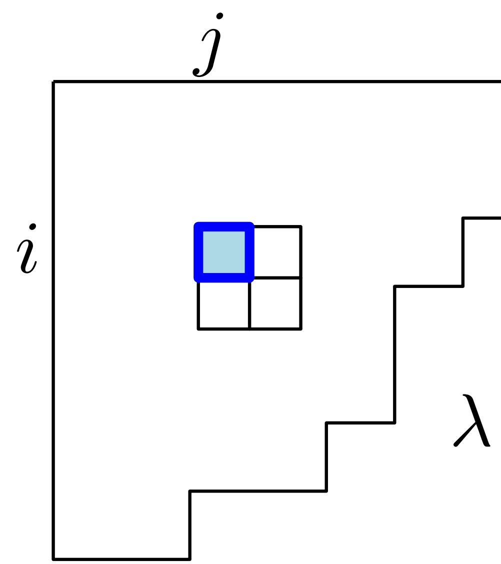

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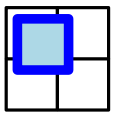

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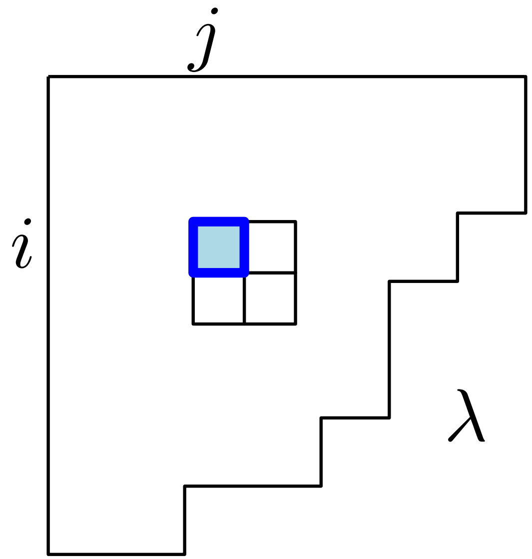

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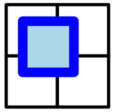

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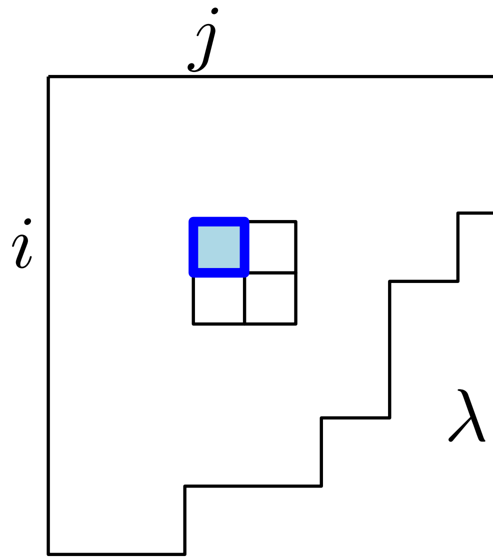

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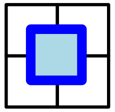

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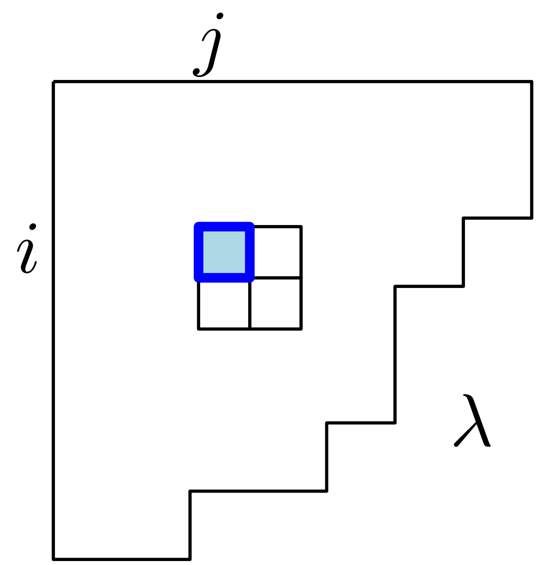

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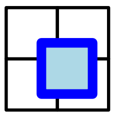

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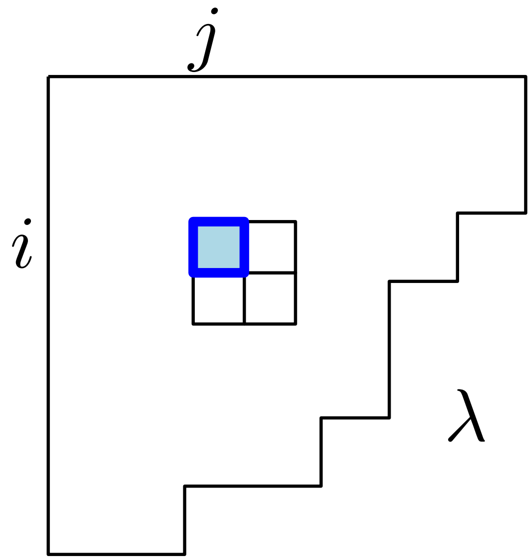

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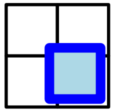

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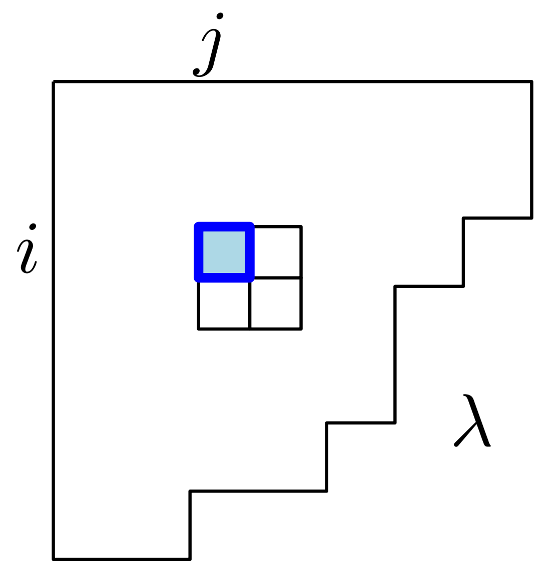

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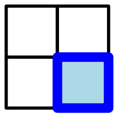

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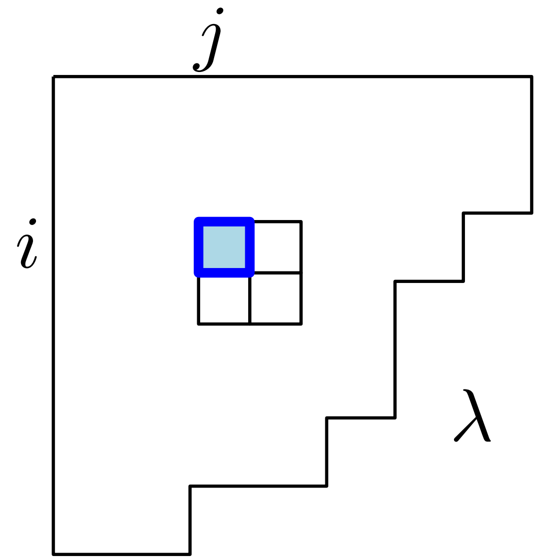

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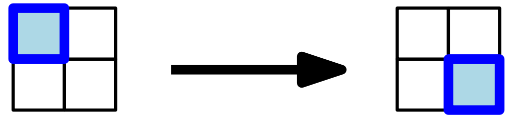

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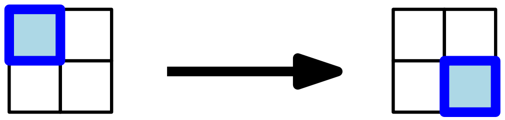

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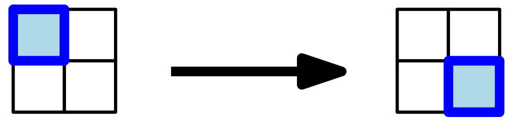

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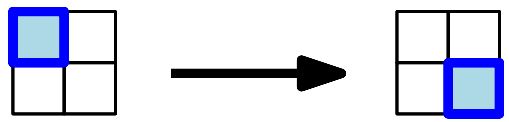

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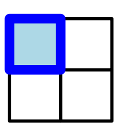
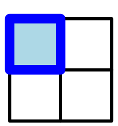
$\mathcal{E}($  $) = \{$  $\}$

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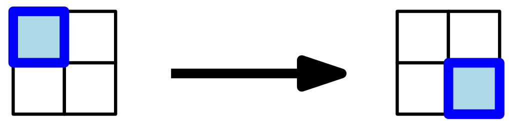

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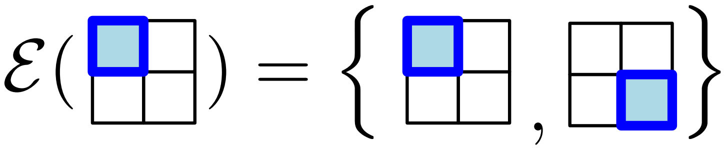

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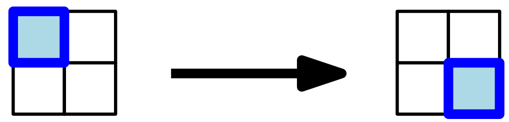

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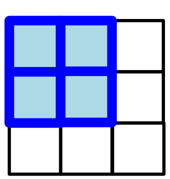
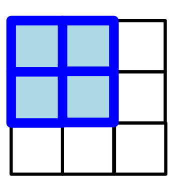
$\mathcal{E}($  $)$: 

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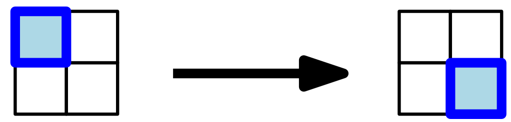

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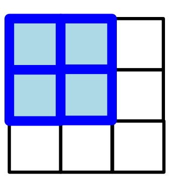
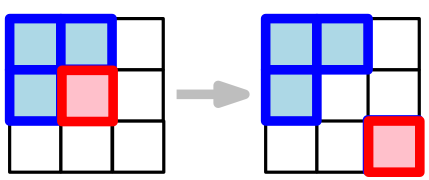
$\mathcal{E}($  $)$: 

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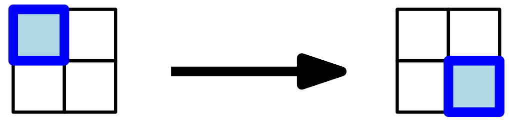

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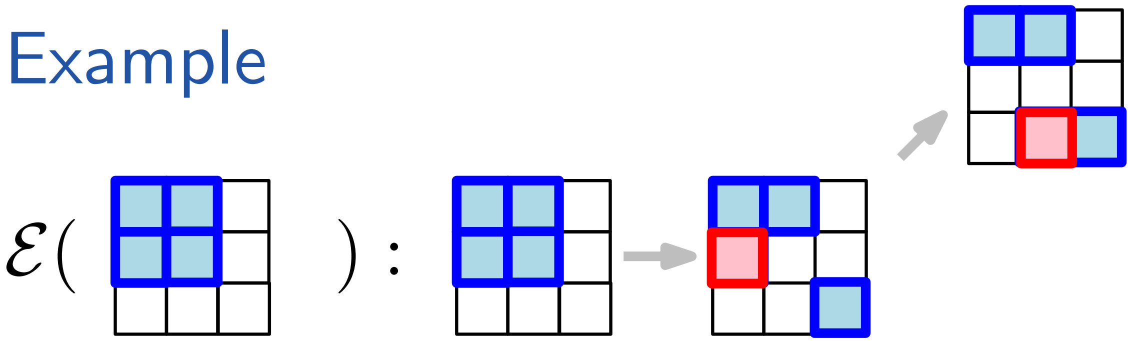

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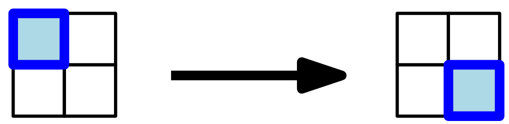

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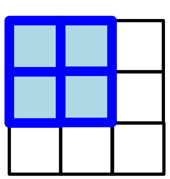
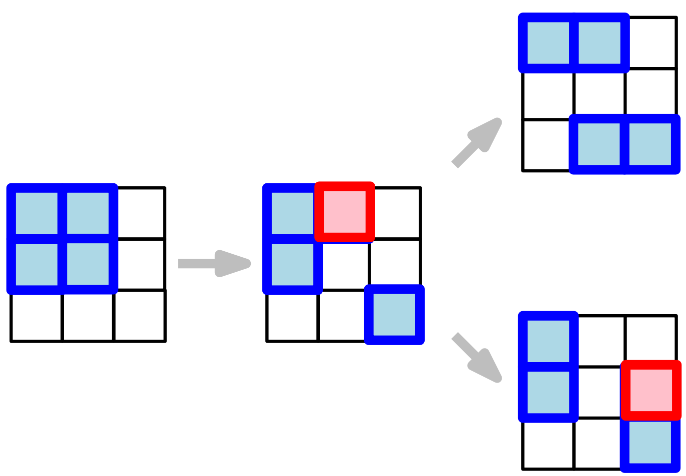
$\mathcal{E}($  $)$: 

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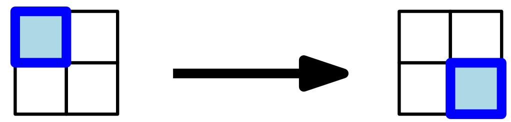

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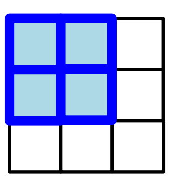
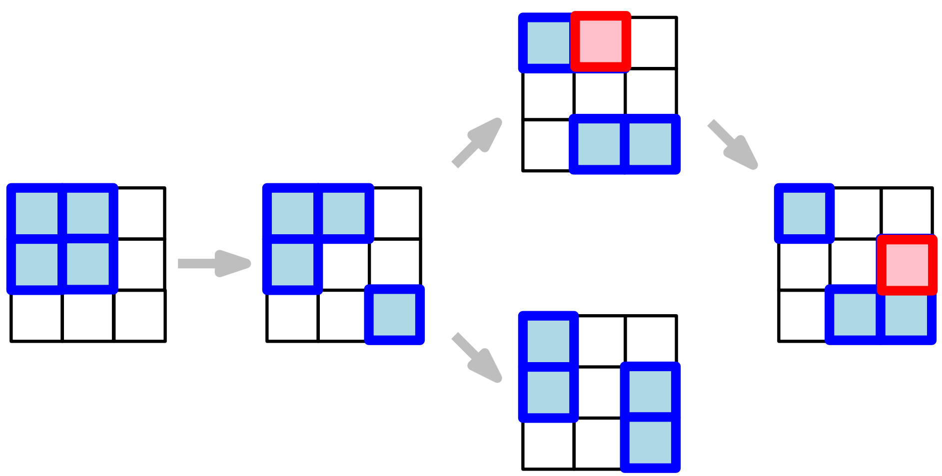
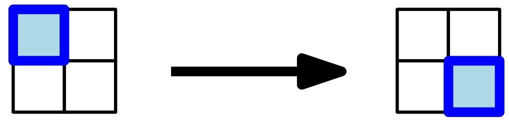
$\mathcal{E}($  $)$:



Excited diagrams of λ/μ (cont.)

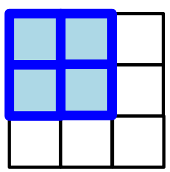
An **excited move** on an excited cell (i,j) in $S \subseteq \lambda$:
replace (i,j) in S by $(i+1, j+1)$

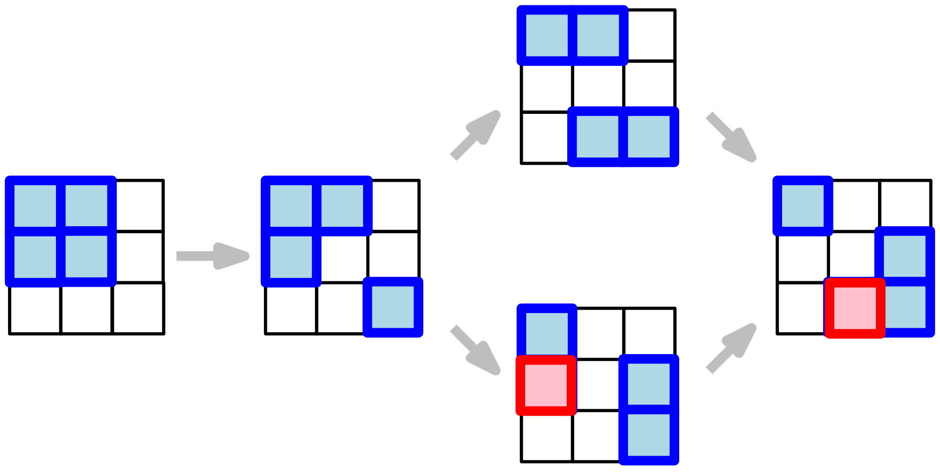


Definition: (Ikeda-Naruse 07, Knutson-Miller-Yong 05, Kreiman 05)

Excited diagrams $\mathcal{E}(\lambda/\mu)$: diagrams obtained from μ by applying iteratively excited moves on excited cells.

Example

$\mathcal{E}($  $)$:



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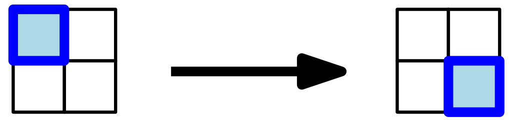

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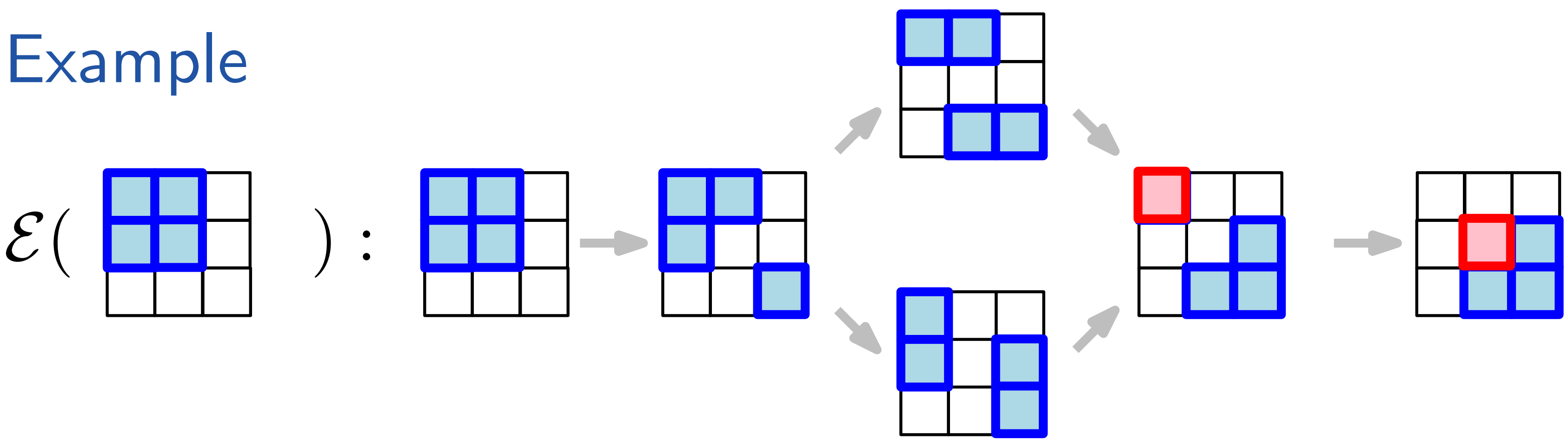

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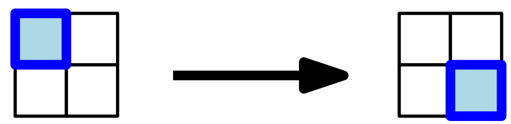

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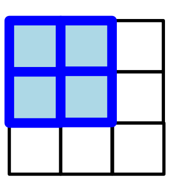
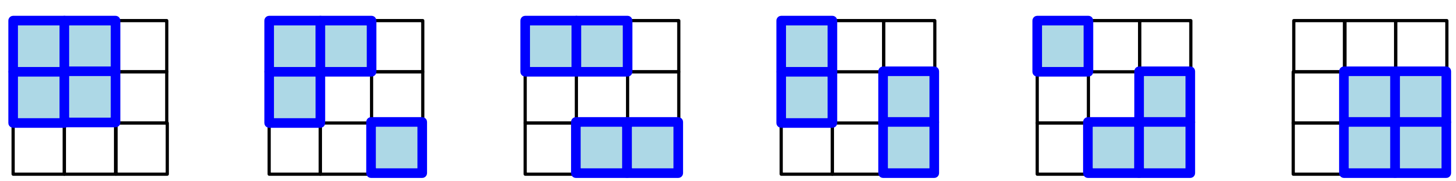
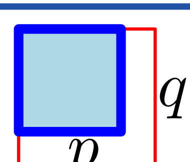
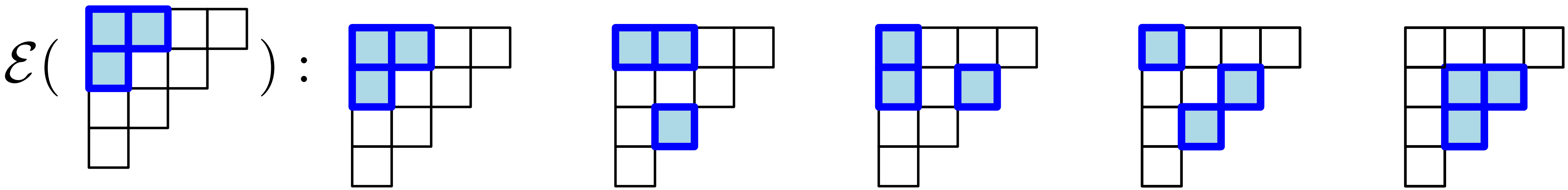
$\mathcal{E}($  $)$: 

Proposition $|\mathcal{E}($ 

$)| = \binom{p+q-2}{q-1}$.

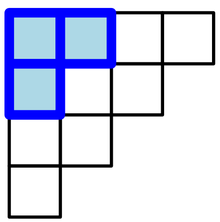
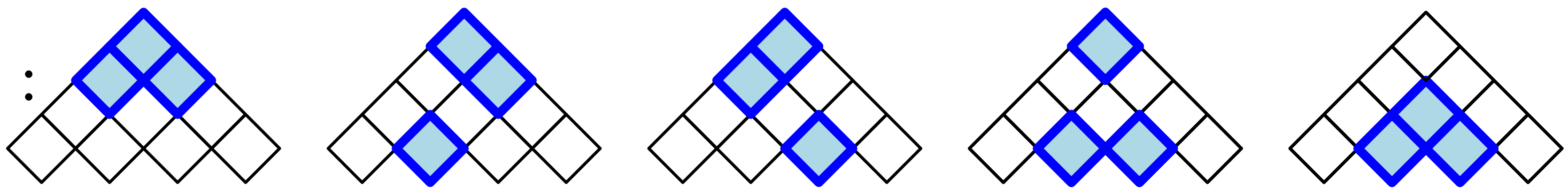
Excited diagrams of λ/μ (cont.)

Example



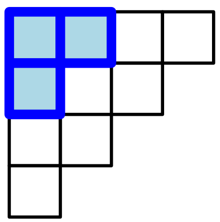
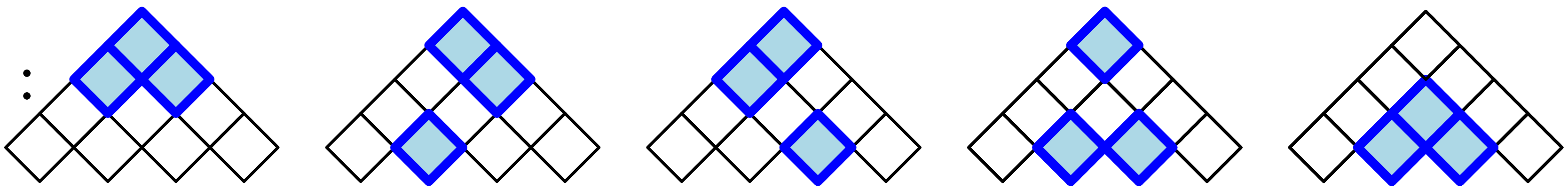
Excited diagrams of λ/μ (cont.)

Example

$\mathcal{E}\Big($  $\Big):$ 

Excited diagrams of λ/μ (cont.)

Example

$\mathcal{E}($  $)$: 

Proposition $|\mathcal{E}(\delta_{n+2}/\delta_n)| = \frac{1}{n+1}\binom{2n}{n}$.

Number of excited diagrams

Theorem (Wachs 85)

$$|\mathcal{E}(\lambda/\mu)| = \det\left[\binom{\mu_i+\vartheta_i-i}{\vartheta_i-i+j}\right]_{i,j=1}^{\ell(\mu)}$$

ϑ_i is row in which diagonal of (i,μ_i) intersects boundary of λ.

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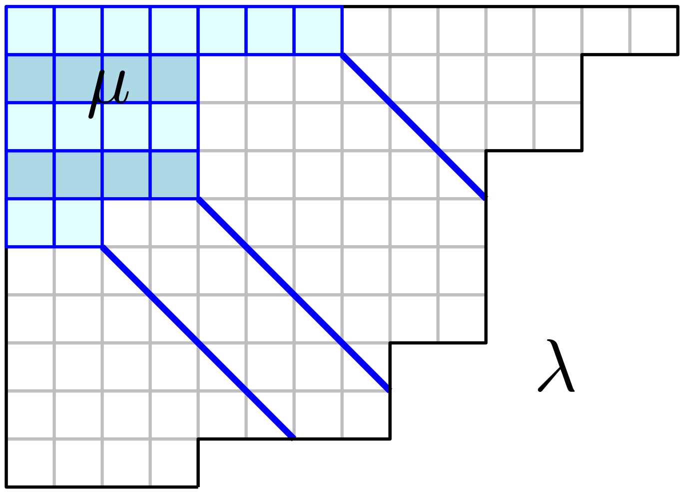

Naruse's "hook-length" formula for $f^{\lambda/\mu}$

Theorem (Naruse 2014)

$$f^{\lambda/\mu} = |\lambda/\mu|! \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{(i,j) \in \lambda \setminus D} \frac{1}{h(i,j)},$$

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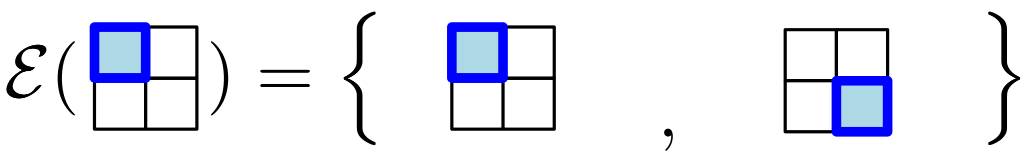

3	2
2	1

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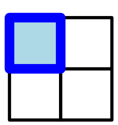
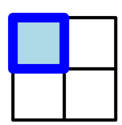
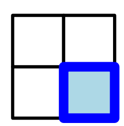
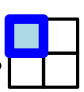
$\mathcal{E}($  $) = \{$  ,  $\}$

3	2
2	1

f  $= 3! \cdot \left(\frac{1}{1 \cdot 2 \cdot 2} + \frac{1}{2 \cdot 2 \cdot 3} \right)$

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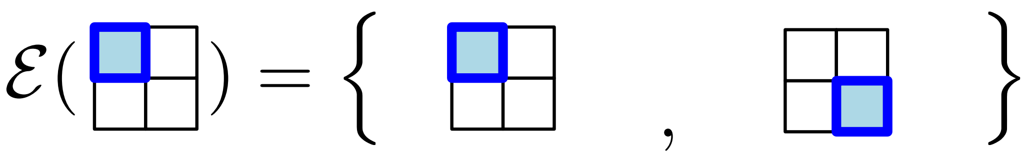
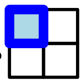
Example



3	2
2	1



$$f = 3! \cdot \left(\frac{1}{1\cdot 2\cdot 2} + \frac{1}{2\cdot 2\cdot 3} \right) = 3! \left(\frac{1}{4} + \frac{1}{12} \right) = 2.$$

Naruse's "hook-length" formula for $f^{\lambda/\mu}$

Theorem (Naruse 2014)

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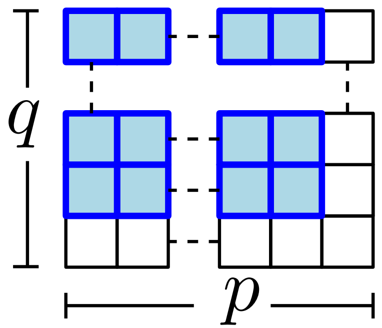

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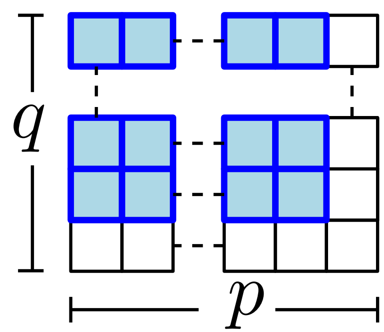

5	4	3
4	3	2
3	2	1

$$\binom{p+q-2}{q-1} = (p+q-2)! \sum_{\mathsf{p}:(q,1)\to(1,p)} \prod_{(i,j)\in p} \frac{1}{i+j-1}$$

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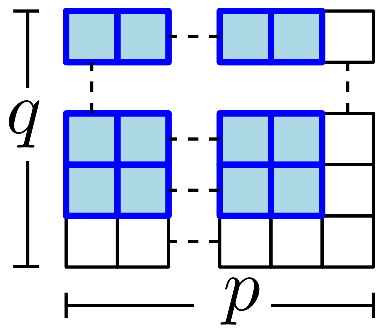

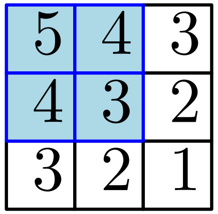

$$\frac{\binom{p+q-2}{q-1}}{(p+q-2)!} = \sum_{\mathsf{p}:(q,1)\to(1,p)} \prod_{(i,j)\in p} \frac{1}{i+j-1}$$

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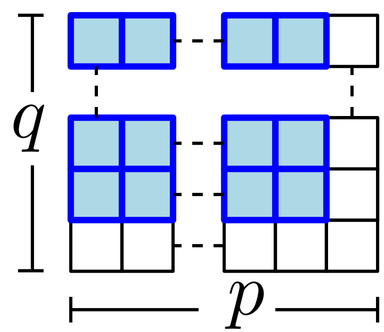

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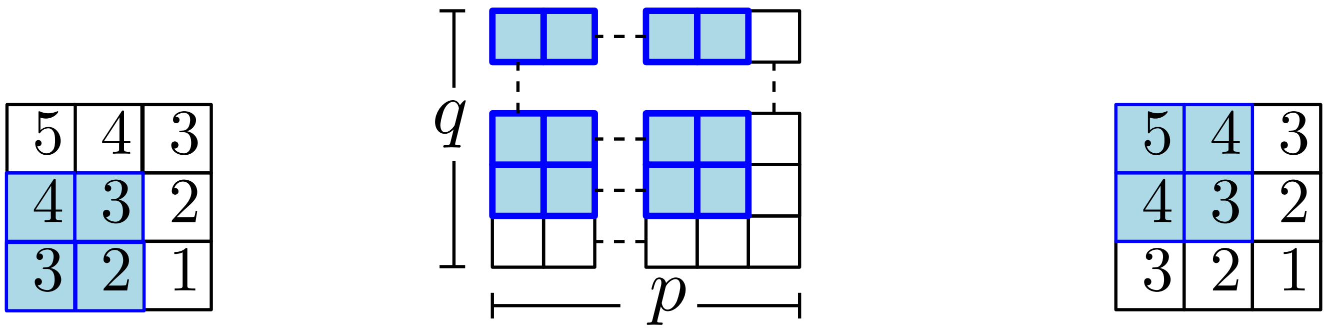

$$\sum_{\mathsf{p}:(1,1)\to(q,p)} \prod_{(i,j)\in \mathsf{p}} \frac{1}{i+j-1} = \sum_{\mathsf{p}:(q,1)\to(1,p)} \prod_{(i,j)\in p} \frac{1}{i+j-1}$$

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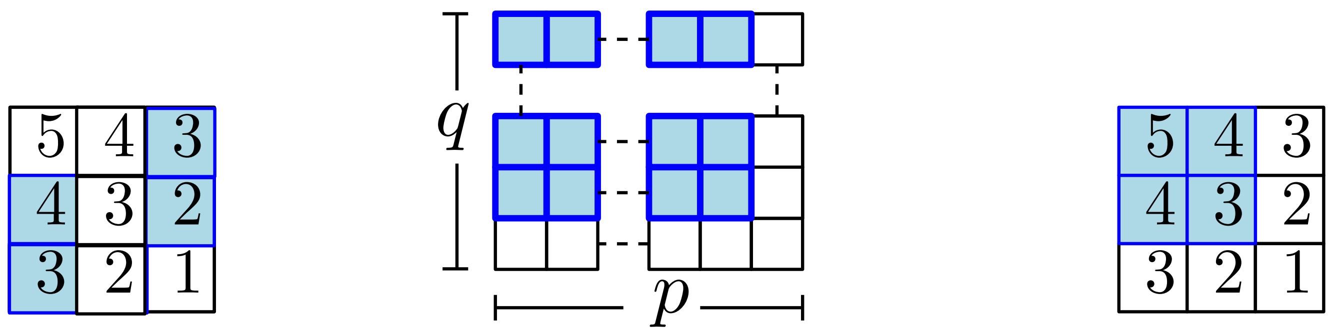


$$\sum_{\mathsf{p}:(1,1)\to(q,p)} \prod_{(i,j)\in \mathsf{p}} \frac{1}{i+j-1} = \sum_{\mathsf{p}:(q,1)\to(1,p)} \prod_{(i,j)\in p} \frac{1}{i+j-1}$$

Outline

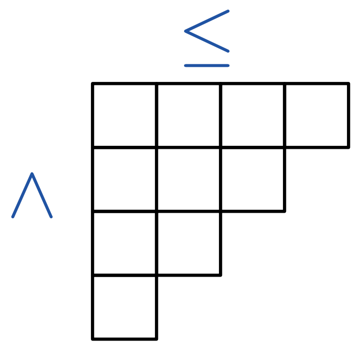
$$f^\lambda = \frac{|\lambda|!}{\prod_{u \in \lambda} h(u)}$$

Naruse's formula for $f^{\lambda/\mu}$

q-analogues

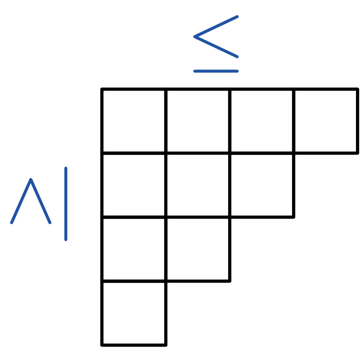
Semistandard tableaux and reverse plane partitions

semistandard tableaux SSYT



0	0	1	2
1	1	2	
2	3		
4			

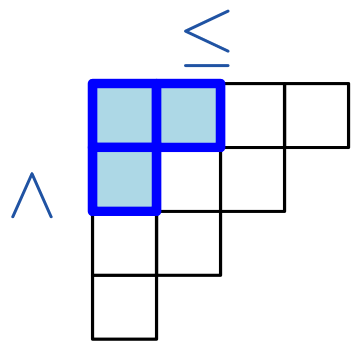
reverse plane partition

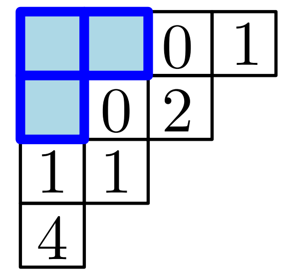


0	0	2	3
0	1	2	
1	2		
1			

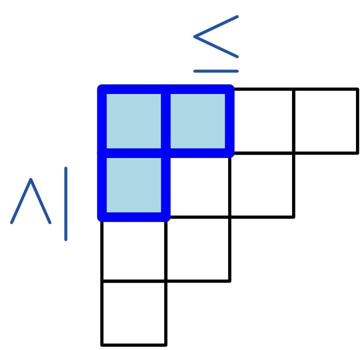
Semistandard tableaux and reverse plane partitions

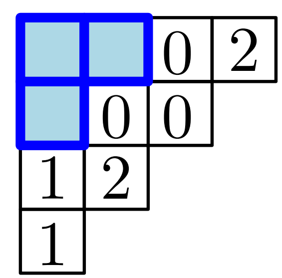
semistandard tableaux SSYT





reverse plane partitions RPP





From SSYT to reverse plane partitions

Theorem (Stanley 1971)

$$\sum_{T \in \mathrm{SSYT}(\lambda)} q^{|T|} = q^{b(\lambda)} \prod_{(i,j) \in \lambda} \frac{1}{1 - q^{h(i,j)}},$$

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note the equivalence:

$$\sum_{T \in \mathrm{SSYT}(\lambda)} q^{|T|} = q^{b(\lambda)} \sum_{\pi \in \mathrm{RPP}(\lambda)} q^{|\pi|}.$$

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$$\underset{\text{SSYT}}{\begin{array}{|c|c|c|}\hline 0 & 2 & 4 \\ \hline 1 & 5 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array}} \quad - \quad \begin{array}{|c|c|c|}\hline 0 & 0 & 0 \\ \hline 1 & 1 \\ \cline{1-2} 2 \\ \cline{1-1}\end{array} \quad = \quad \underset{\text{RPP}}{\begin{array}{|c|c|c|}\hline 0 & 2 & 4 \\ \hline 0 & 4 \\ \cline{1-2} 1 \\ \cline{1-1}\end{array}}$$

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SSYT RPP

no equivalence for skew shapes:

$$\begin{array}{|c|c|}\hline & 1 \\ \hline 0 & 2 \\ \hline \end{array} \quad - \quad \begin{array}{|c|c|}\hline & 0 \\ \hline 1 & 1 \\ \hline \end{array} \quad = \quad \begin{array}{|c|c|}\hline & 1 \\ \hline -1 & 1 \\ \hline \end{array}$$

From SSYT to reverse plane partitions

Theorem (Stanley 1971)

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note the equivalence:

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From SSYT to reverse plane partitions

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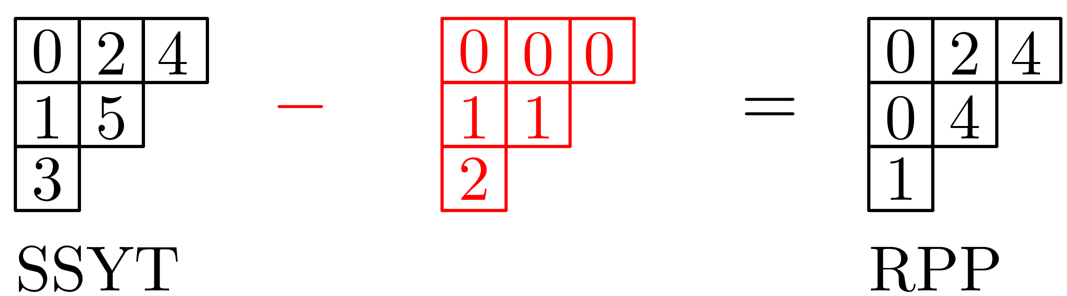

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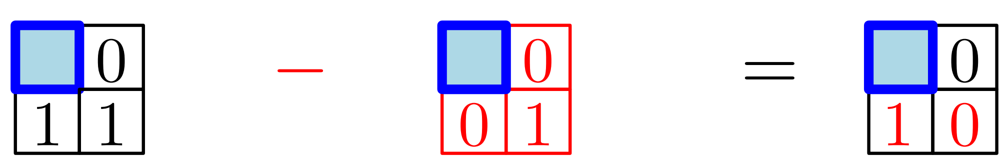
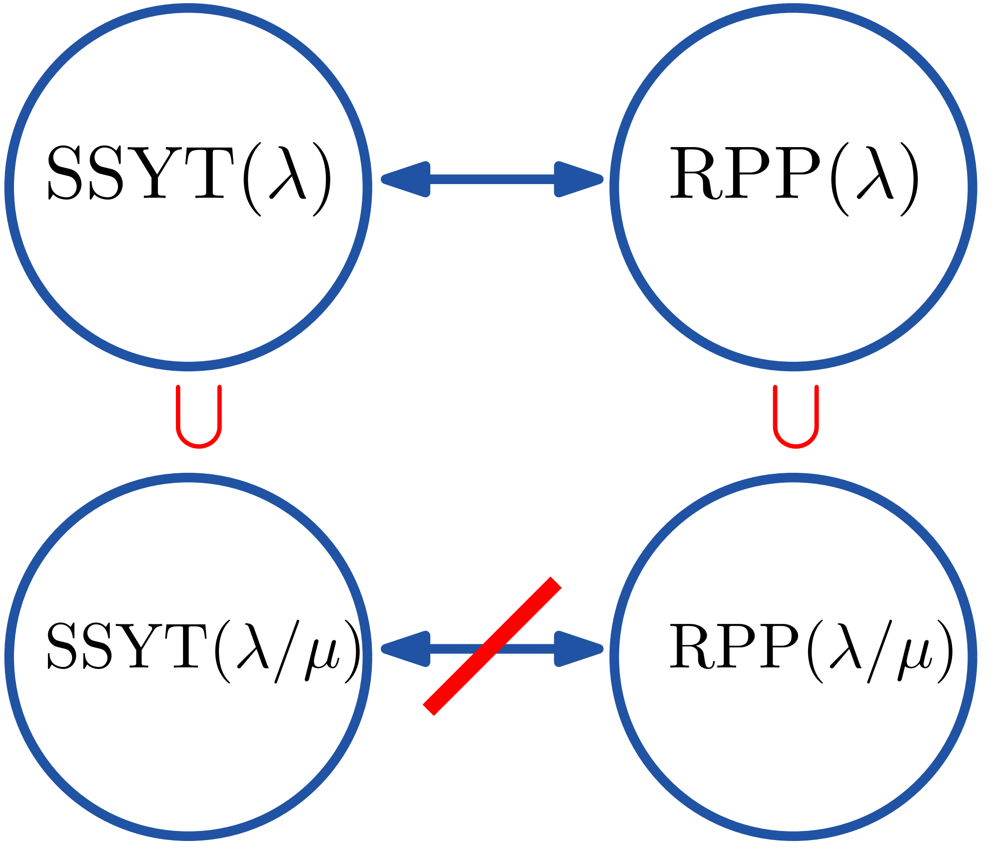


skew SSYT vs skew RPP



q-analogue Naruse's formula skew SSYT

Theorem (M-Pak-Panova 2015)

$$s_{\lambda/\mu}(1, q, , q^2, \ldots) = \sum_{D \in \mathcal{E}(\lambda/\mu)} \left(\prod_{(i,j) \in \lambda \setminus D} \frac{q^{\lambda'_j - i}}{1 - q^{h(i,j)}} \right).$$

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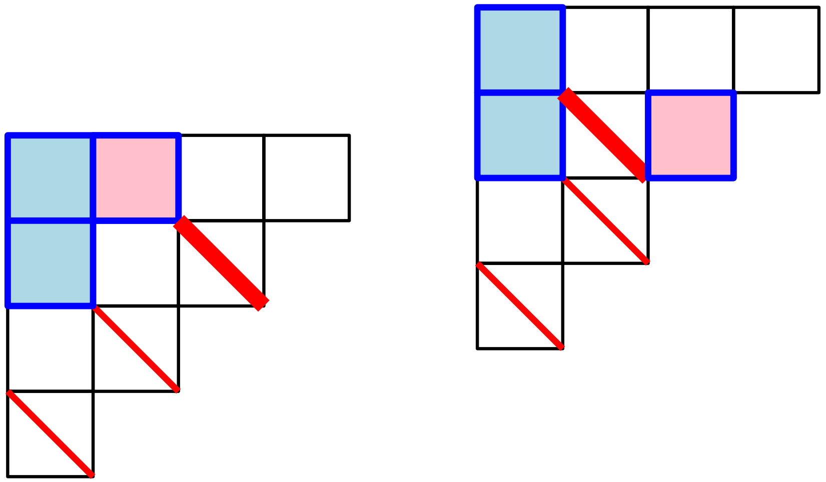

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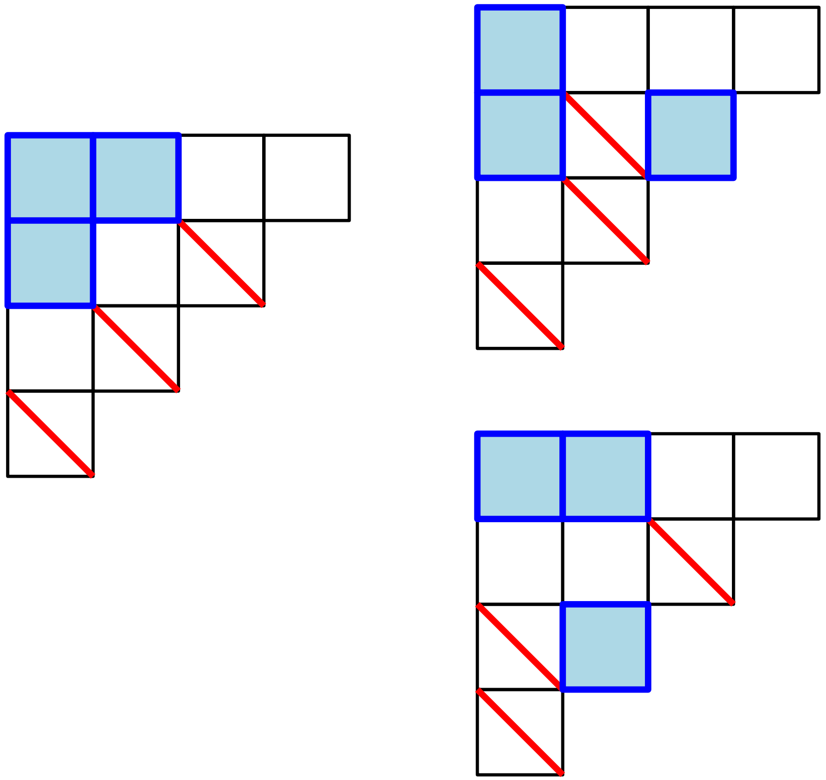

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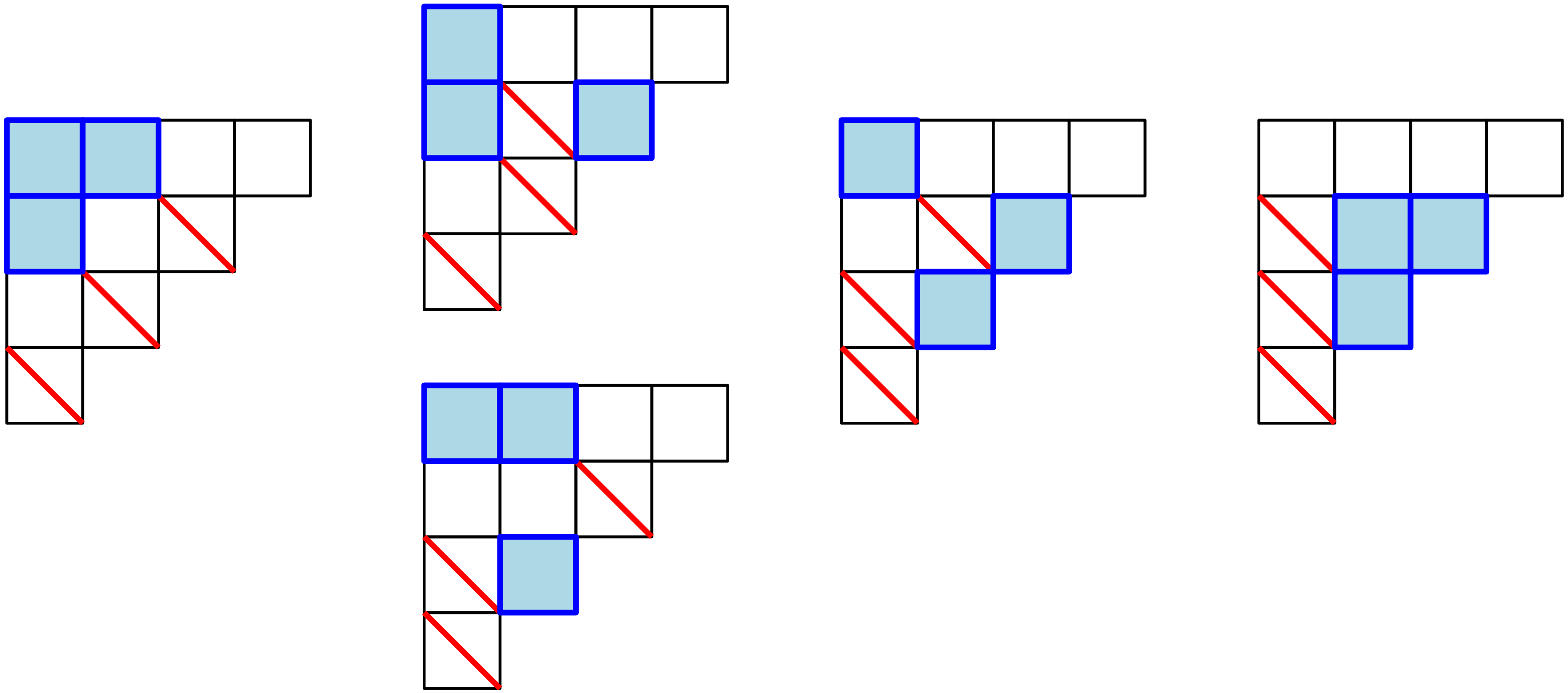

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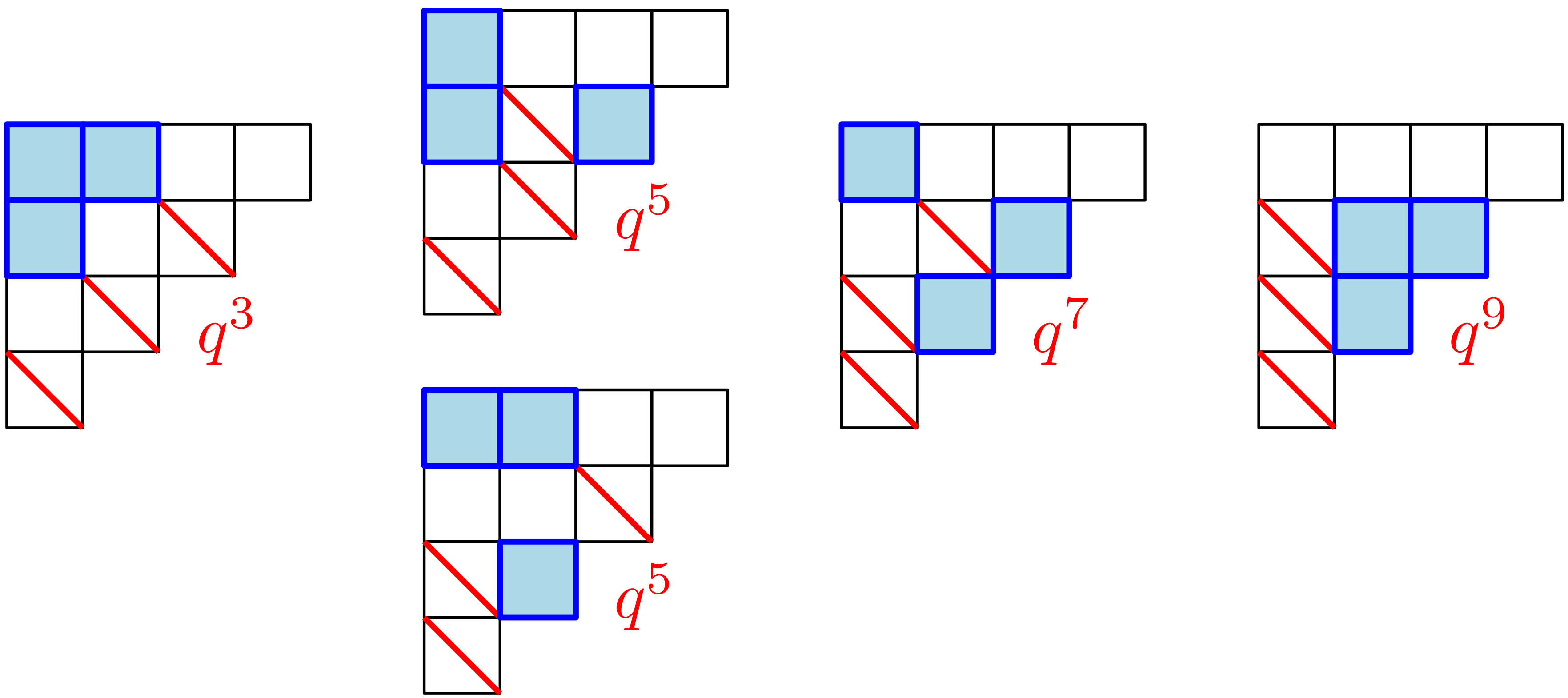

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$$\sum_{\pi \in \mathrm{RPP}(\lambda/\mu)} q^{|\pi|} = \sum_{D \in \mathcal{E}(\lambda/\mu)} q^{\mathrm{hkpeaks}(D)} \prod_{u \notin D} \frac{1}{1-q^{h(u)}}.$$

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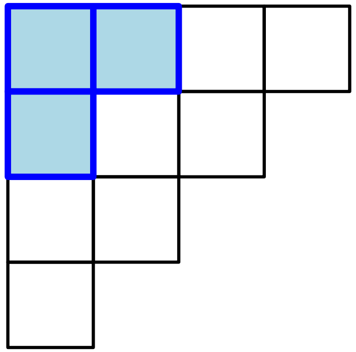

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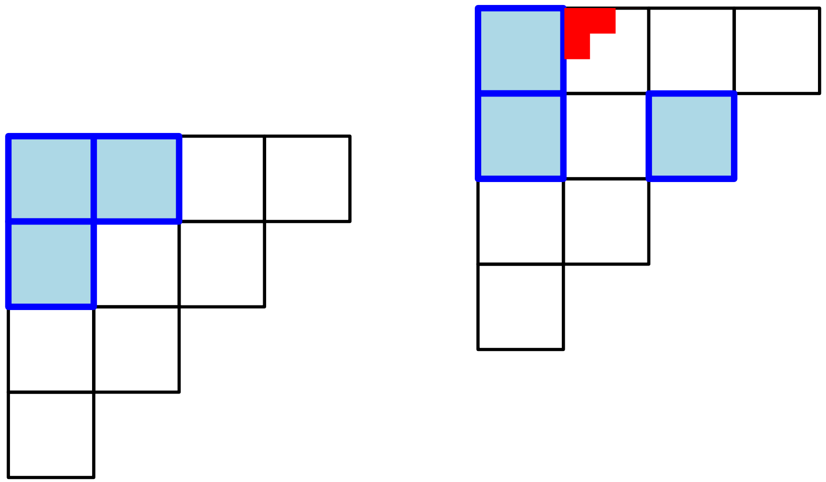

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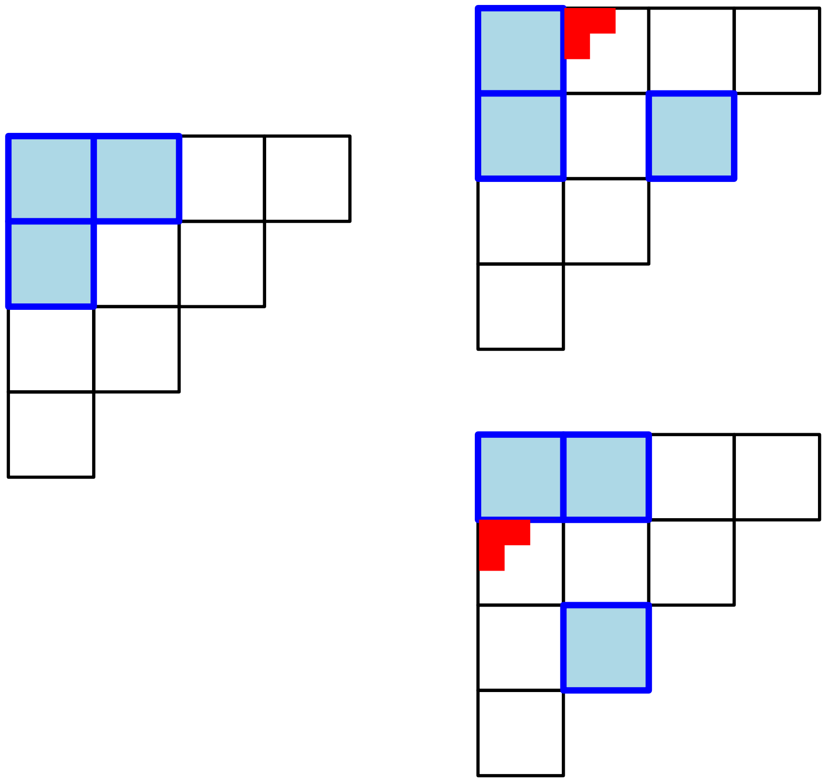

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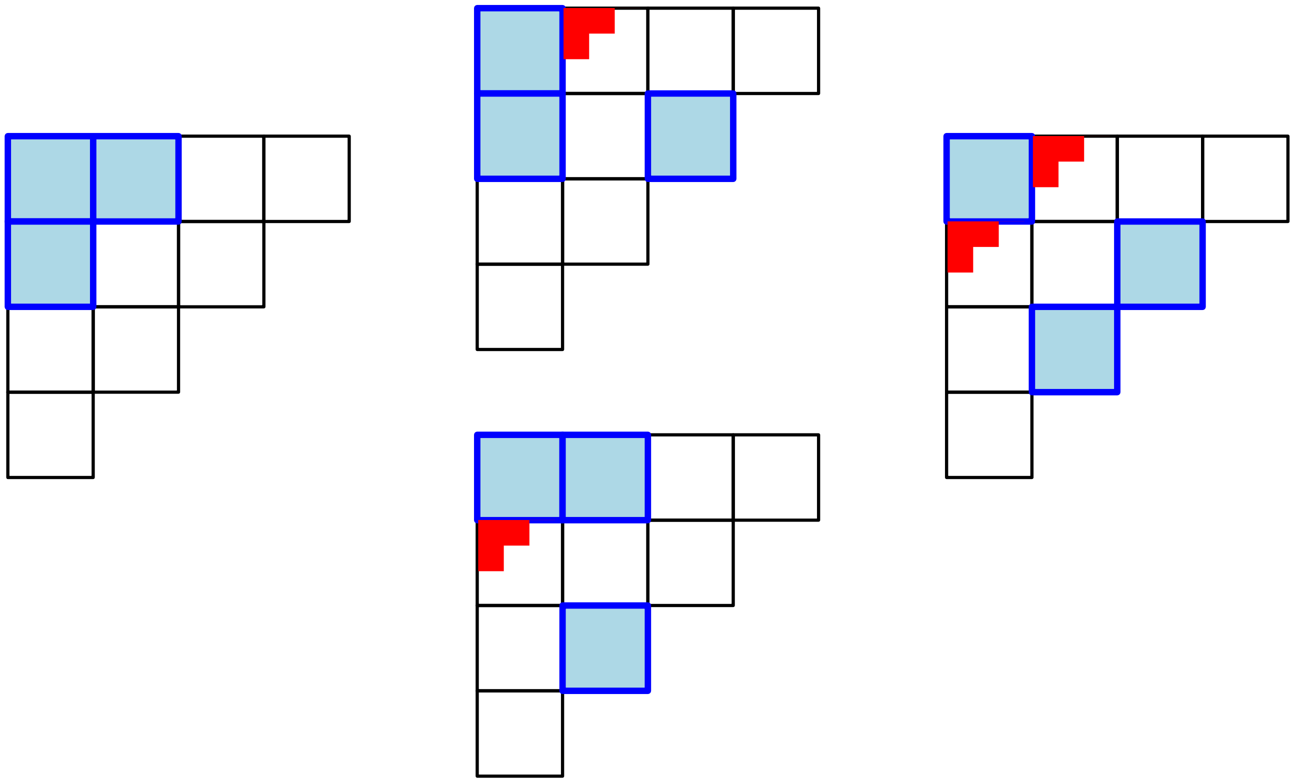

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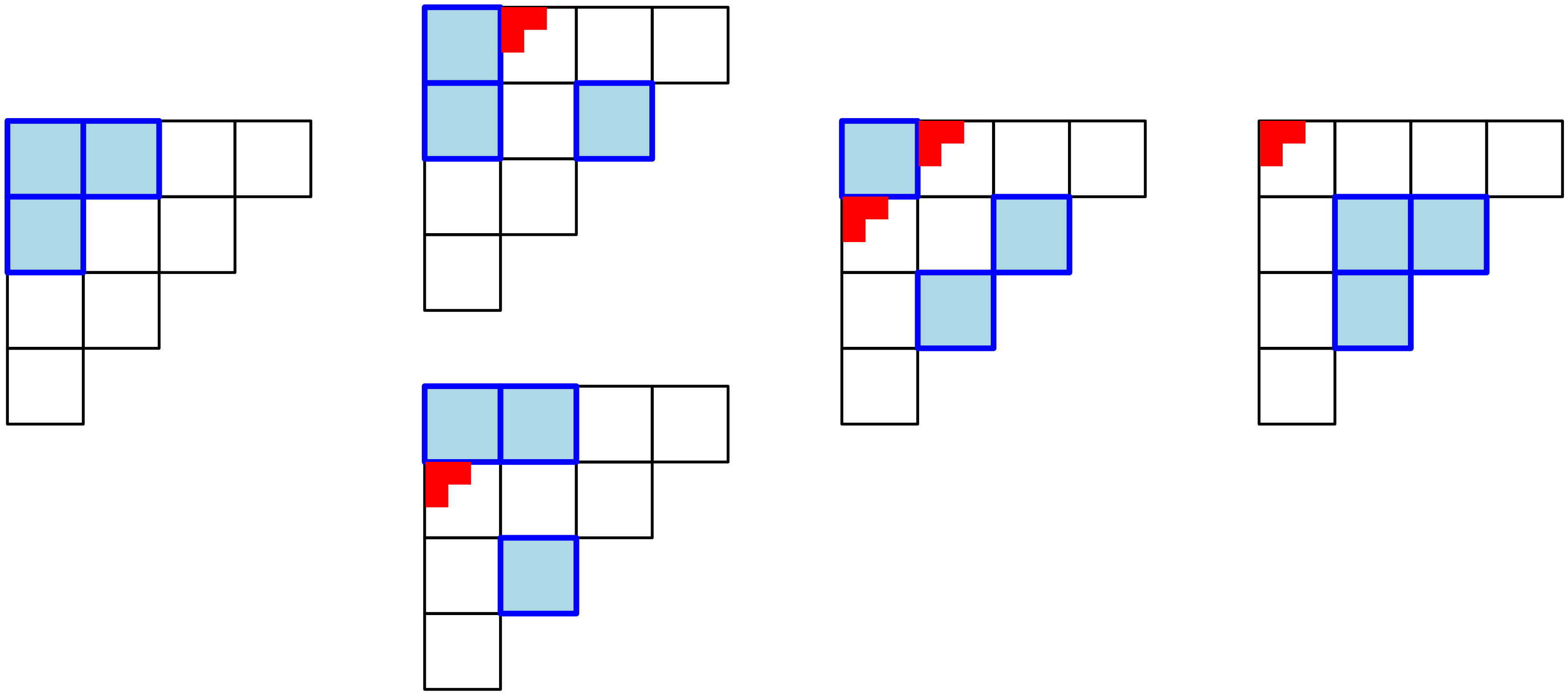

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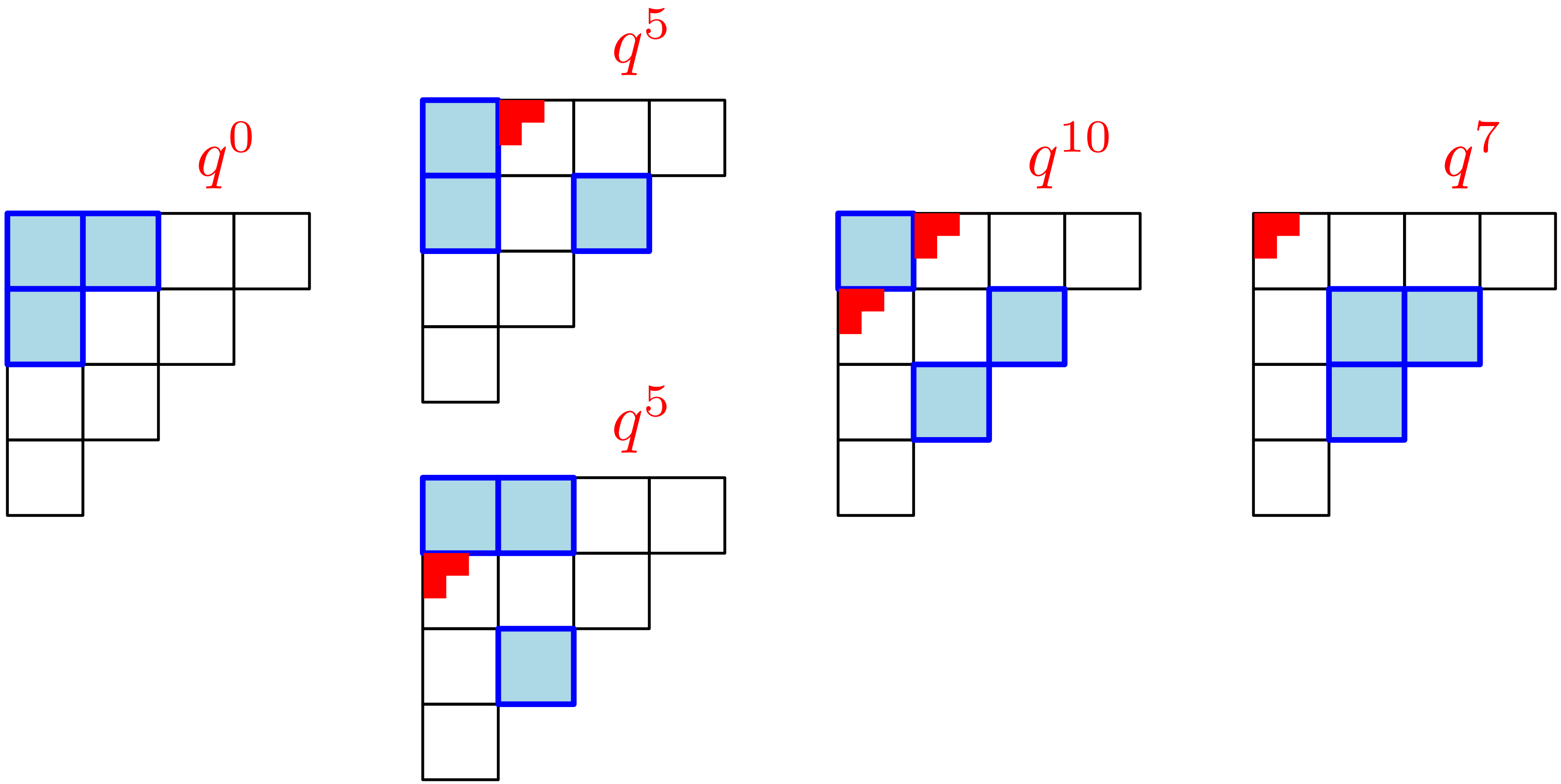

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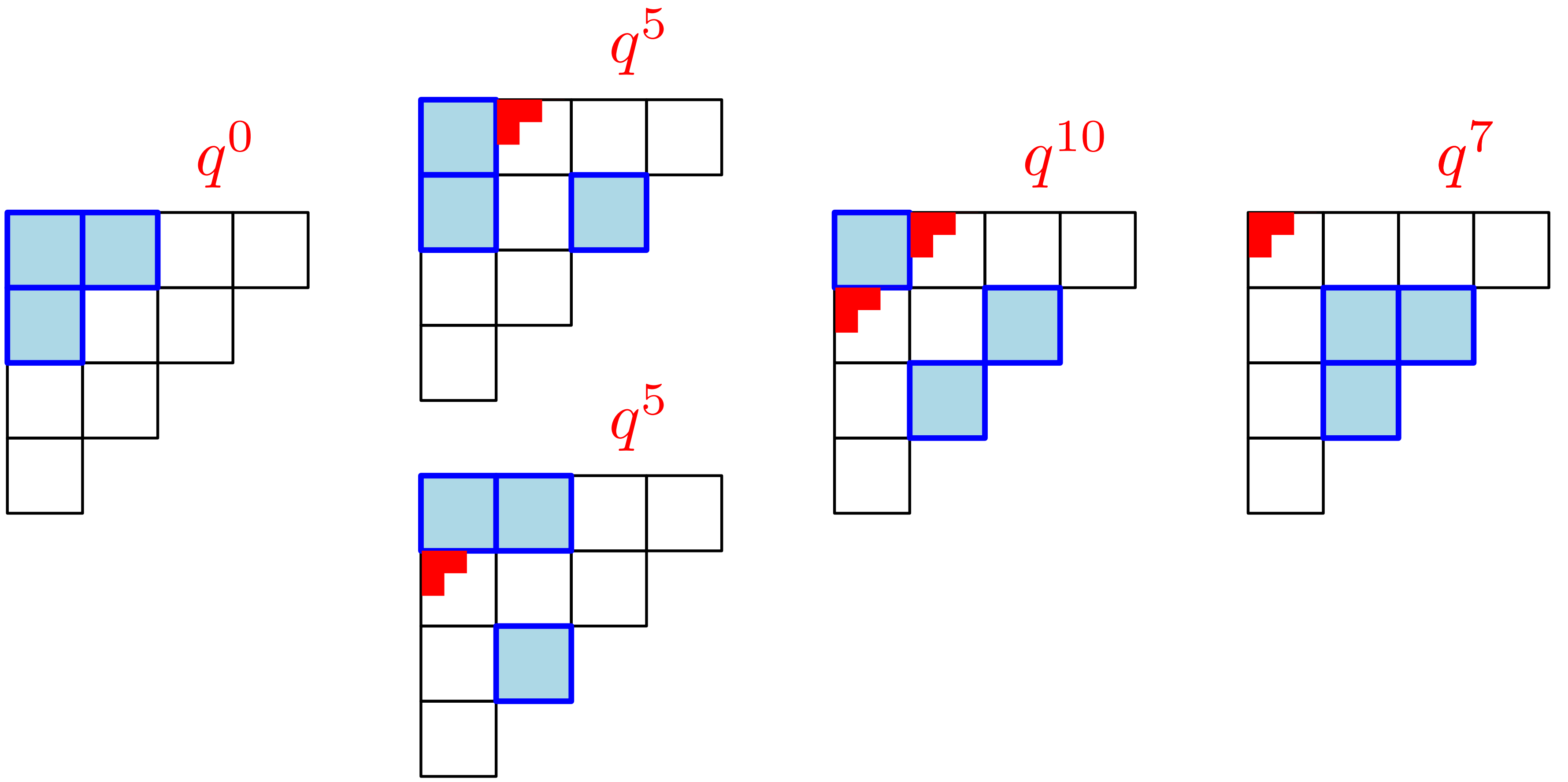


- Naruse-Okada have an analogue for all d-**complete posets**!

Outline

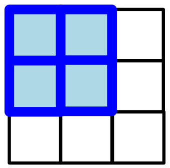
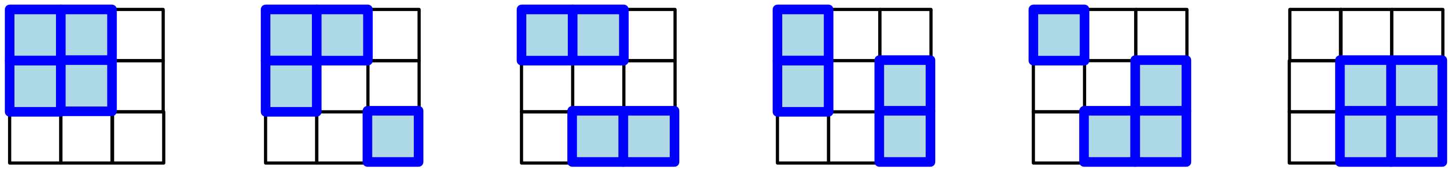
$$f^\lambda = \frac{|\lambda|!}{\prod_{u\in\lambda} h(u)}$$

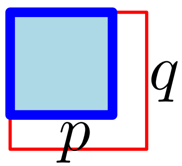
Naruse's formula for $f^{\lambda/\mu}$

- q-analogues
- Applications
  - relation to lozenge tilings
  - bounds and asymptotics for $f^{\lambda/\mu}$
  - family of skew shapes with product formulas

Excited diagrams of a rectangle

Example

$\mathcal{E}($  $)$: 

Proposition $|\mathcal{E}($ 

$)| = \binom{p+q-2}{q-1}$.

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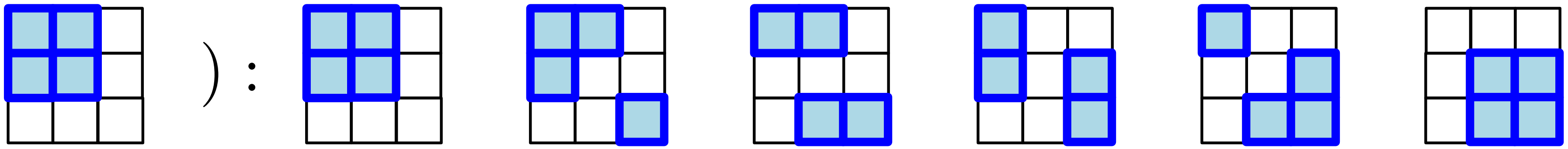
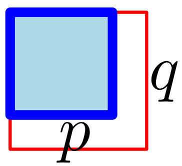
$\mathcal{E}($  $)$:

Proposition $|\mathcal{E}($ 

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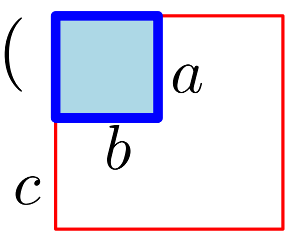
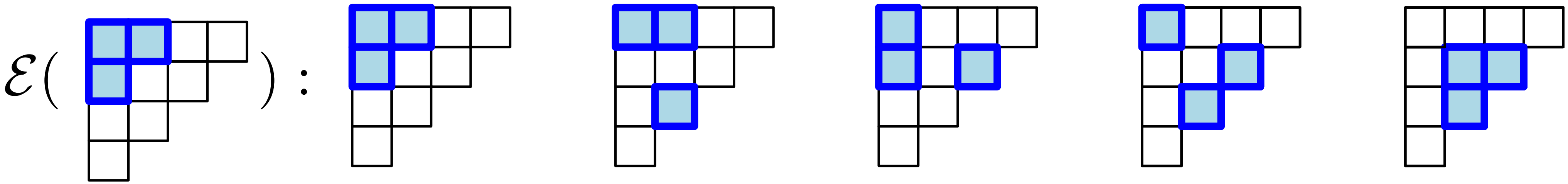
MacMahon box formula

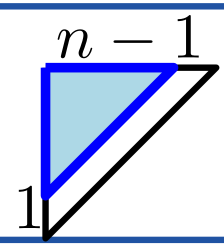
$|\mathcal{E}($ 

$)| = \prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{i+j+k-1}{i+j+k-2}$

Excited diagrams of a staircase/staircase

Example



Proposition $|\mathcal{E}($ 

$)| = \frac{1}{n+1}\binom{2n}{n}$.

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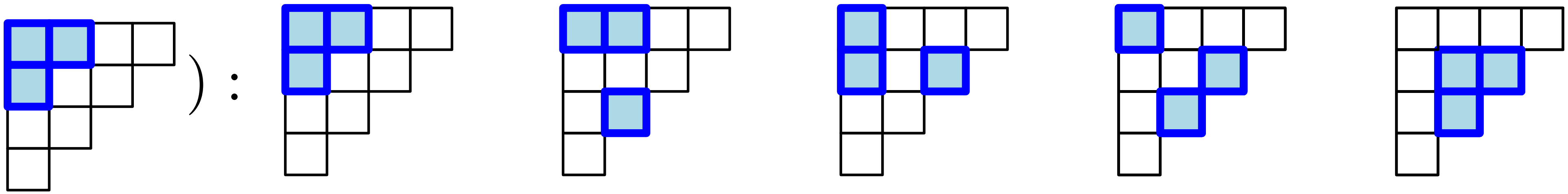
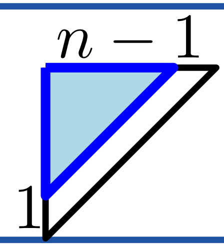
$\mathcal{E}($  $)$:

Proposition $|\mathcal{E}($ 

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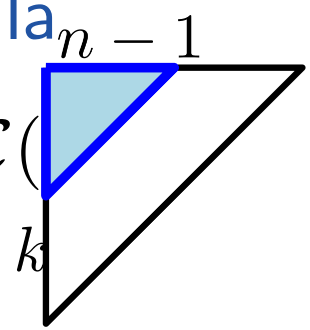
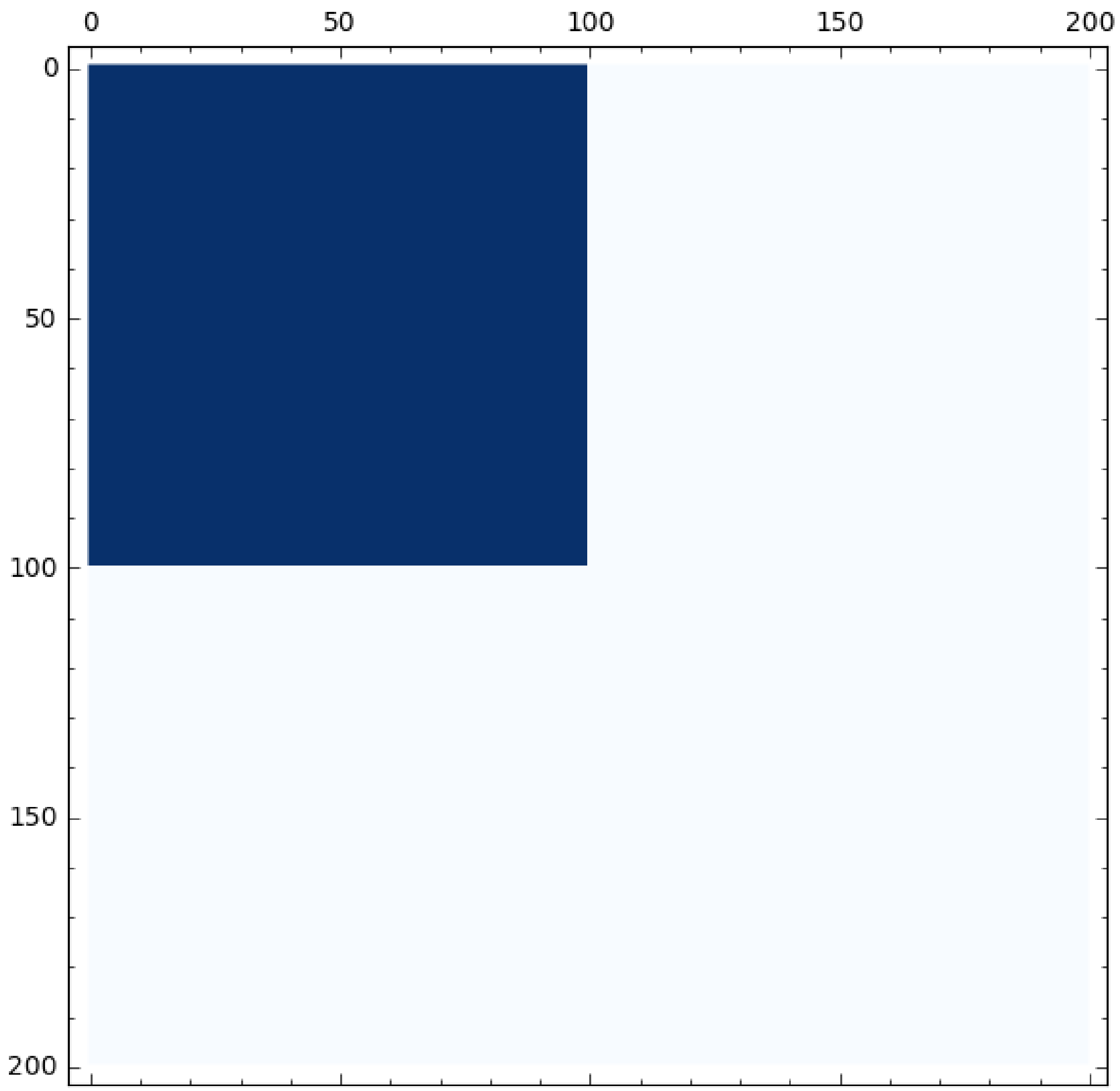
Proctor's formula

$|\mathcal{E}($ 

$$)| = \prod_{1 \le i < j \le n} \frac{2k+i+j-1}{i+j-1},$$

Bigger example of excited diagrams



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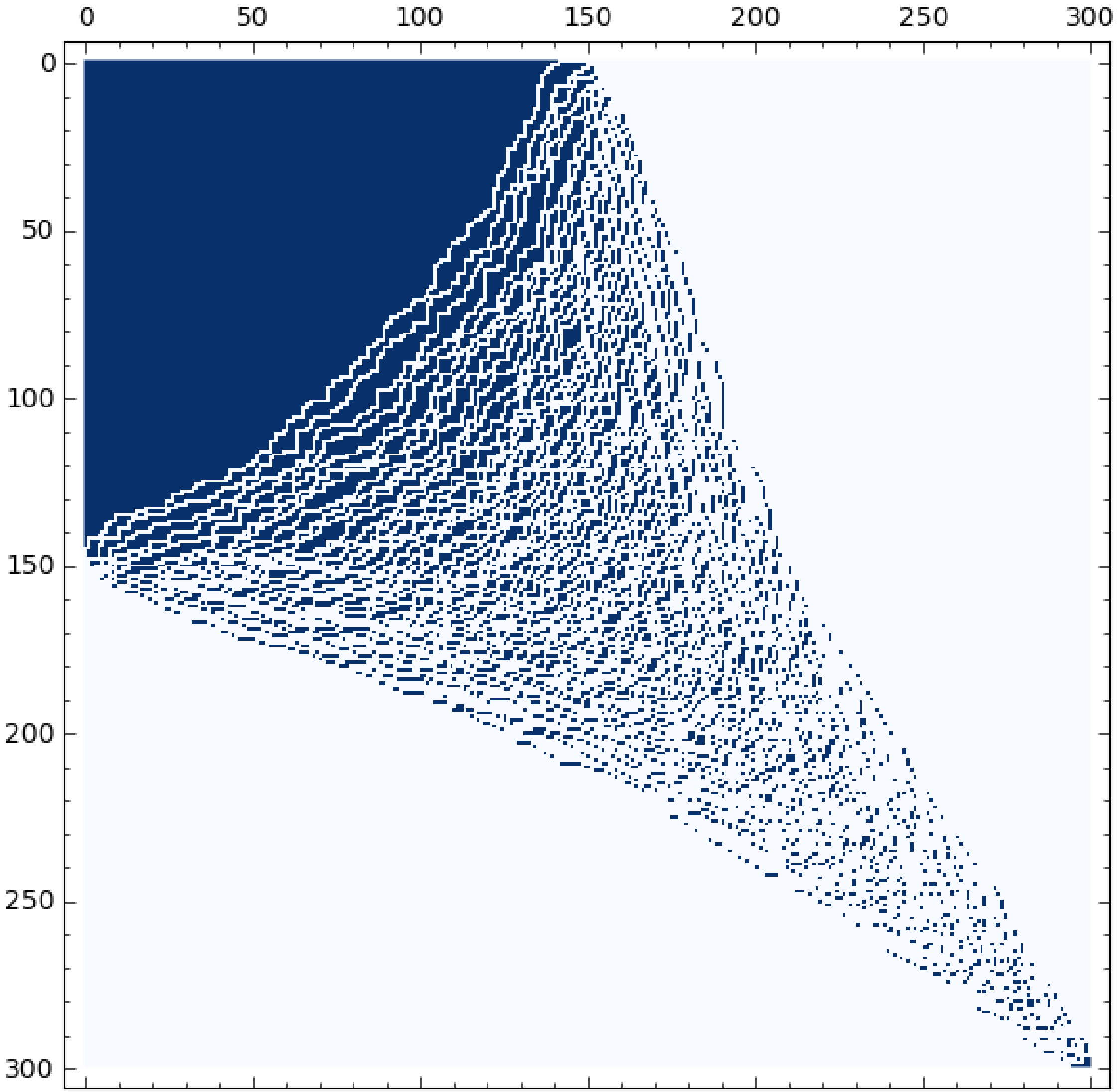

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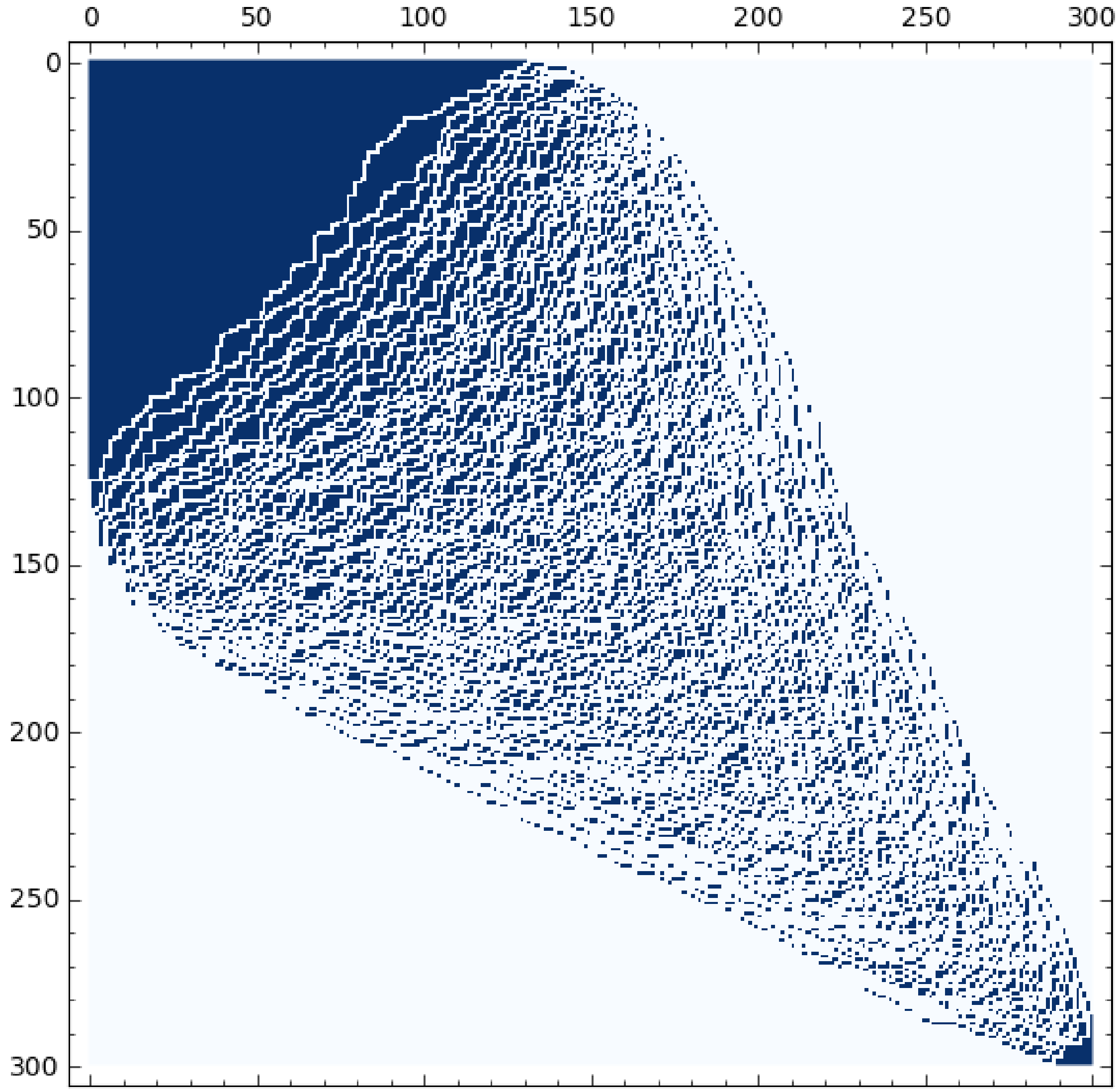

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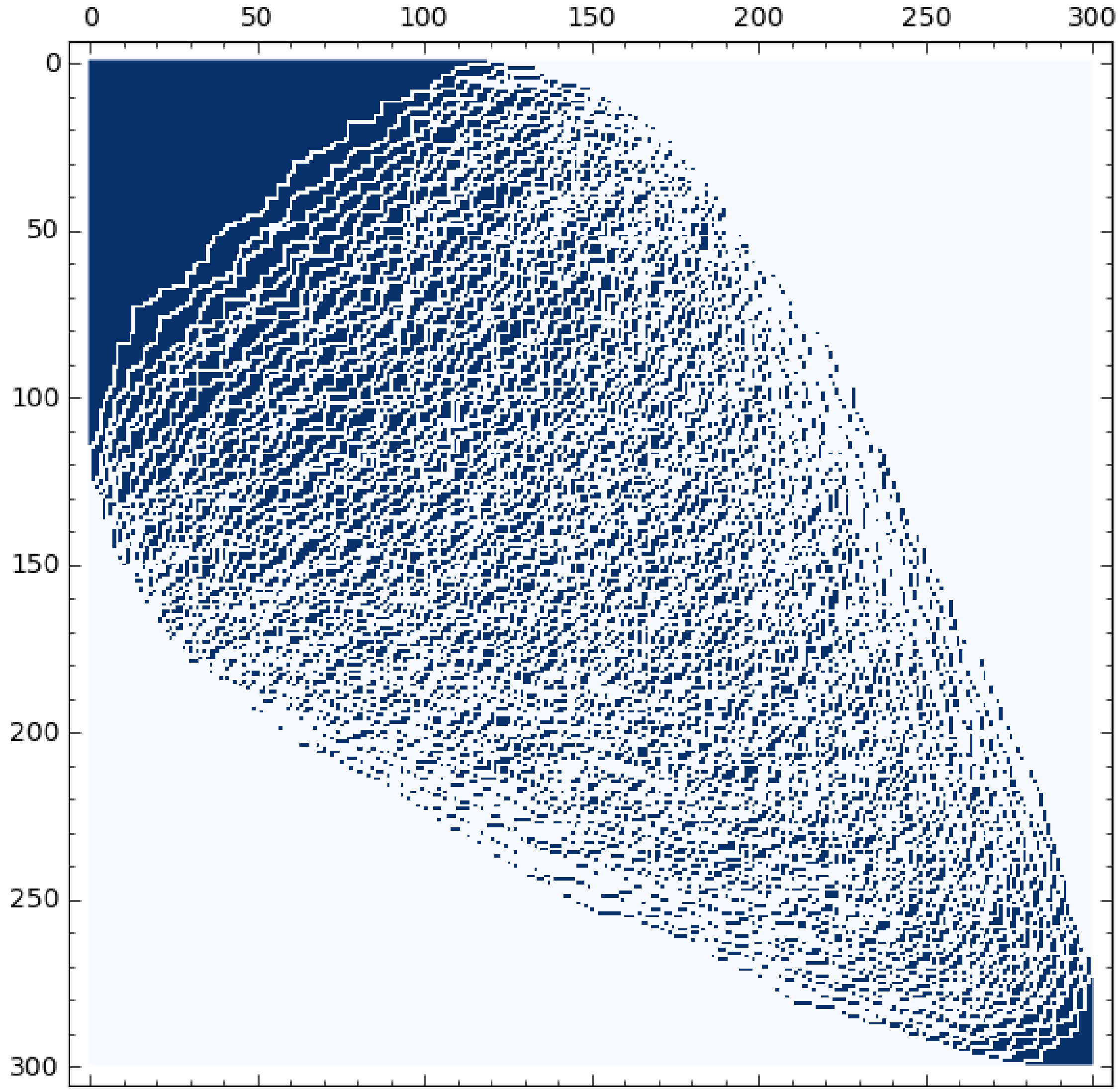

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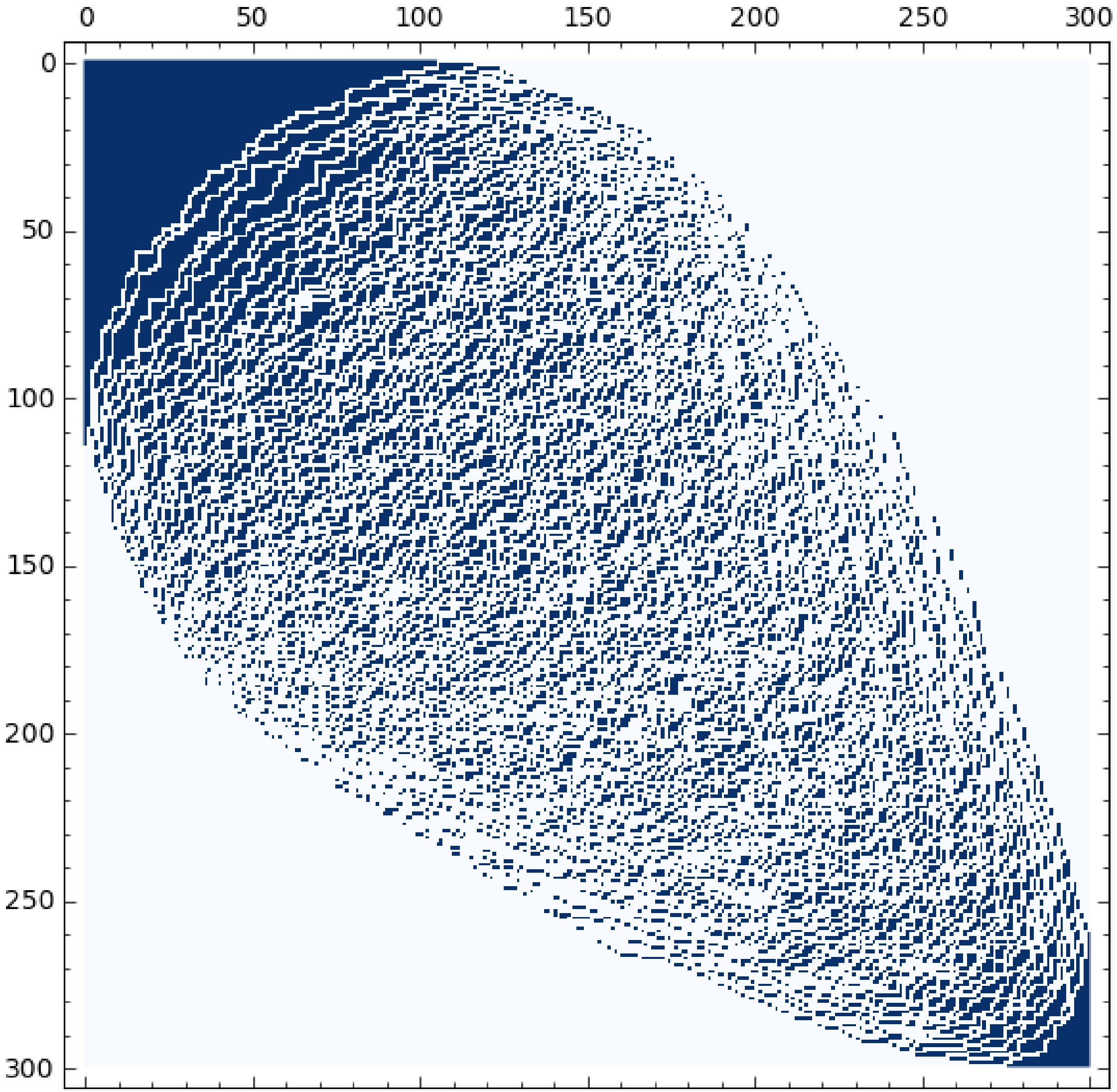

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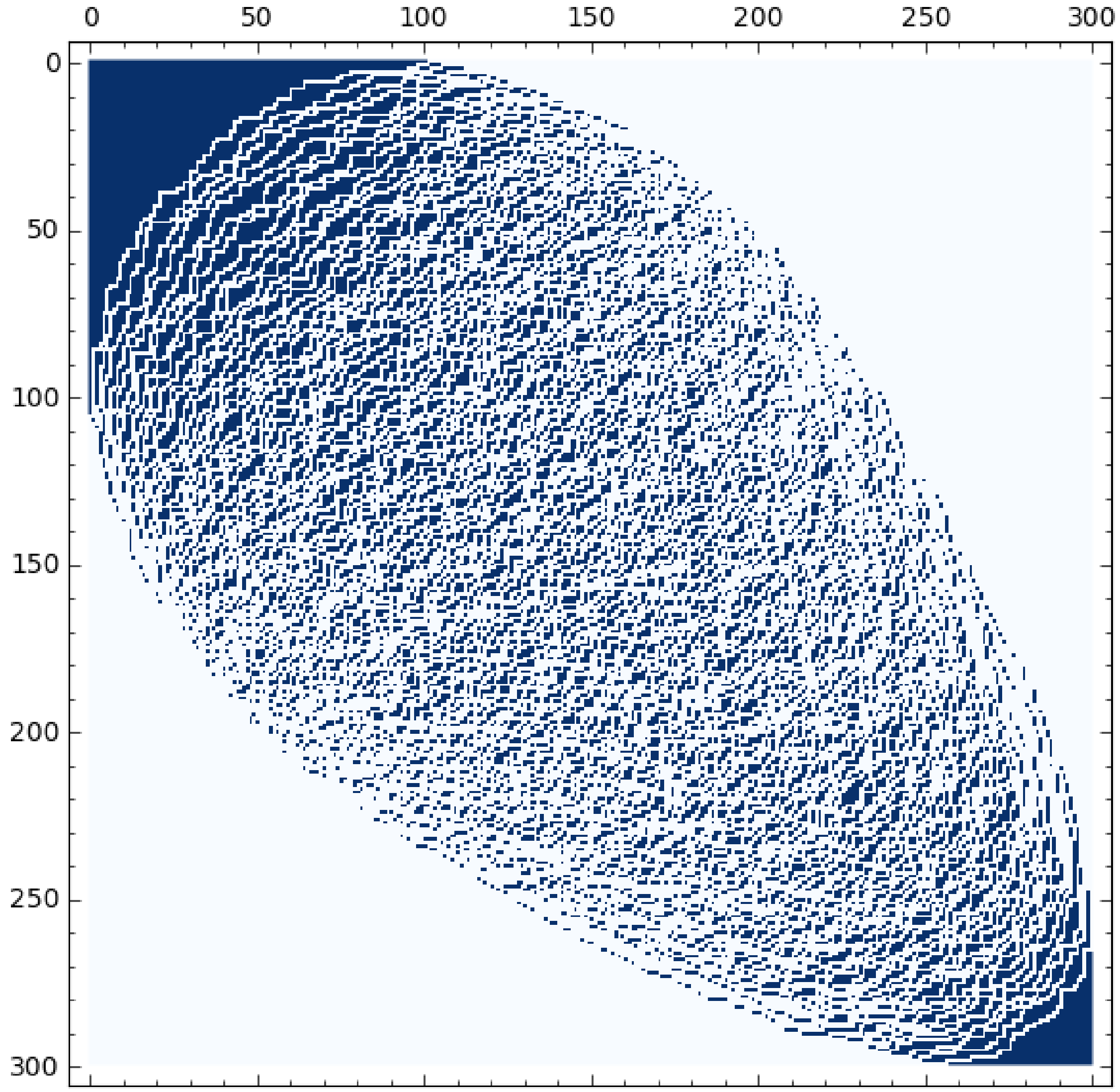

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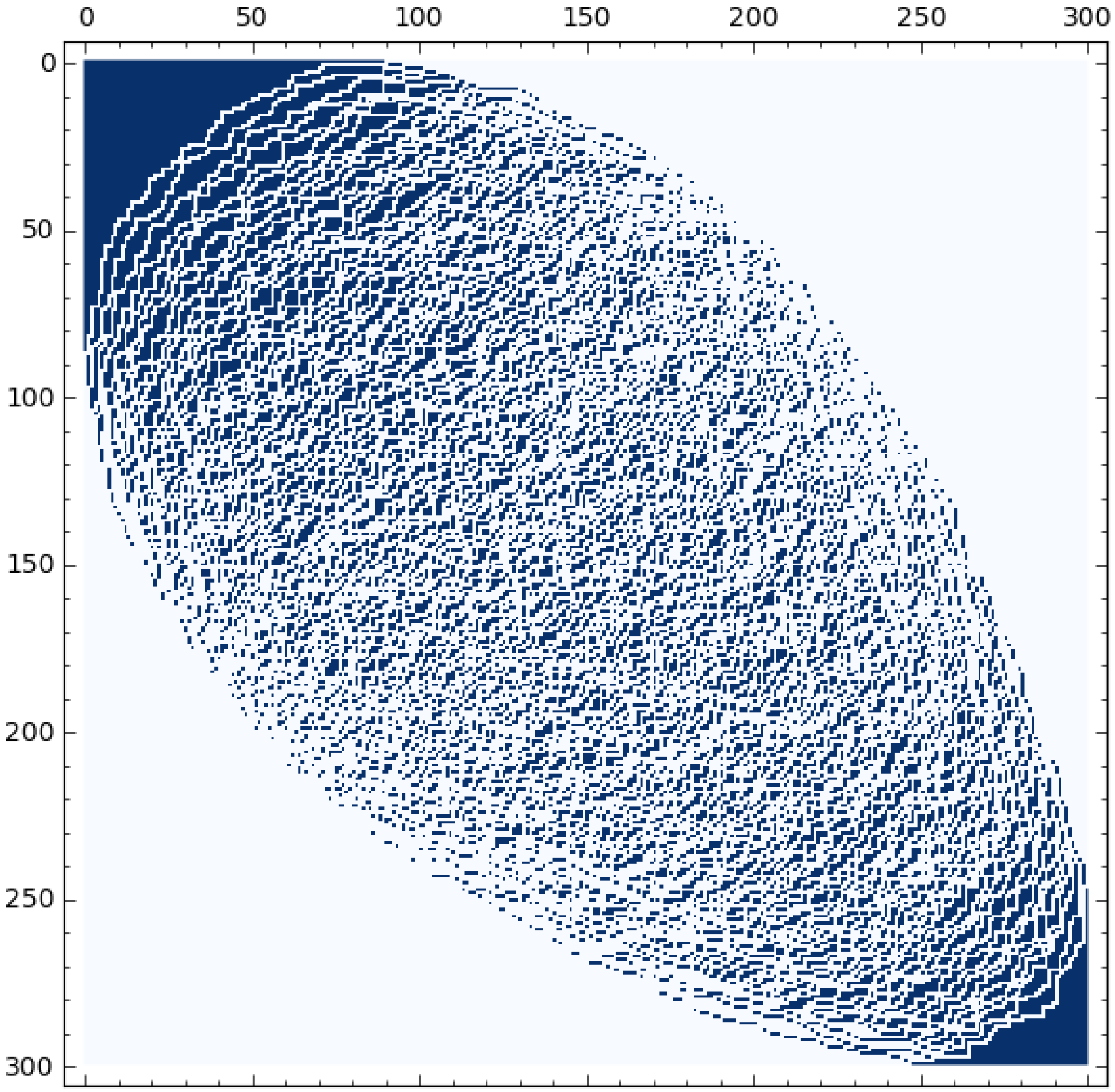

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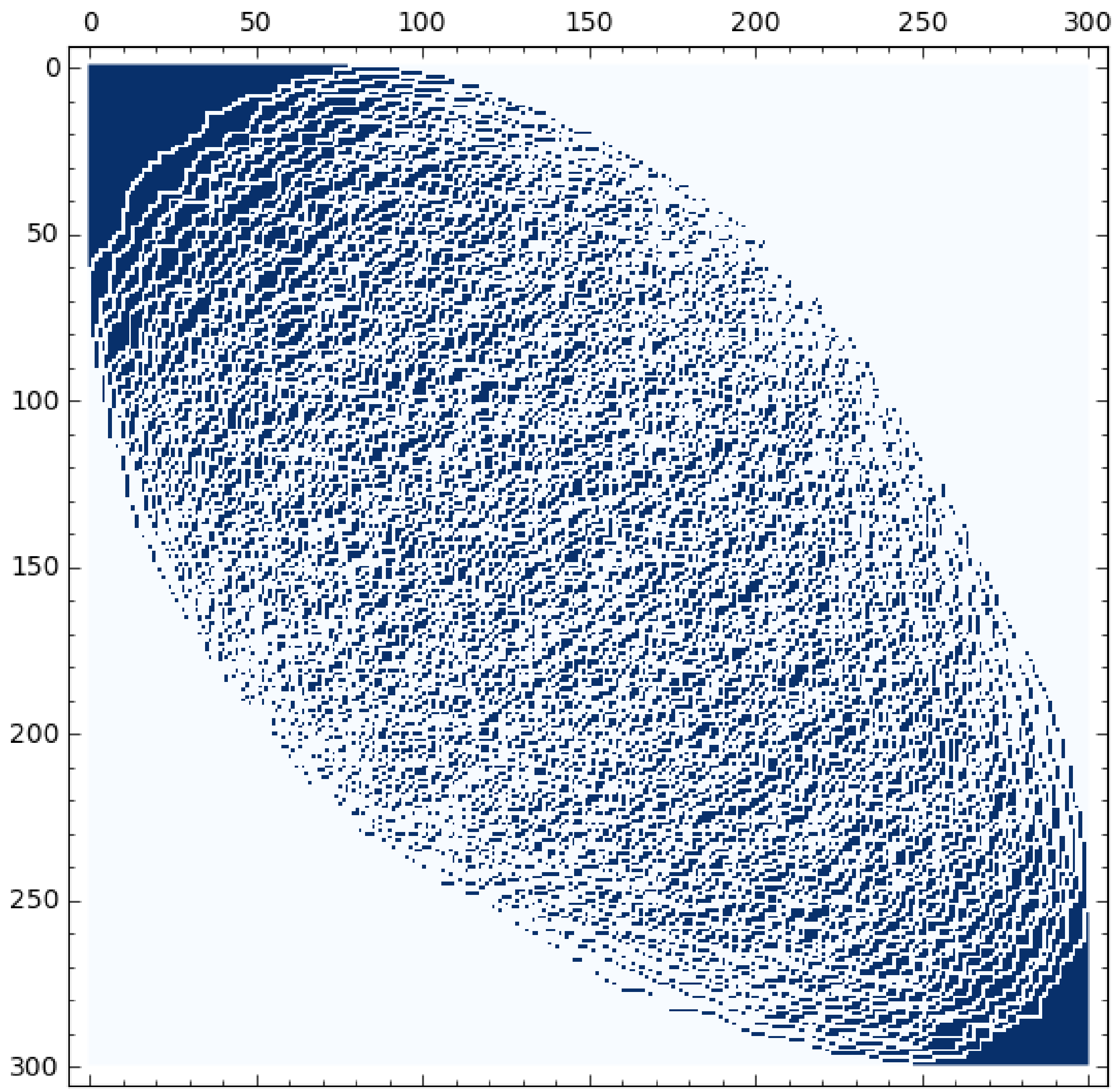

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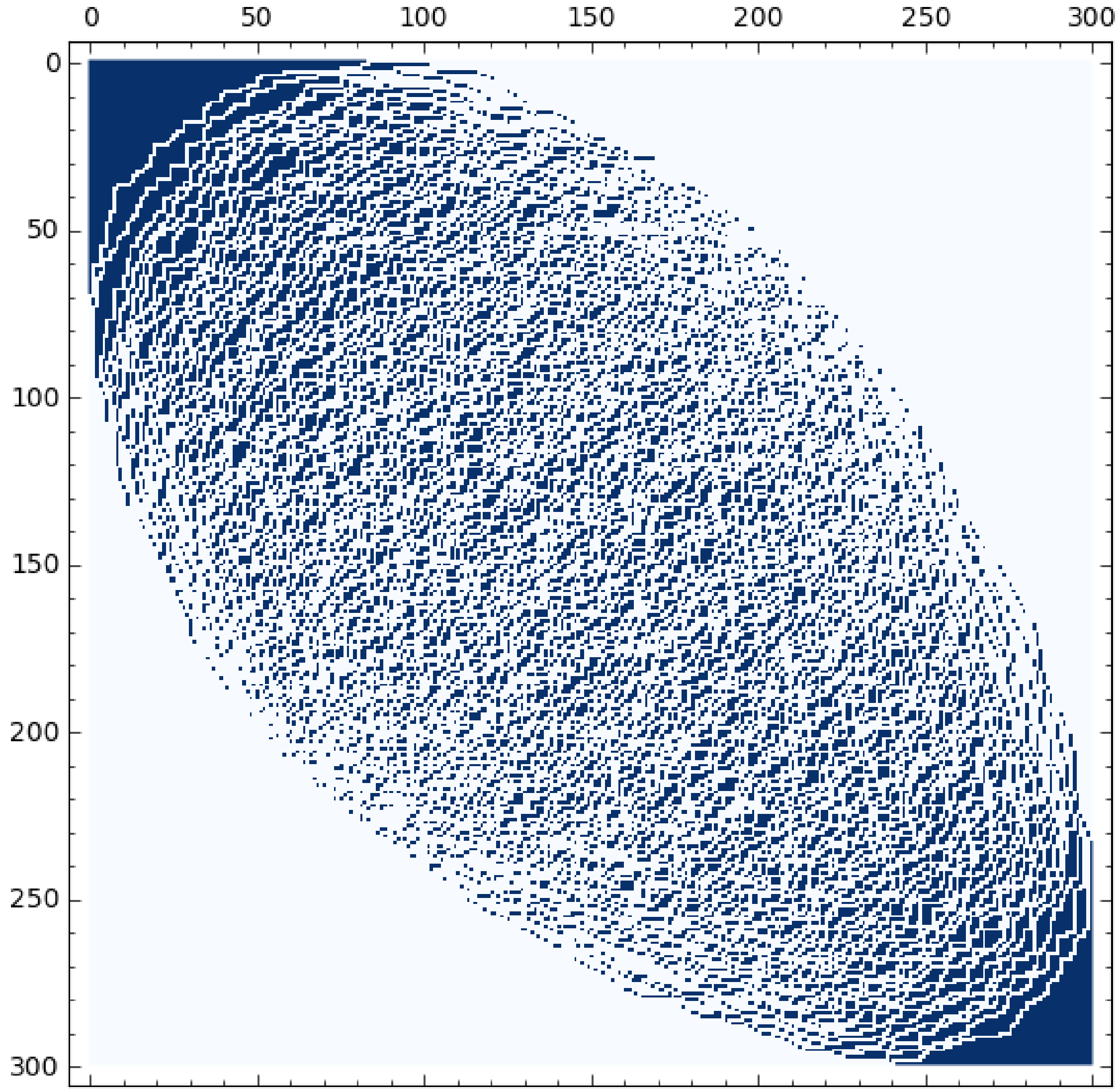

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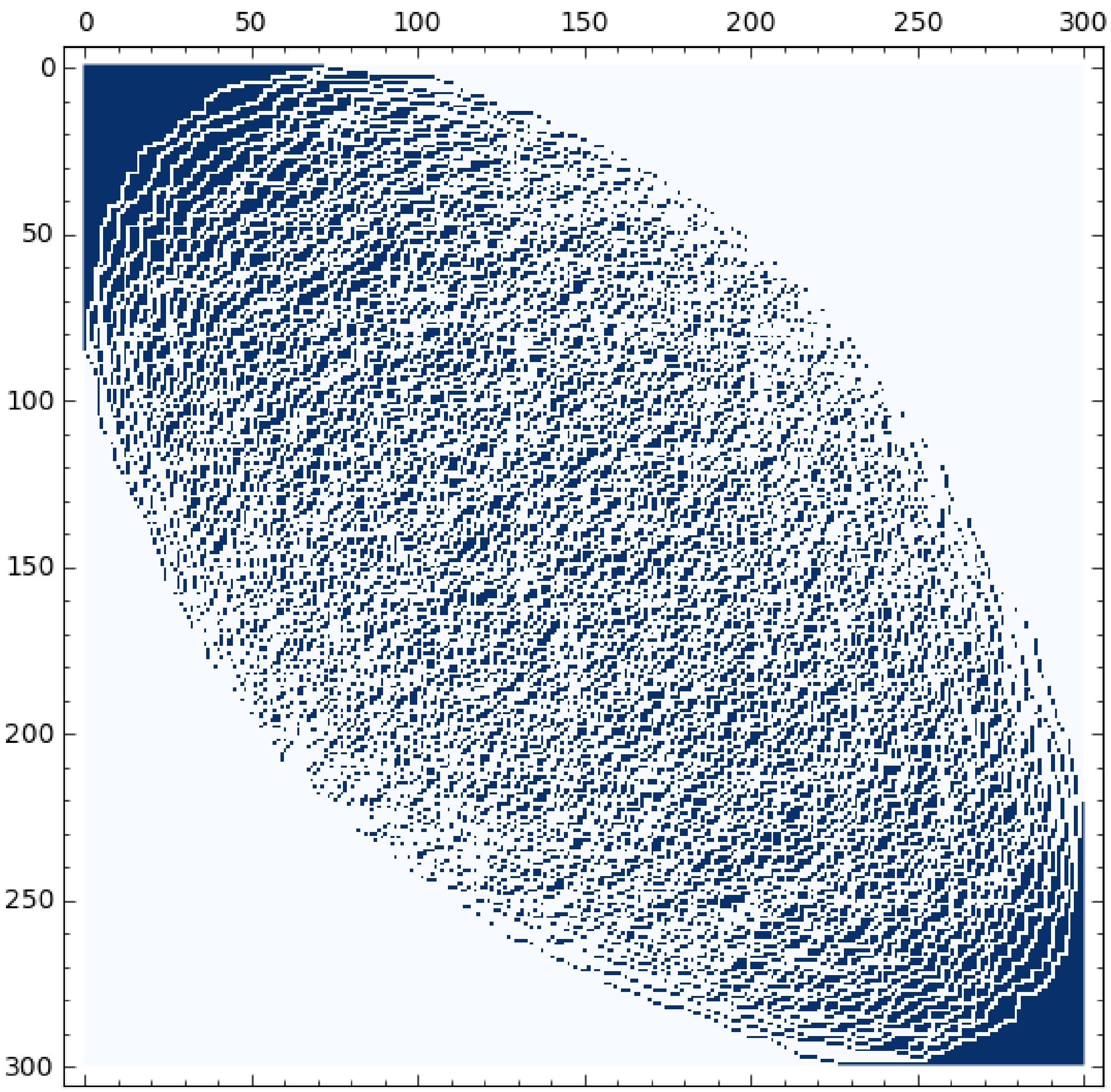

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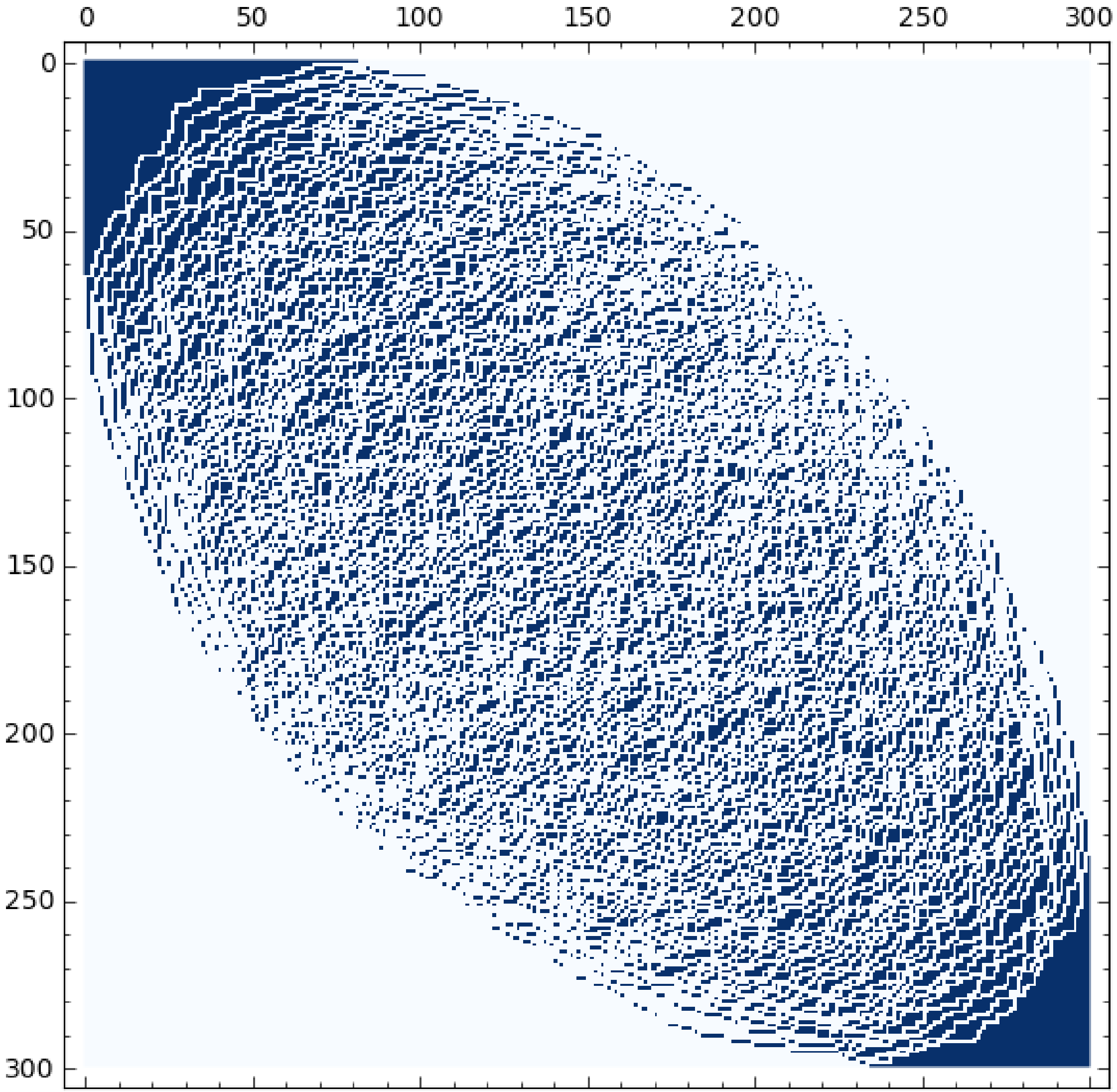


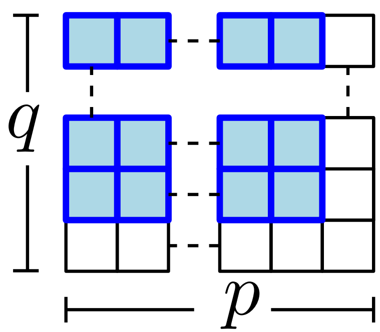
Naruse's "hook-length" formula for $f^{\lambda/\mu}$

Theorem (Naruse 2014)

$$f^{\lambda/\mu} = |\lambda/\mu|! \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{(i,j) \in \lambda \setminus D} \frac{1}{h(i,j)},$$

where $\mathcal{E}(\lambda/\mu)$ is the set of **excited diagrams** of λ/μ.

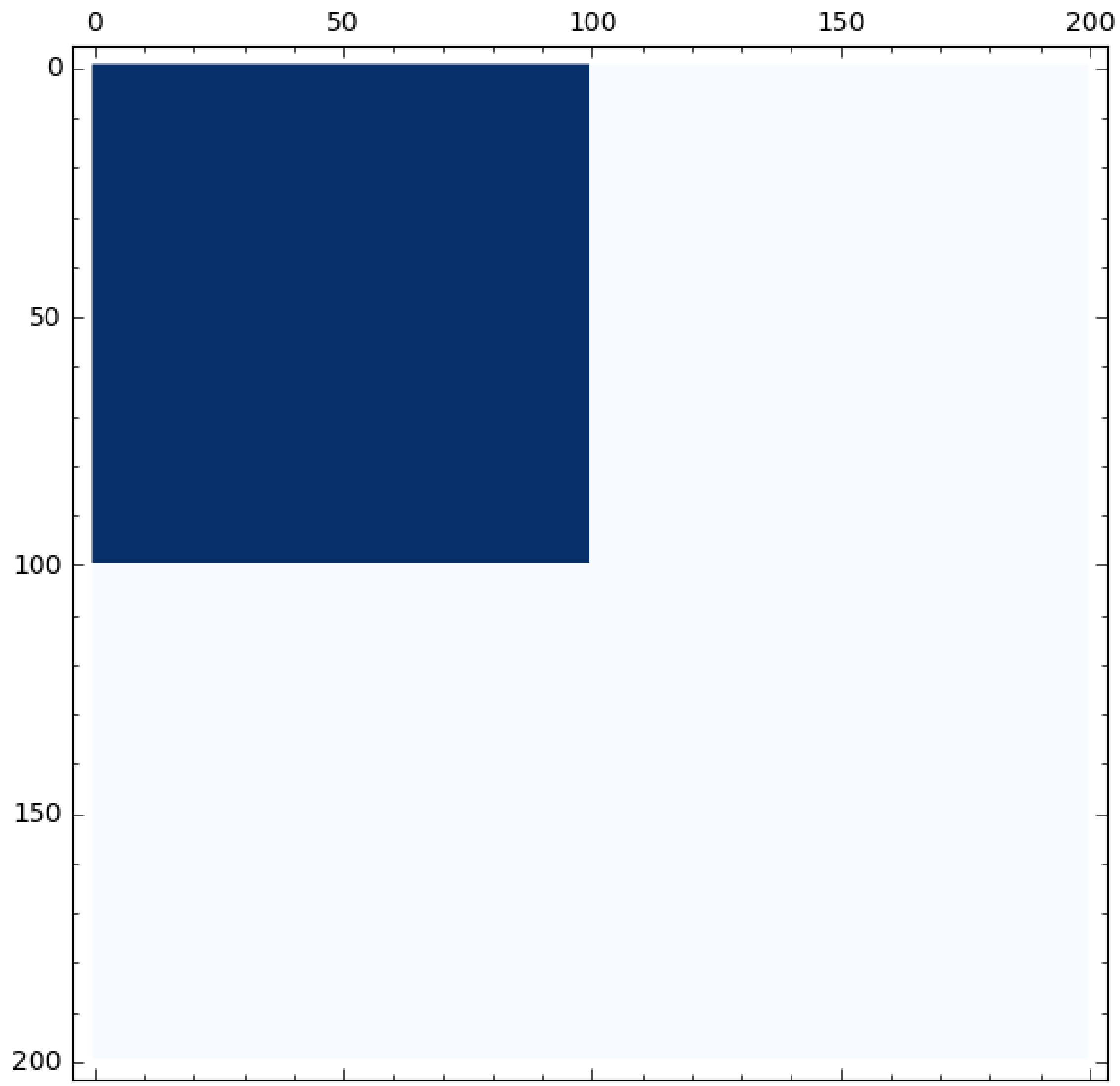
Example



5	4	3
4	3	2
3	2	1

$$\binom{p+q-2}{q-1} = (p+q-2)! \sum_{\mathsf{p}:(q,1)\to(1,p)} \prod_{(i,j)\in p} \frac{1}{i+j-1}$$

Excited diagrams weighted by hooks



Weighted excited diagrams: $\sum_D \prod_{u \in \lambda \setminus D} \frac{1}{h(u)}$

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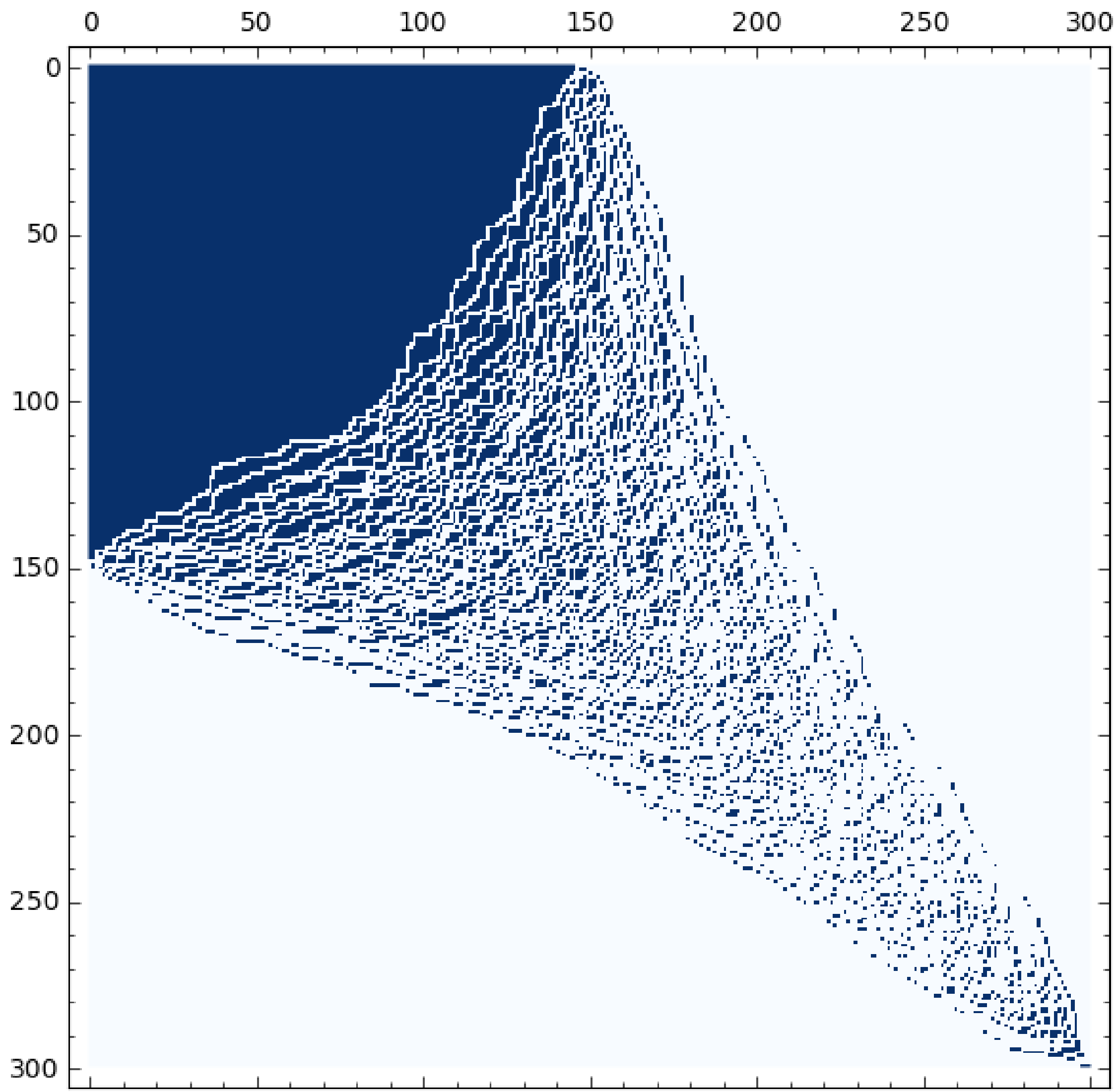

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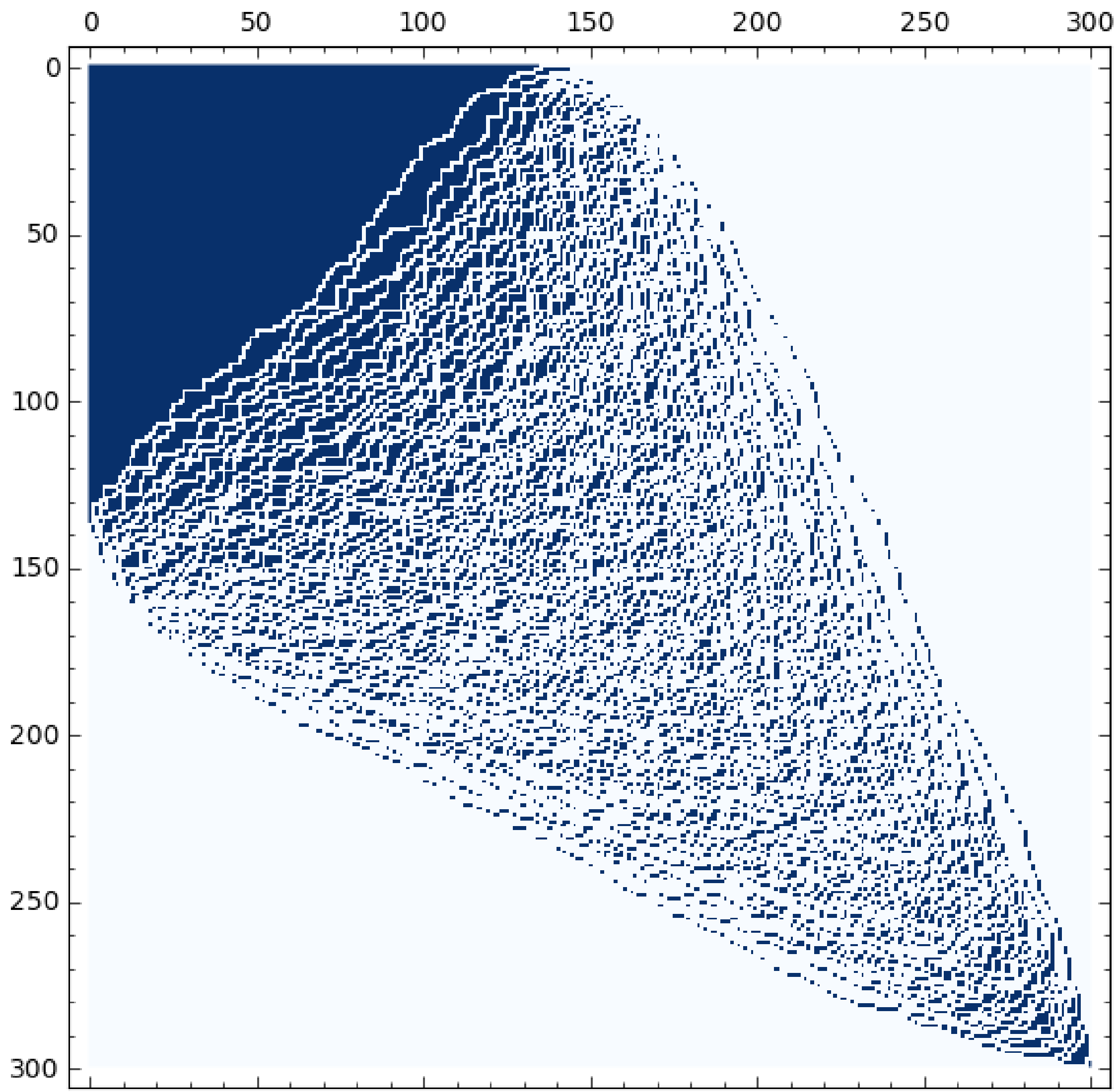

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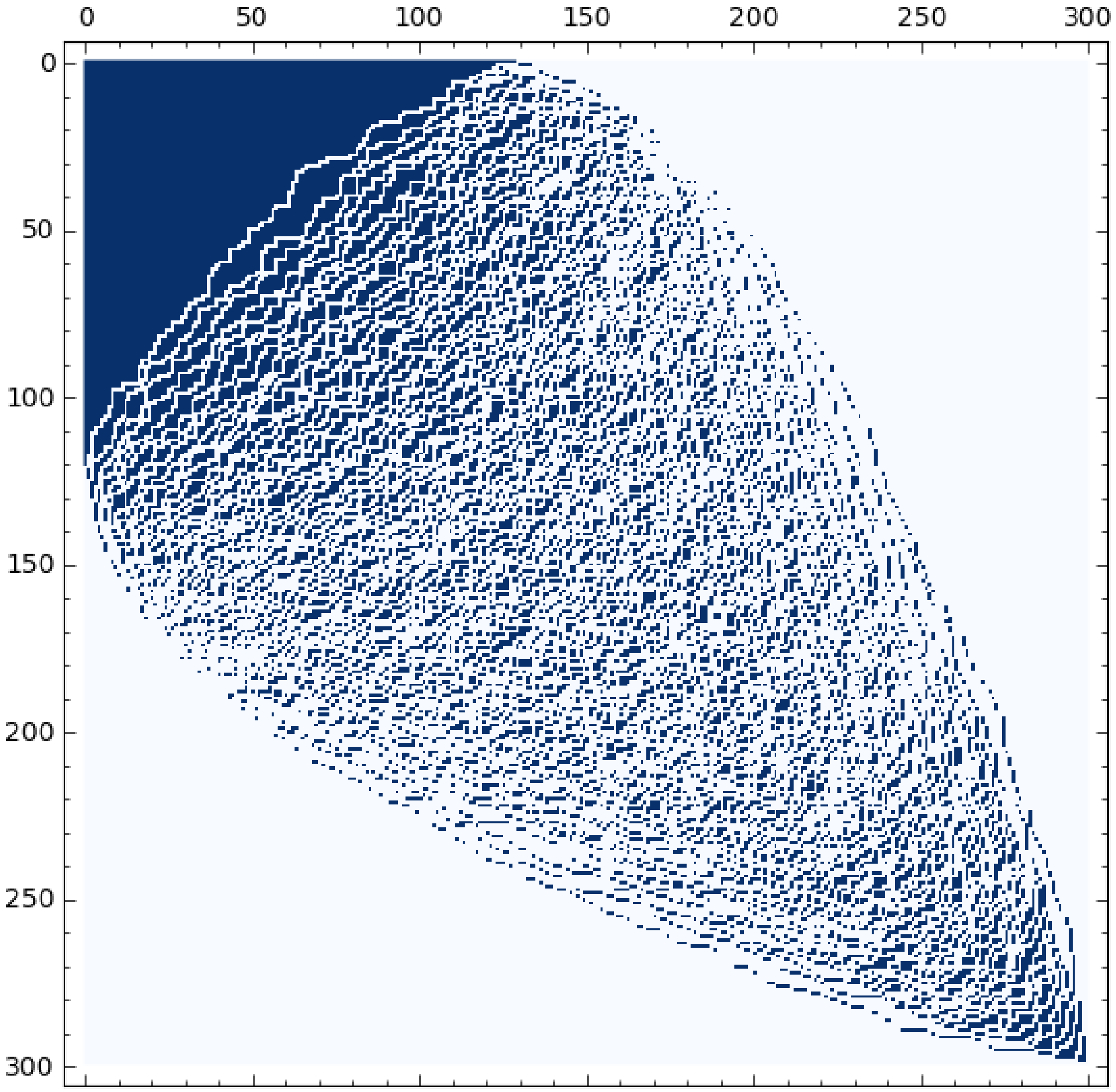

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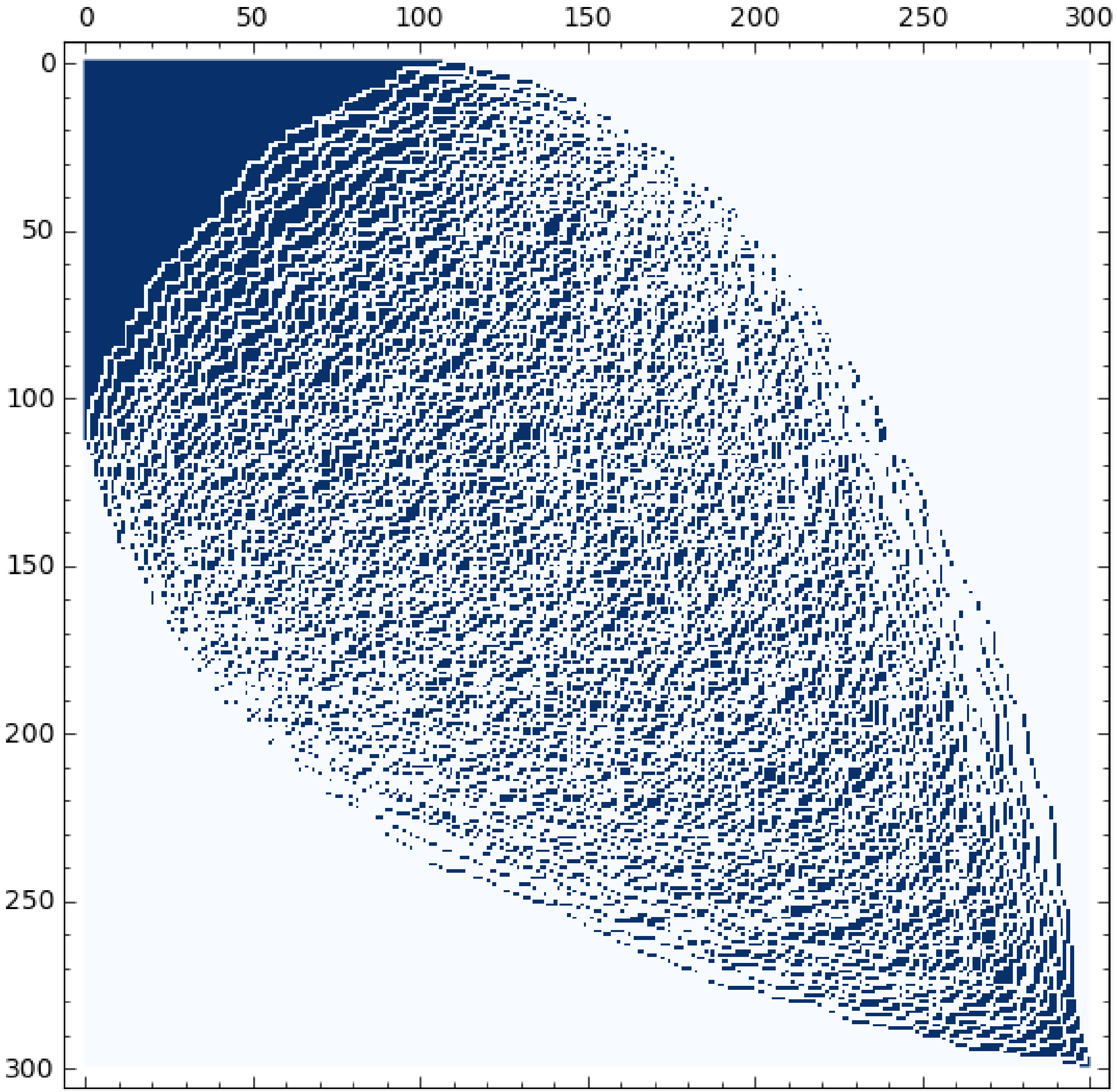

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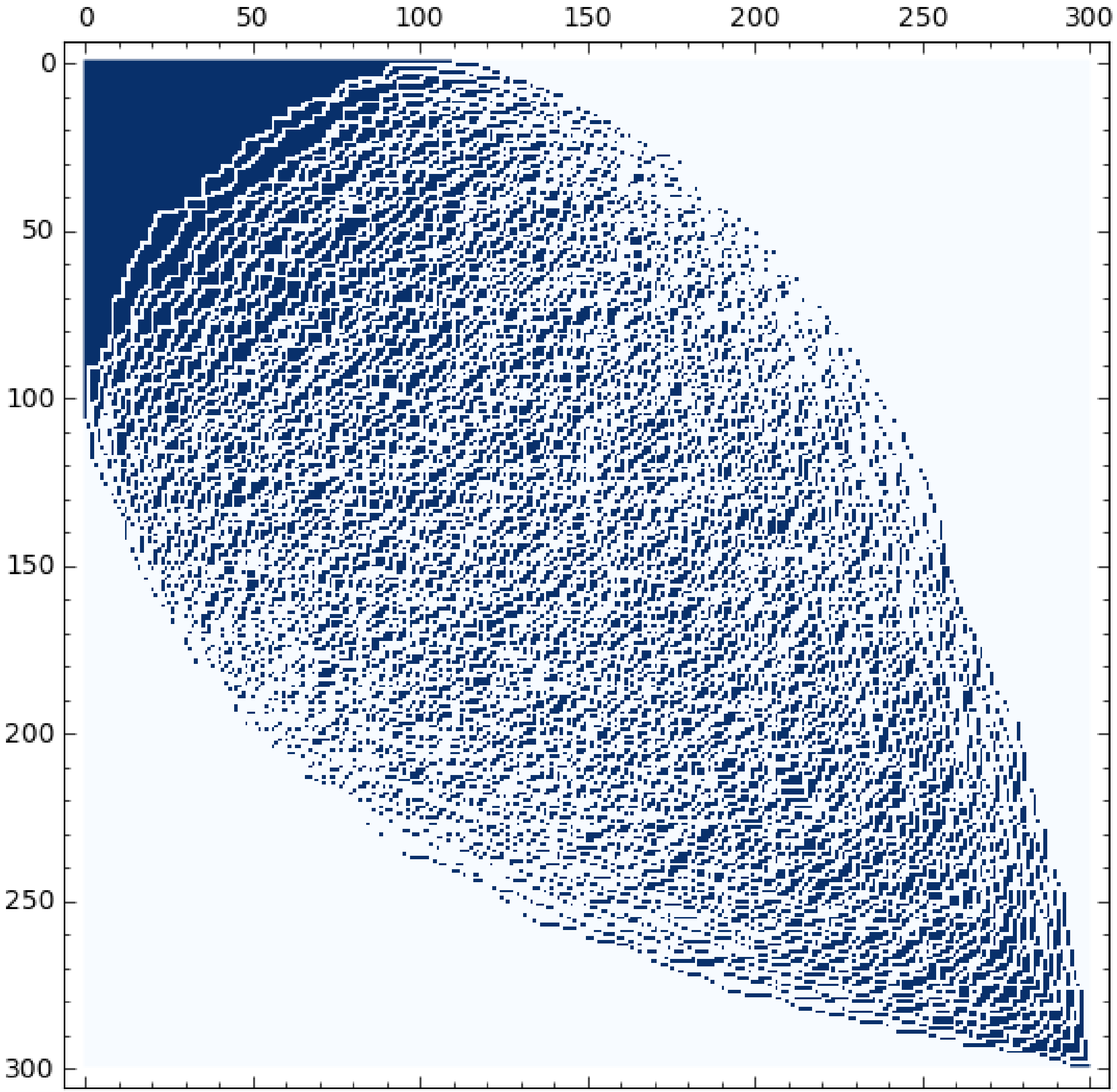

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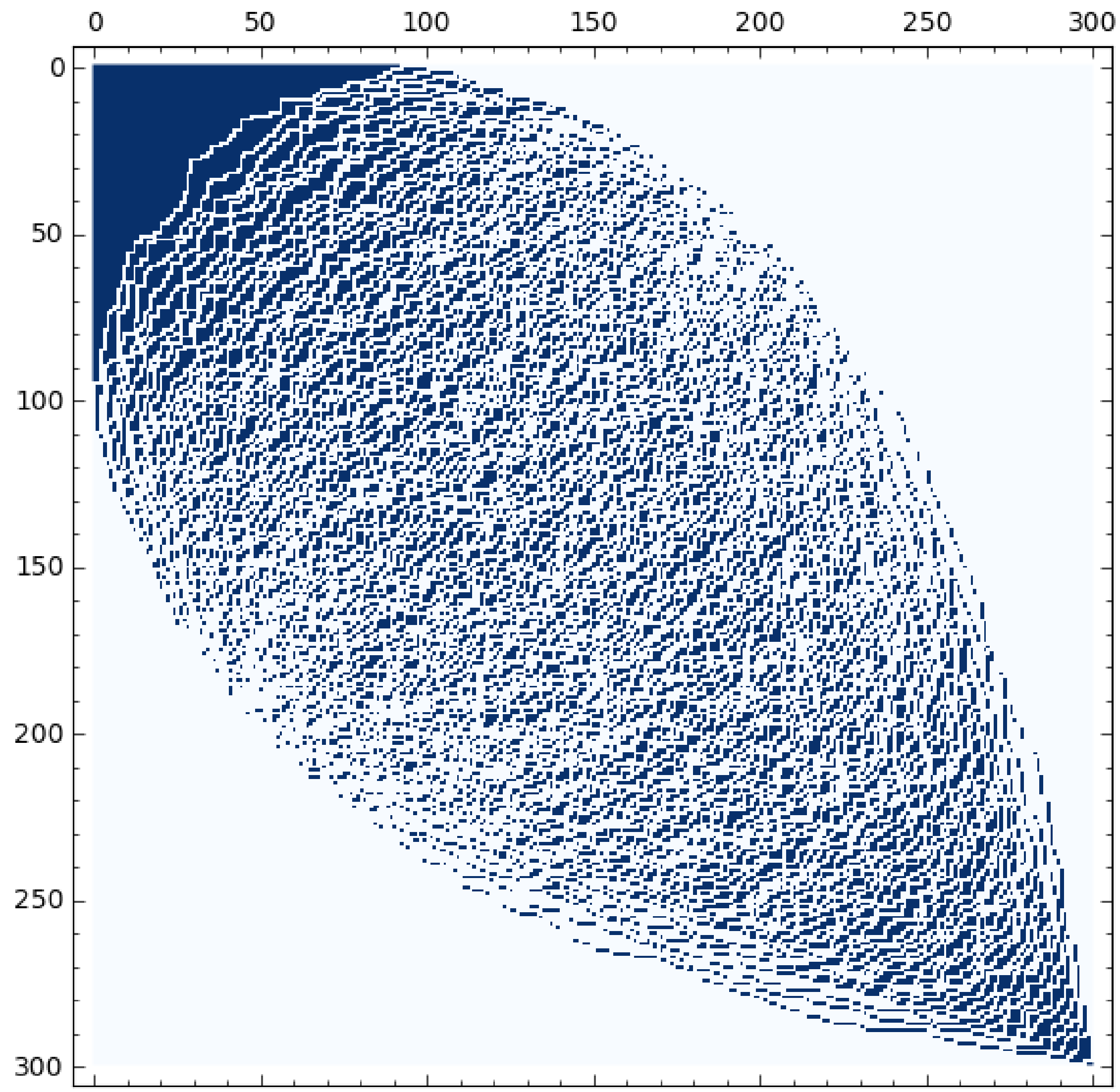

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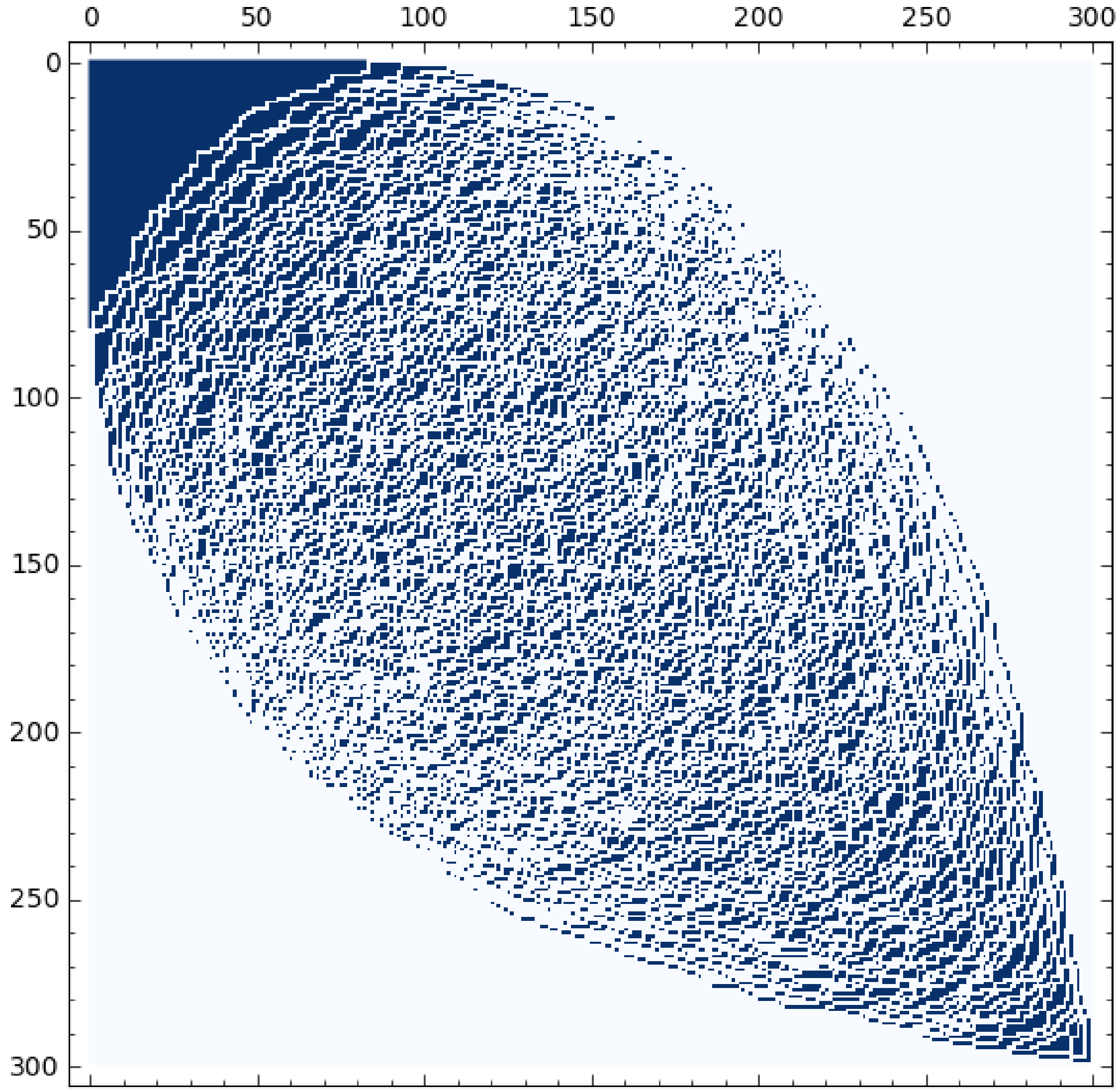

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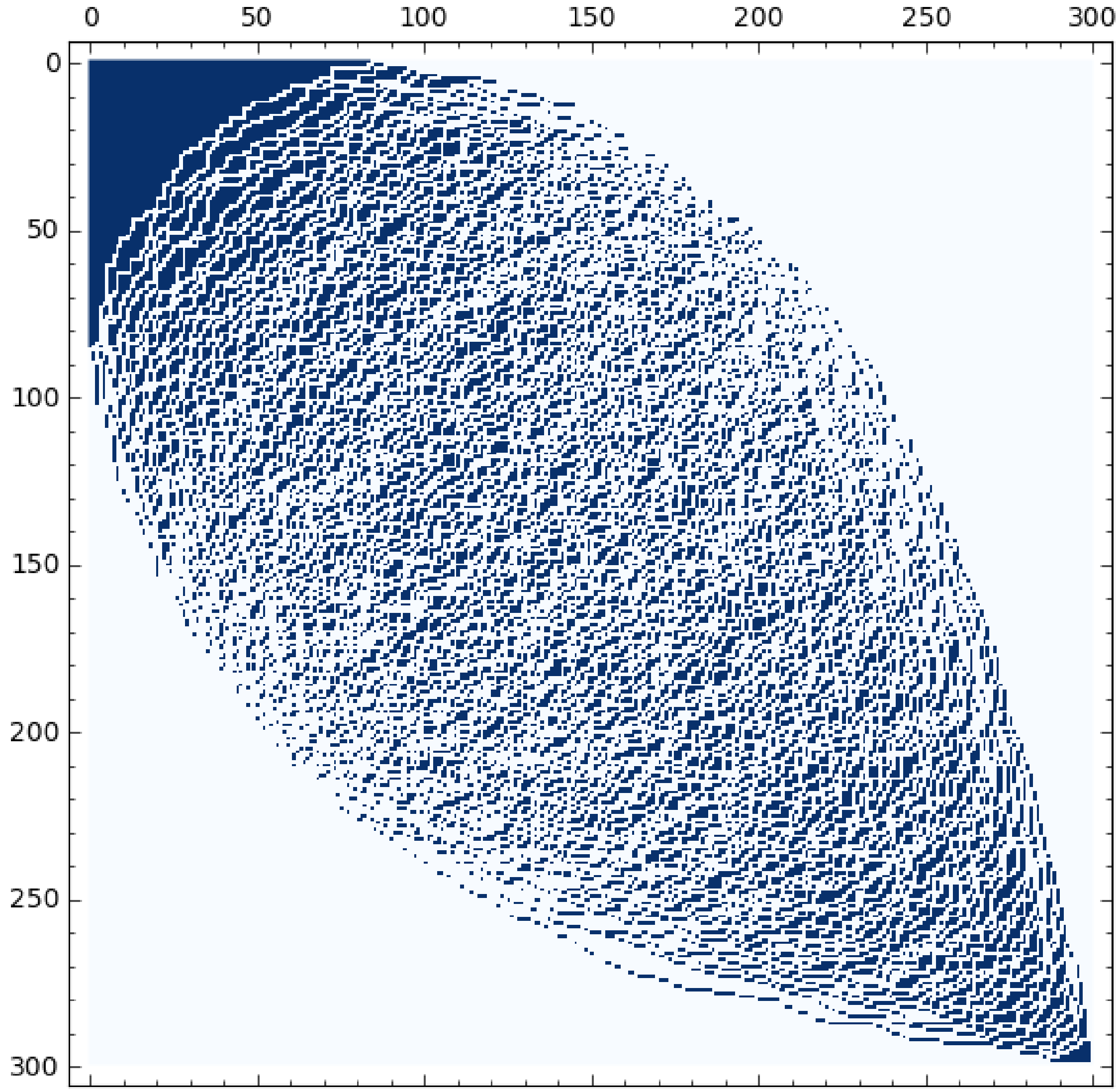

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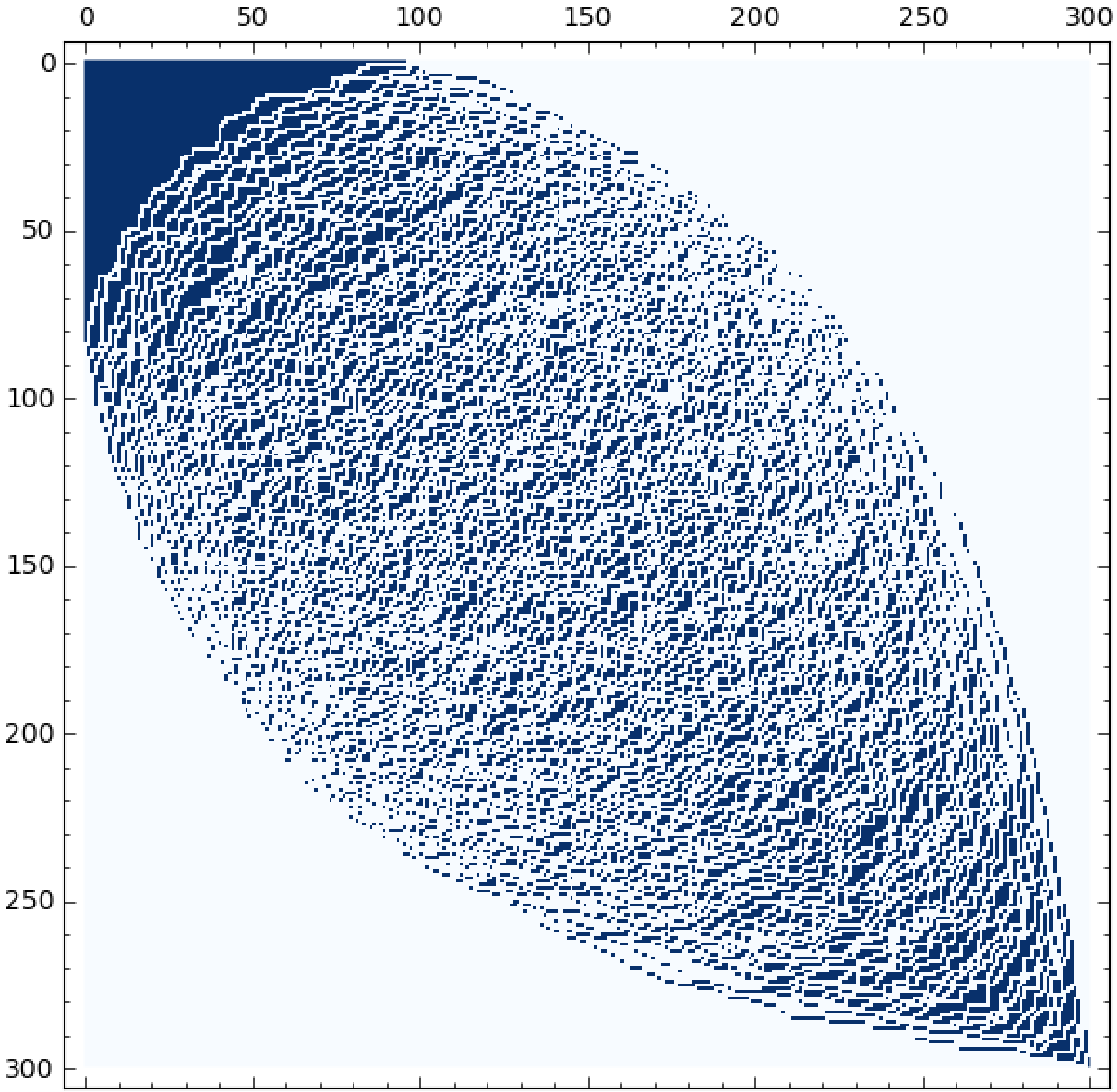

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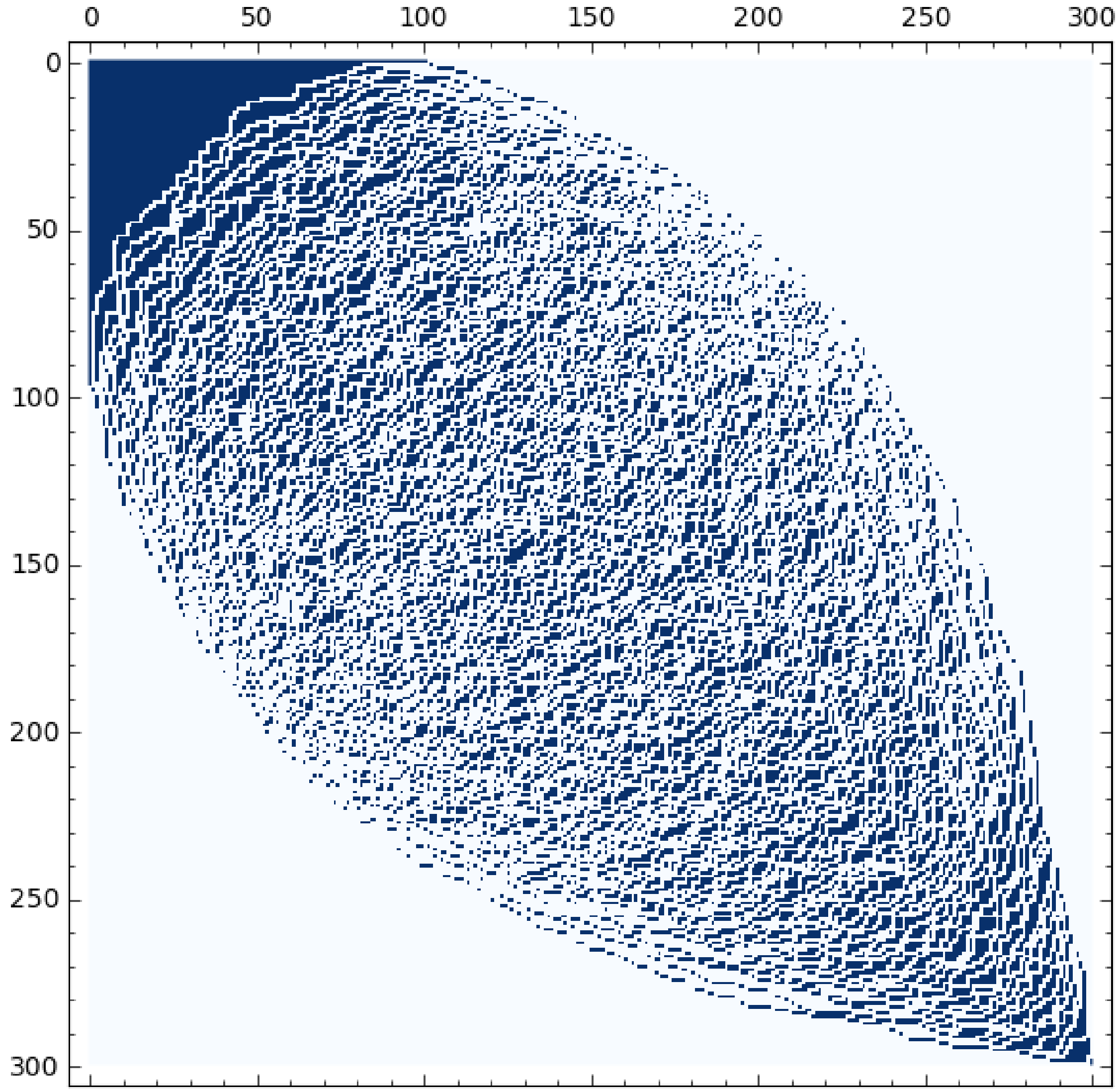

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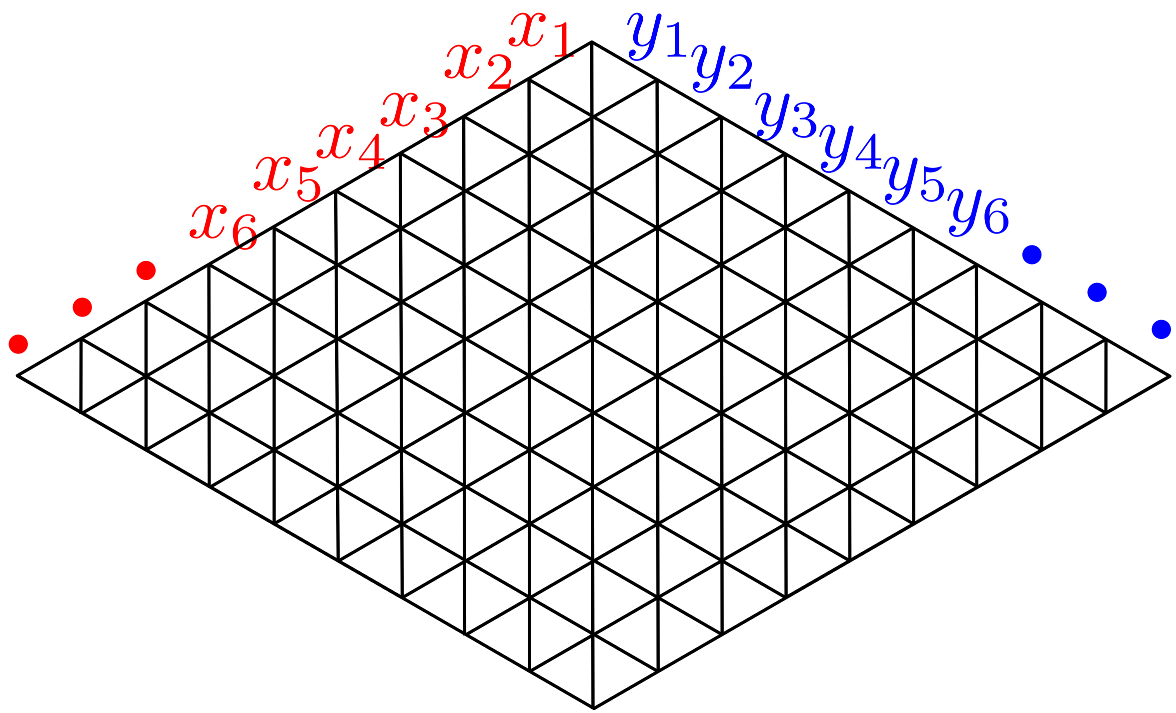
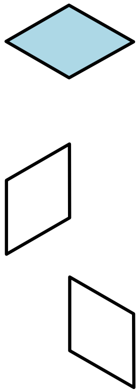
Relation to lozenge tilings

Excited diagrams can be seen as **lozenge tilings**:
There is a bijection between excited diagrams and certain lozenge tilings of a region with bottom bounded by μ.





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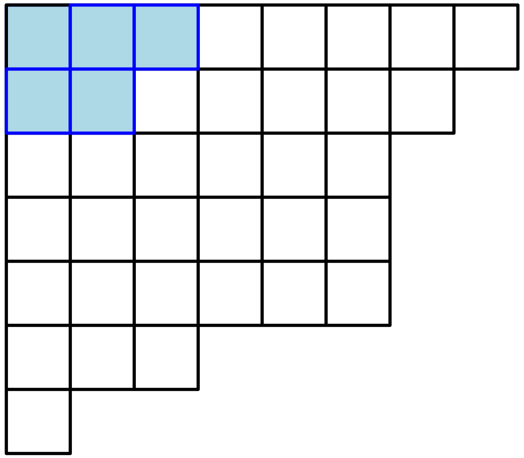

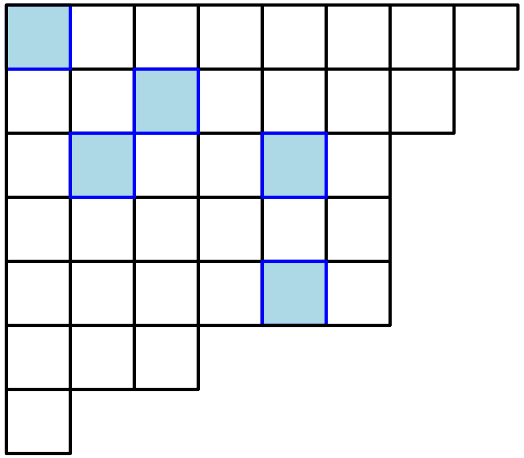

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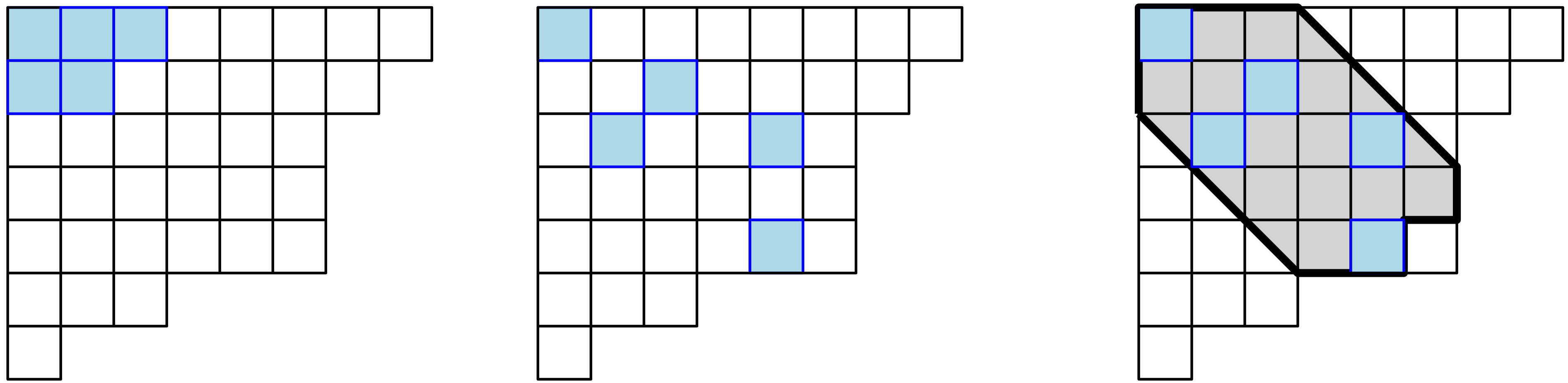

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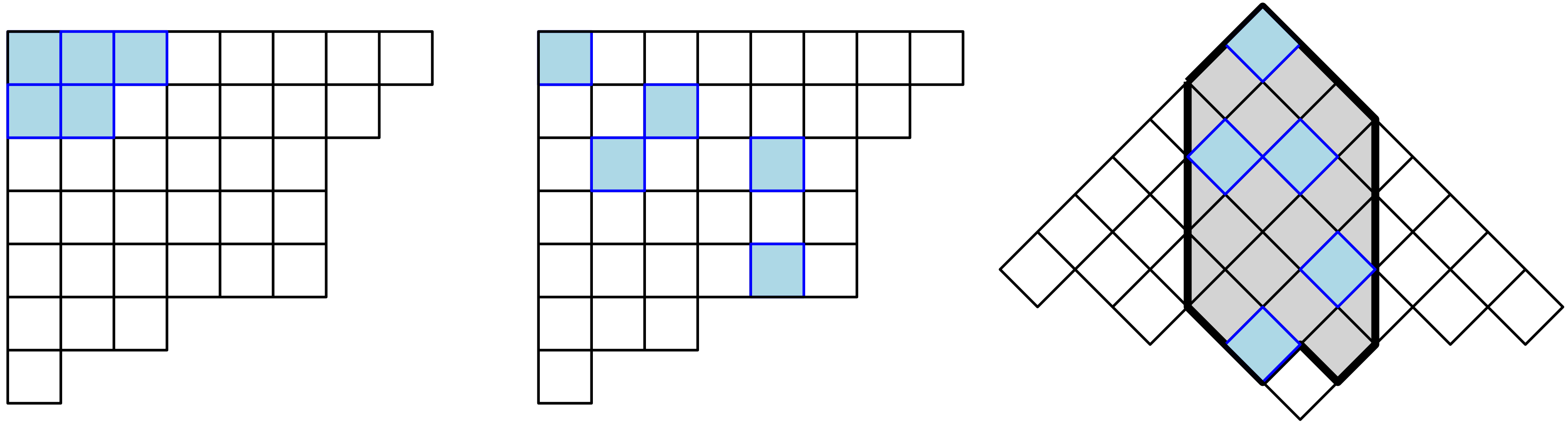

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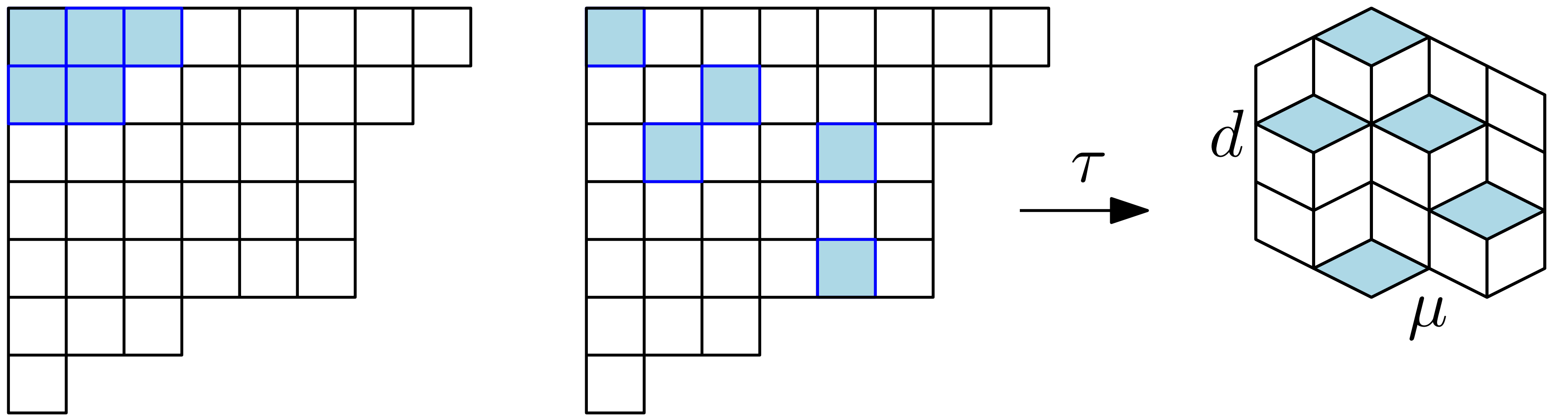

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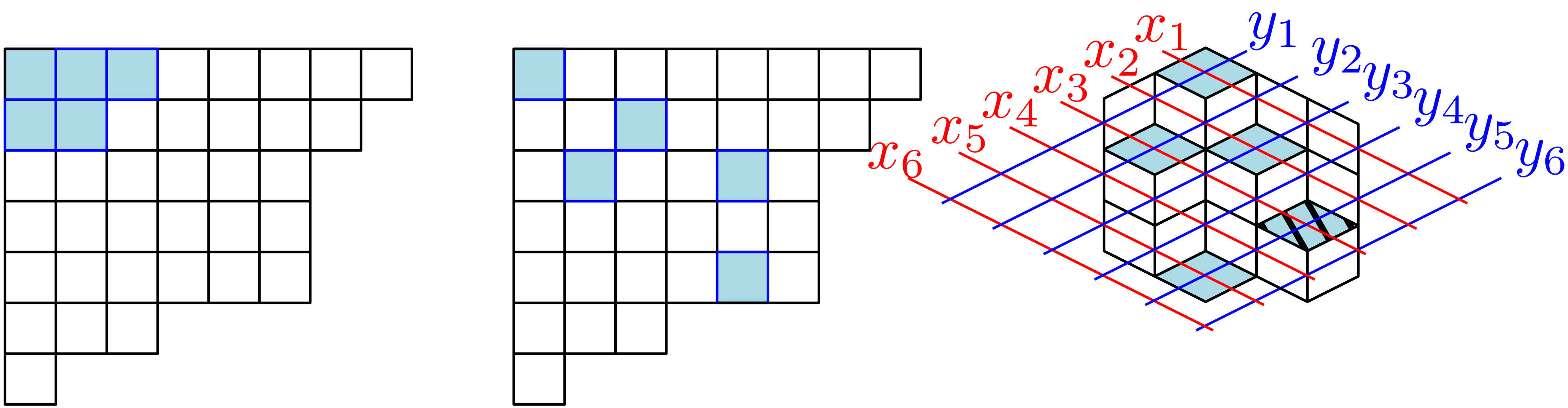
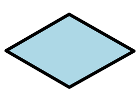
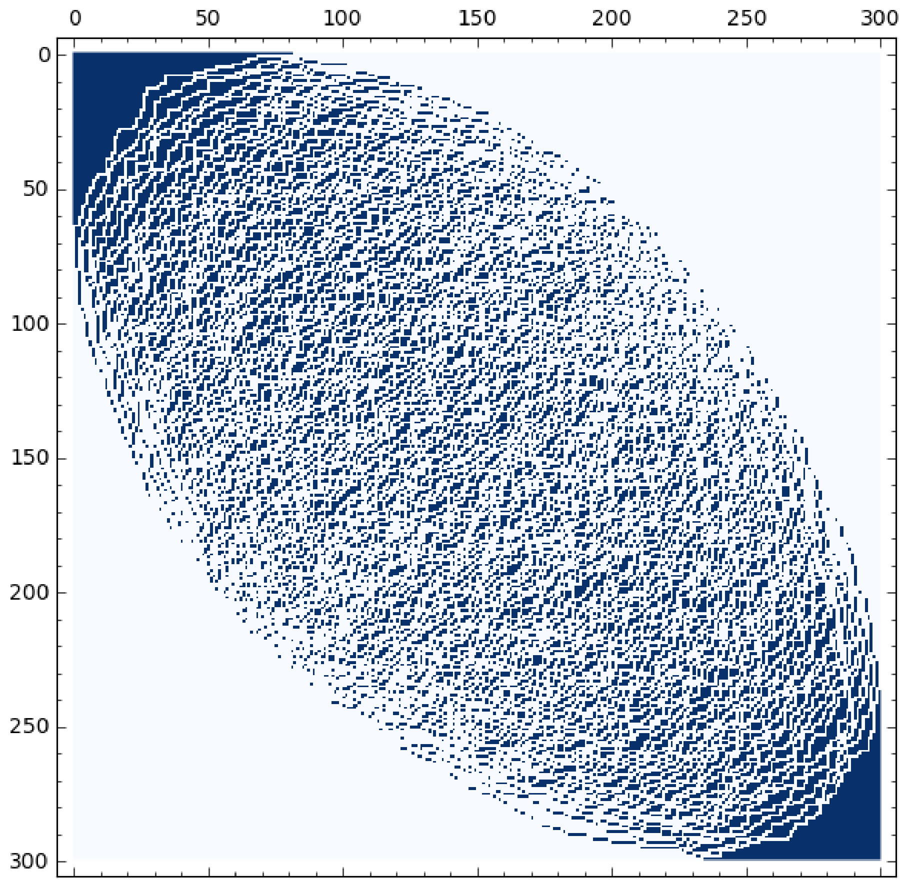


hook-length weight of excited diagrams is a weight on
horizontal lozenges 

Simulations revisited (no weight)



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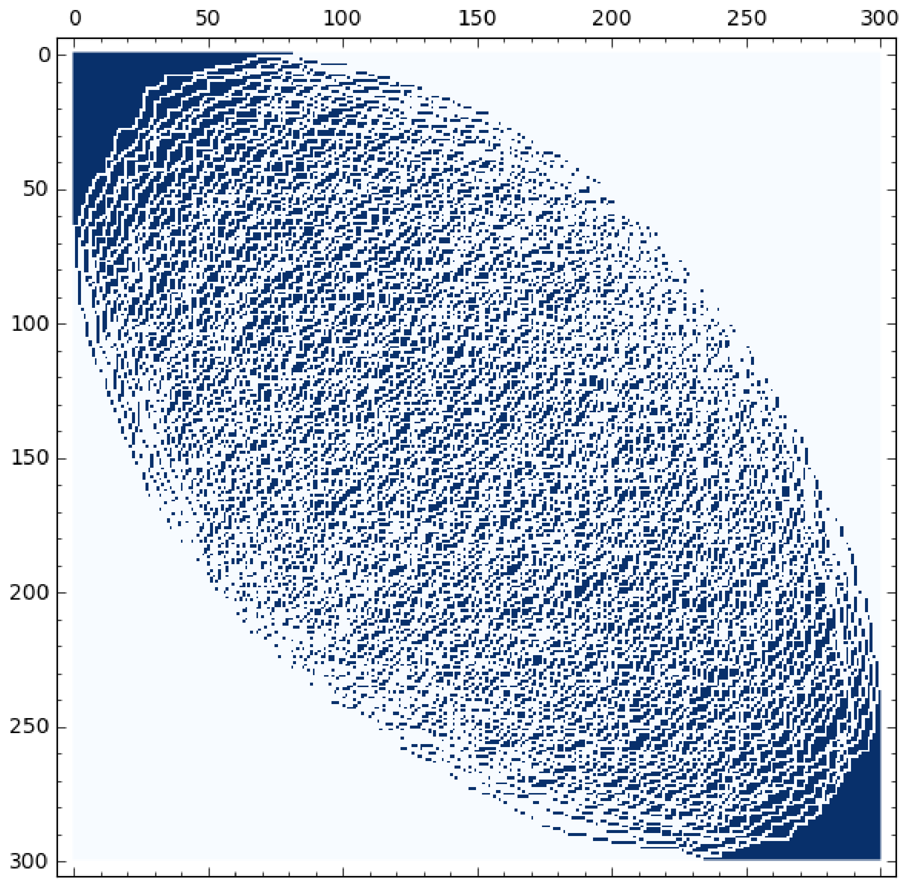

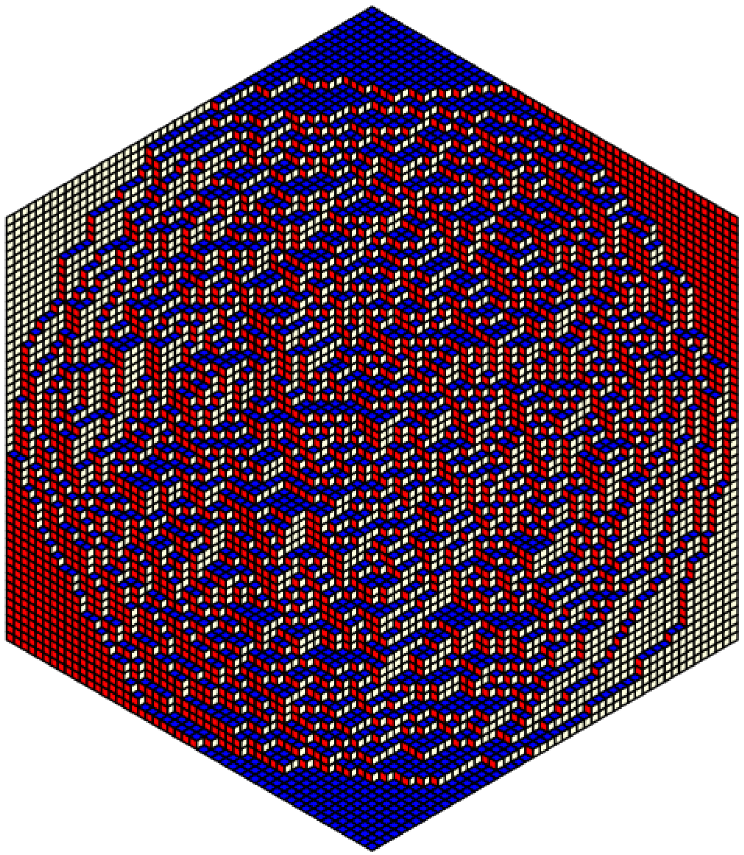

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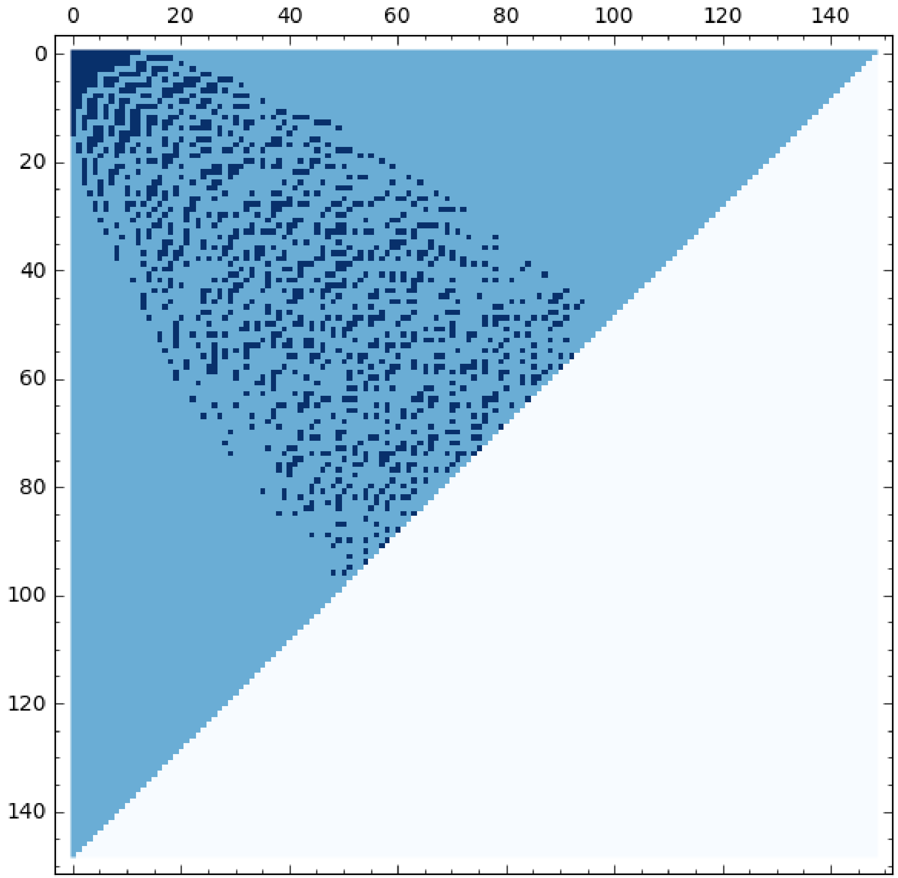
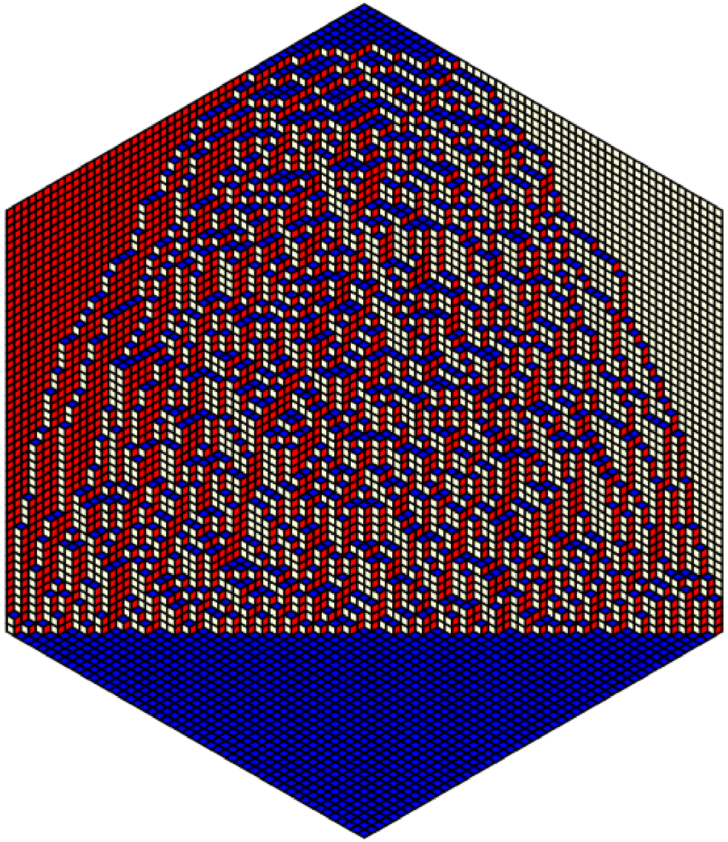
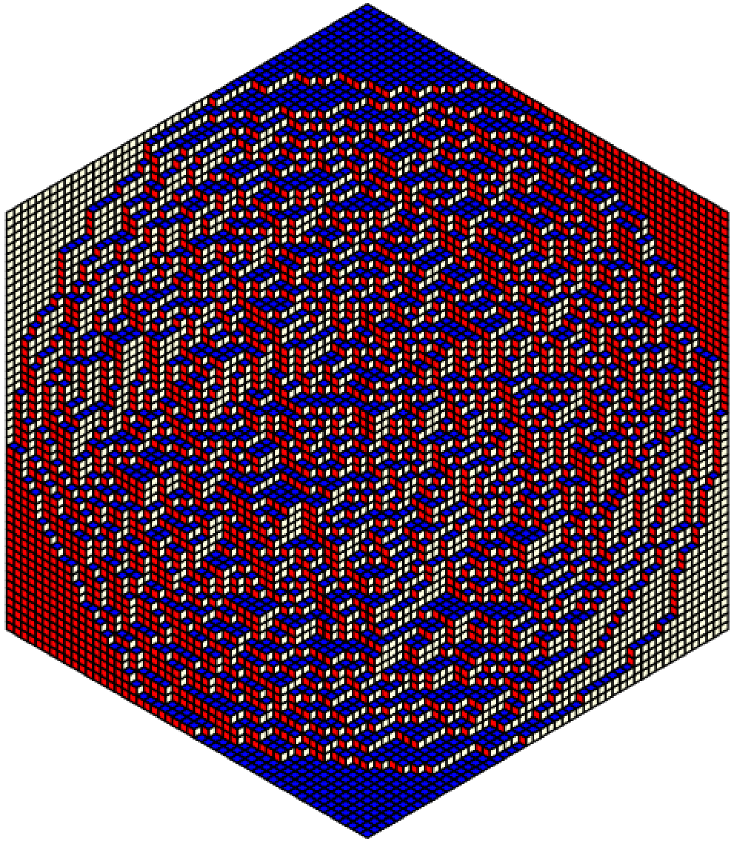
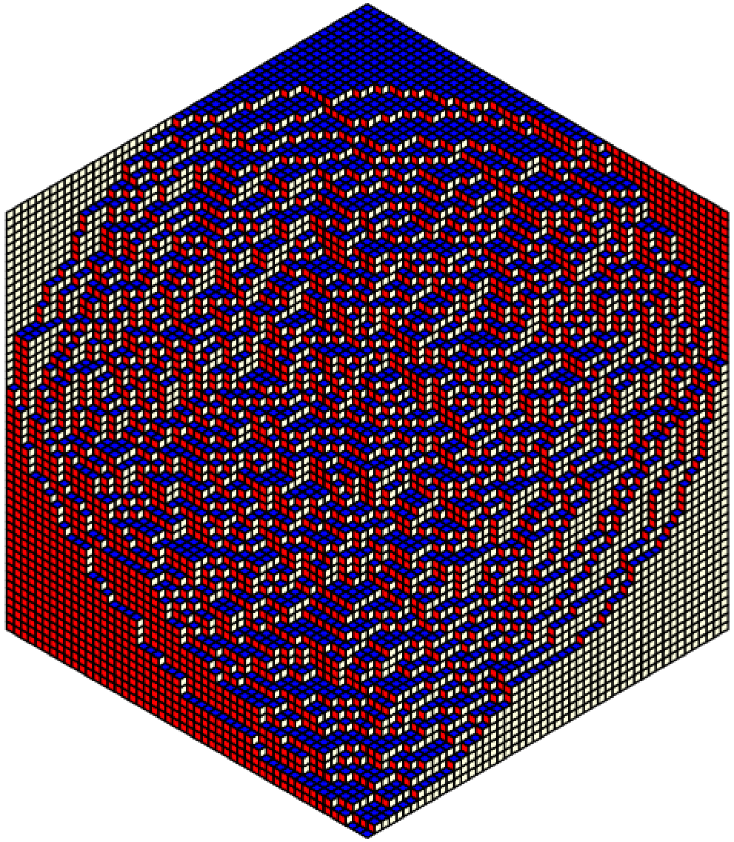




Simulations revisited: unweighted vs weighted





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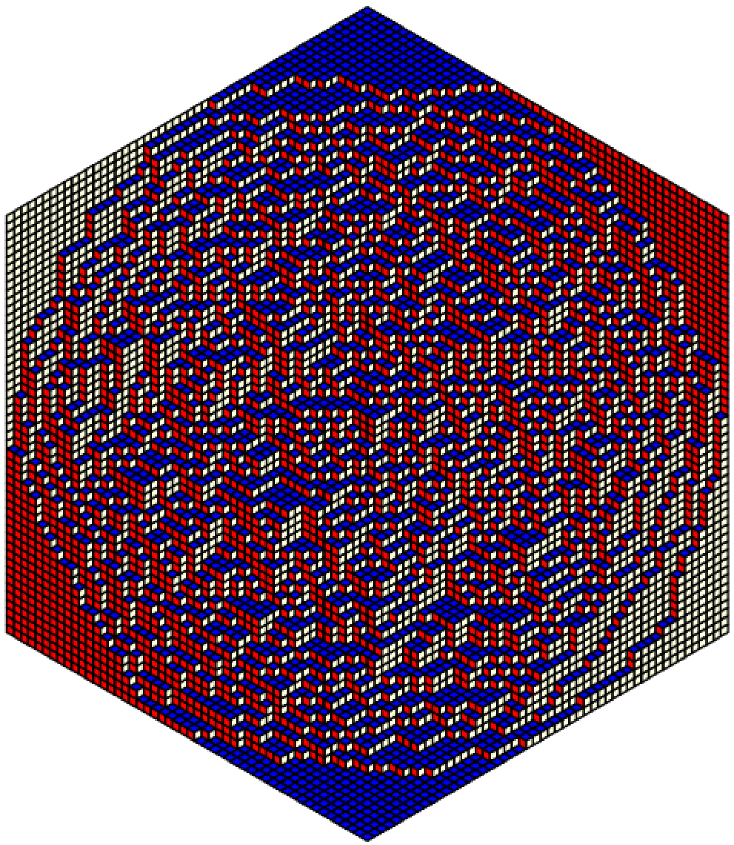
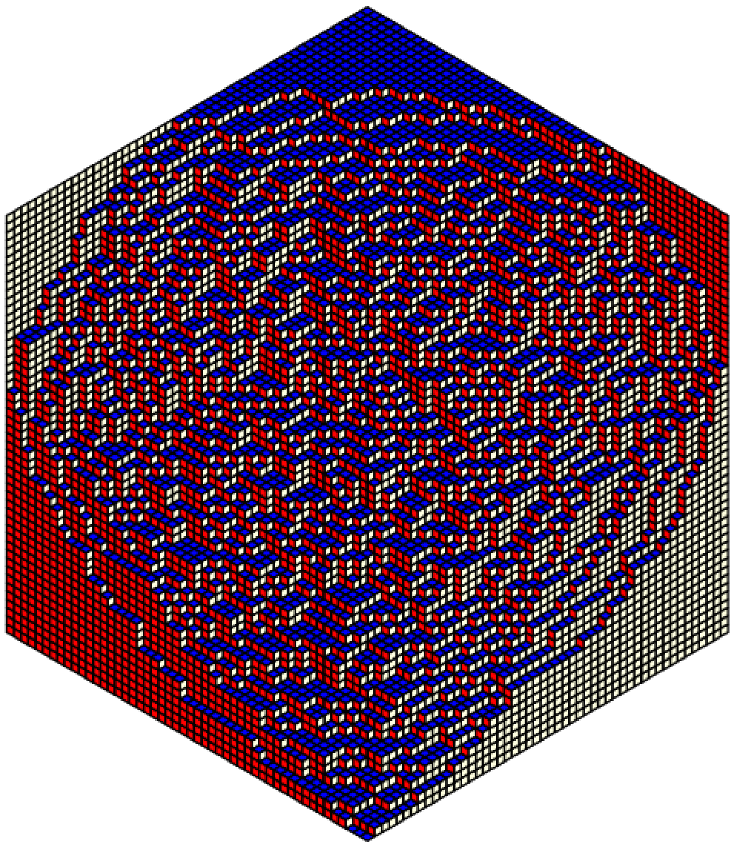
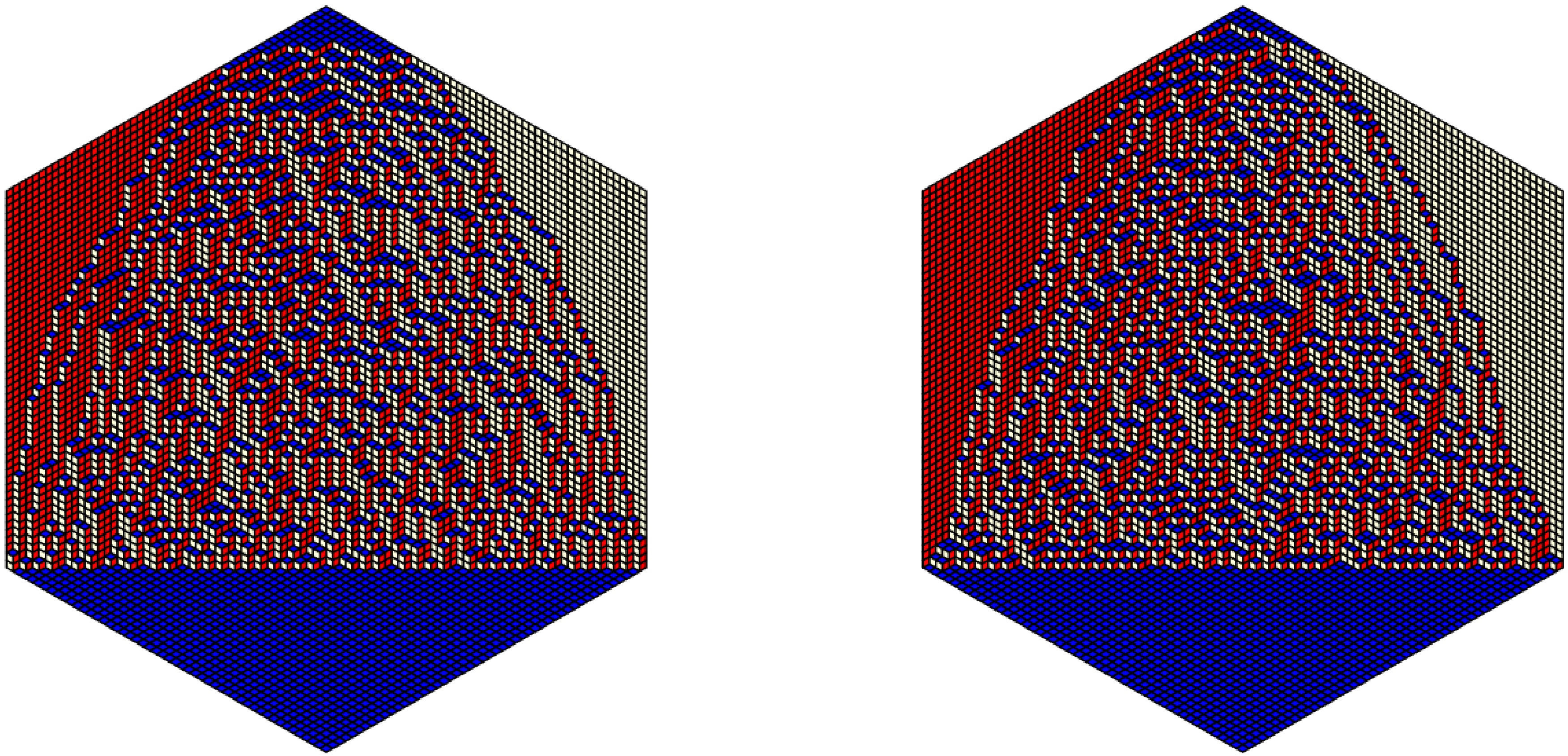




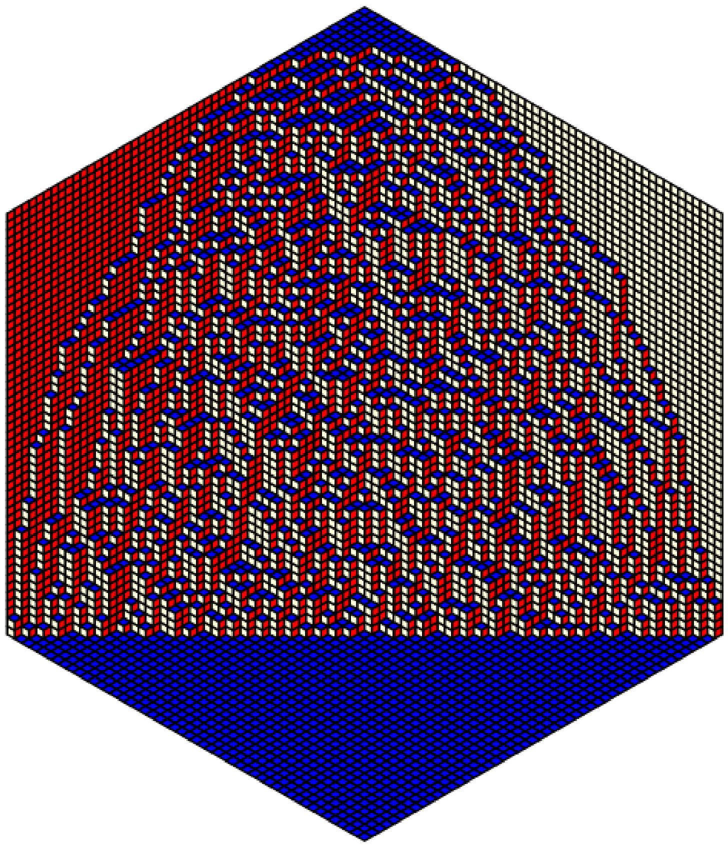
For the rectangle when tilings are weighted by hook-lengths, there is also limiting behavior (Borodin–Gorin–Rains 2010).

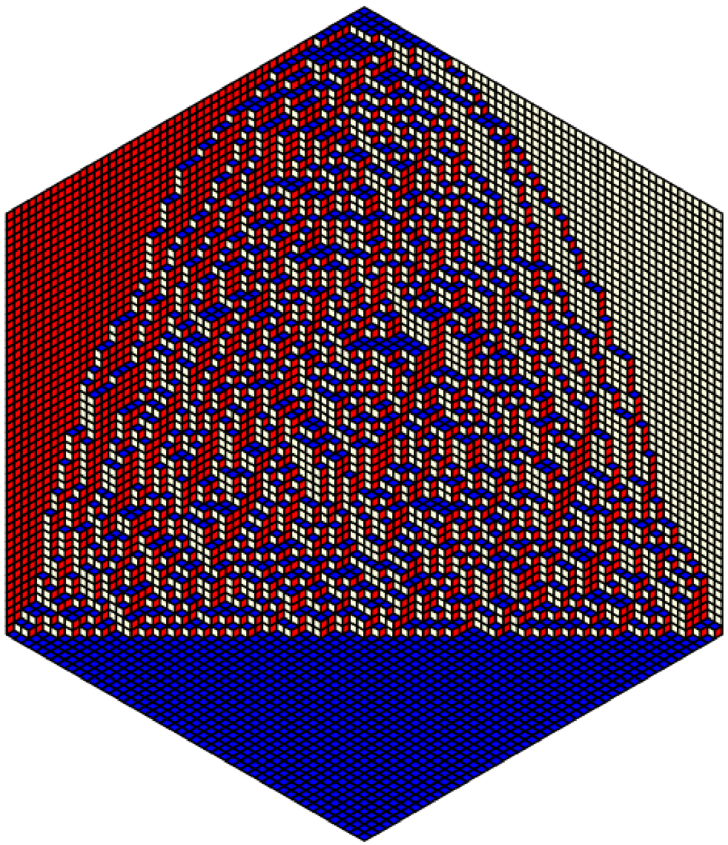
Simulations revisited: unweighted vs weighted



Simulations revisited: unweighted vs weighted

known: when tilings are chosen uniformly at random on domain and mesh size $\to 0$, there is limiting behavior.





Outline

$$f^\lambda = \frac{|\lambda|!}{\prod_{u\in\lambda} h(u)}$$

Naruse's formula for $f^{\lambda/\mu}$

- q-analogues
- Applications
  - relation to lozenge tilings ✓
  - bounds and asymptotics for $f^{\lambda/\mu}$
  - family of skew shapes with product formulas

Asymptotics for $f^{\lambda/\mu}$

$$f^{\lambda/\mu} = n! \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{(i,j) \in \lambda \setminus D} \frac{1}{h(i,j)},$$

Asymptotics for $f^{\lambda/\mu}$

$$f^{\lambda/\mu} = n! \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{(i,j) \in \lambda \setminus D} \frac{1}{h(i,j)},$$

define the *naive hook-length formula*

$$F(\lambda/\mu) := \frac{n!}{\prod_{(i,j)\in\lambda/\mu} h(i,j)}$$

Corollary (M-Pak-Panova 16)

$$F(\lambda/\mu) \leq f^{\lambda/\mu} \leq |\mathcal{E}(\lambda/\mu)| \cdot F(\lambda/\mu)$$

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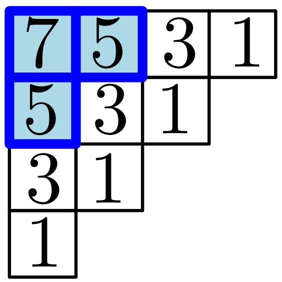
Proof

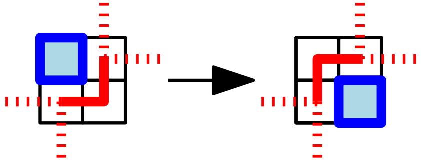
LB: μ is an excited diagram

UB: The diagram that contributes the most is $D = \mu$.



Excited diagrams and non-intersecting paths

Excited diagrams correspond to certain non-intersecting paths in λ (Kreiman 05)



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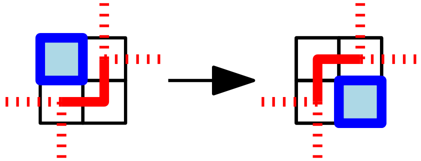

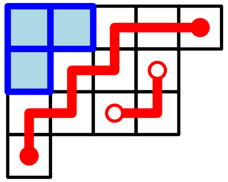

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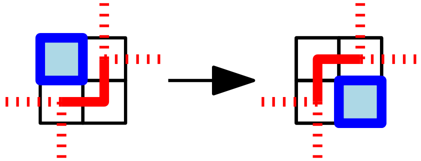

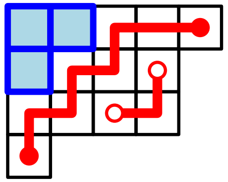

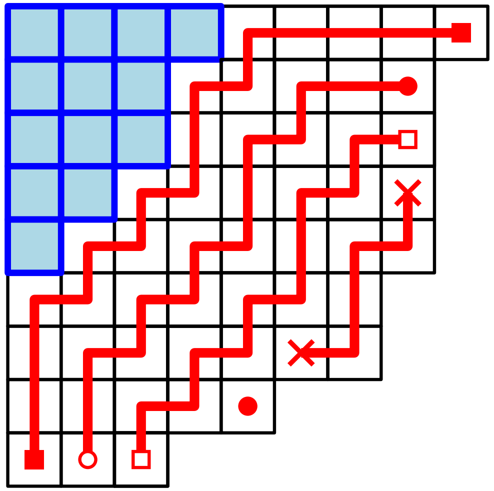

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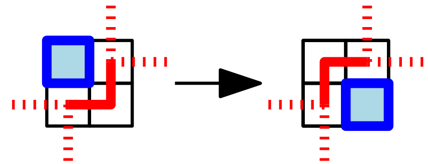

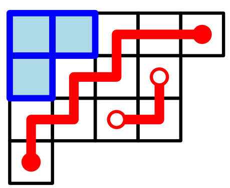

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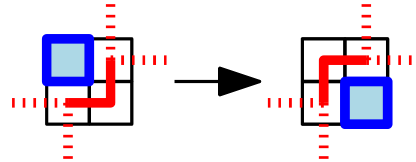
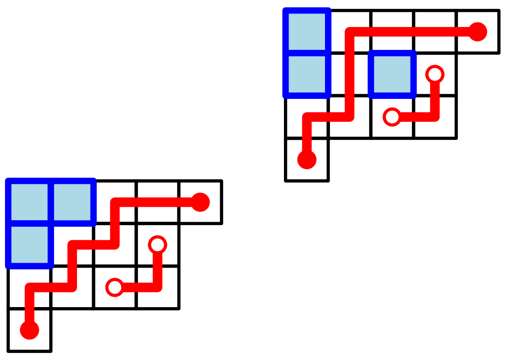
Excited diagrams correspond to certain non-intersecting paths in λ  (Kreiman 05)



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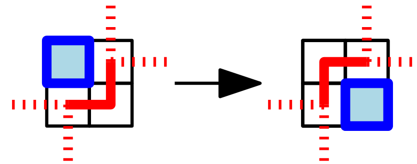

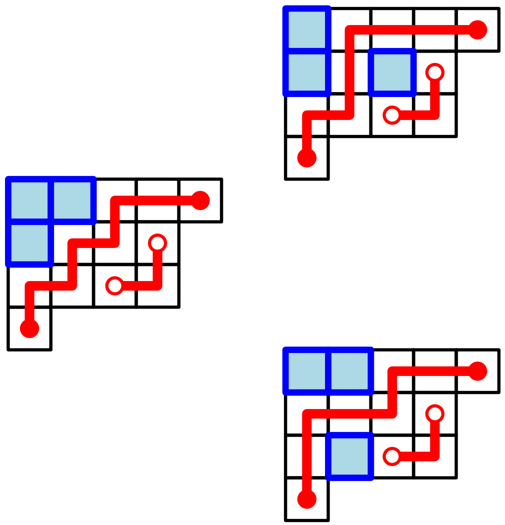

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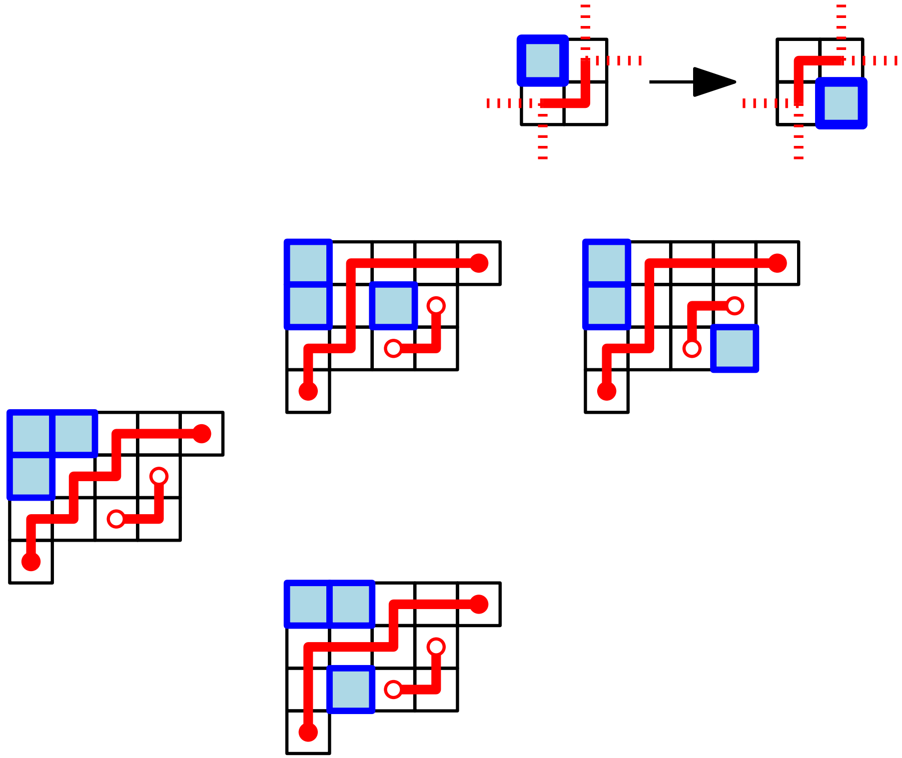

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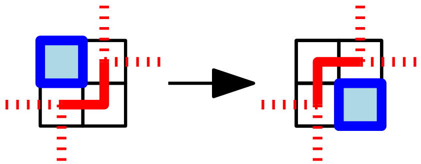

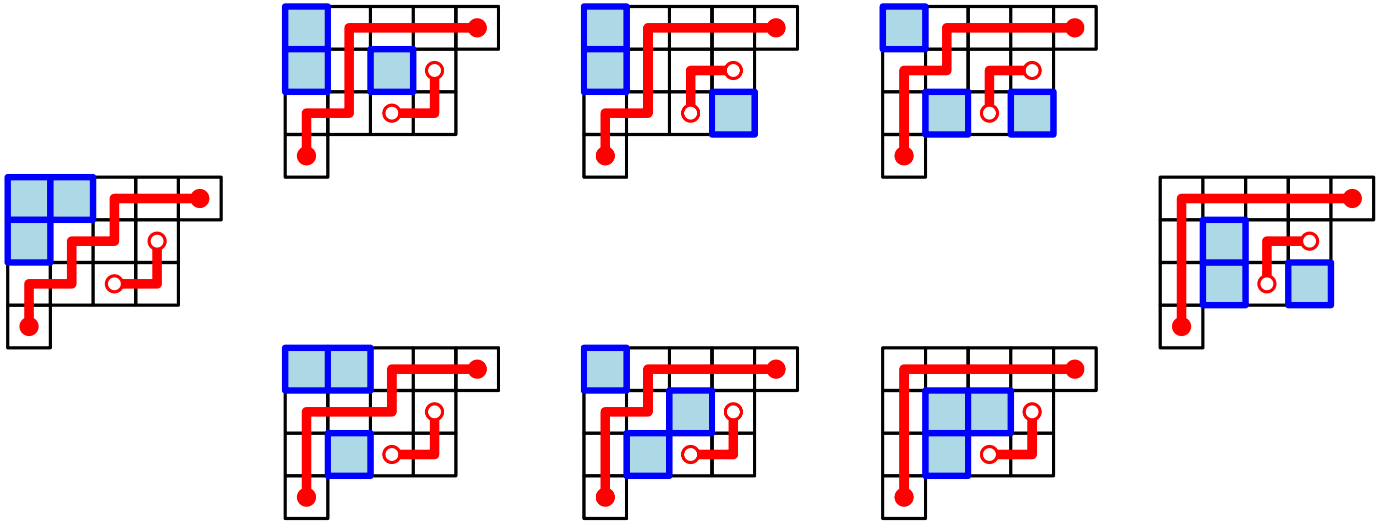

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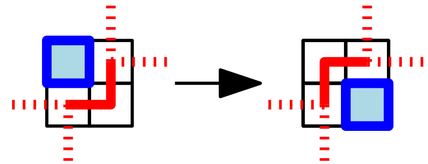
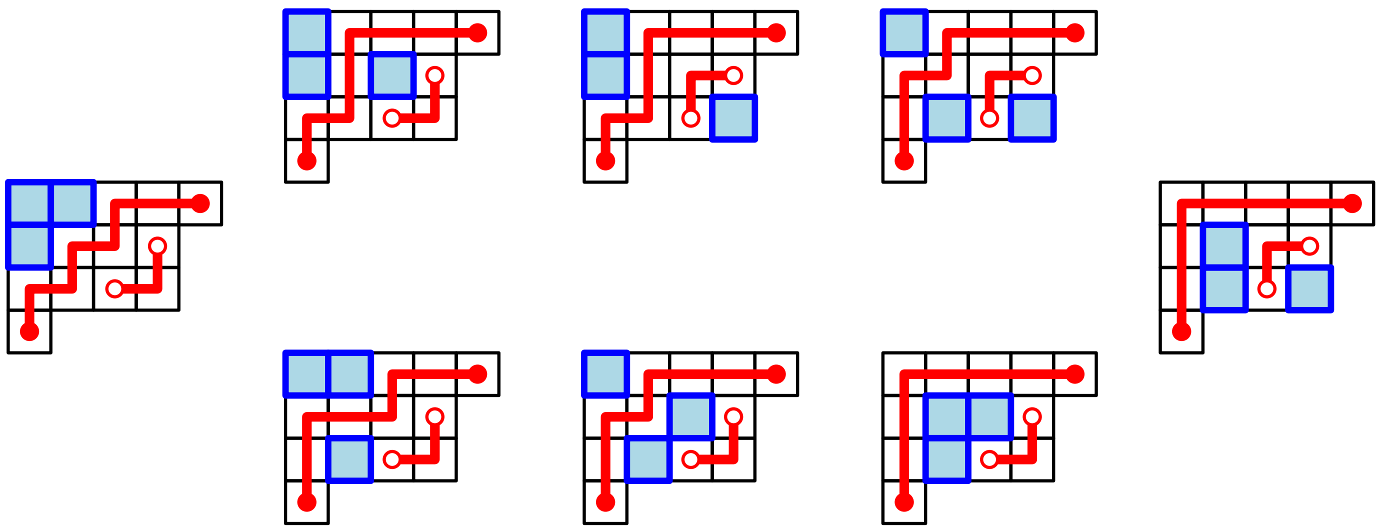


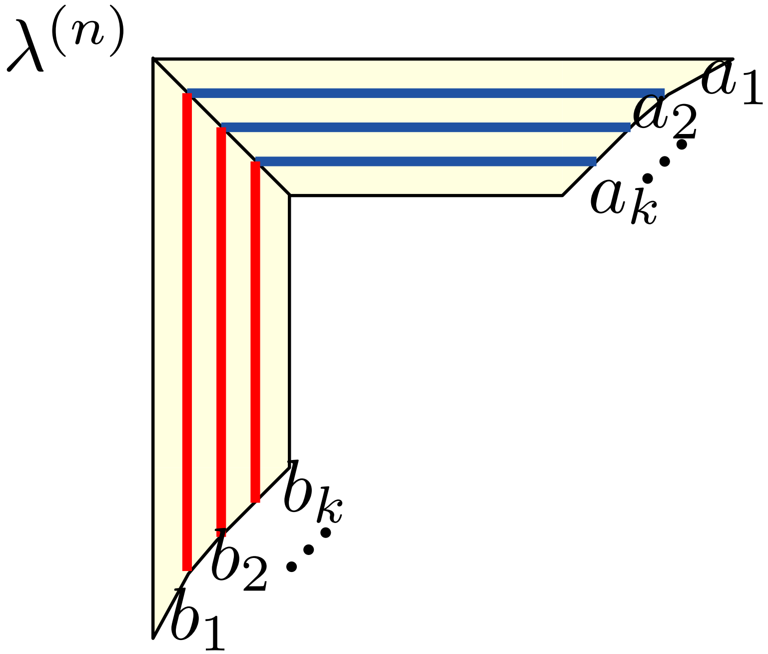


Bounds for $|\mathcal{E}(\lambda/\mu)|$:

1. If $n = |\lambda/\mu|$ then $|\mathcal{E}(\lambda/\mu)| \leq 2^n$
2. If s is the size of Durfee square of λ then $|\mathcal{E}(\lambda/\mu)| \leq n^{2d^2}$

Asymptotics for $f^{\lambda/\mu}$: Thoma-Vershik-Kerov limit shapes

Fix $k \geq 1$, $\lambda^{(n)}$ has **Thoma-Vershik-Kerov (TVK)** limit if its *Frobenius coordinates* scale linearly $a_i/n \to \alpha_i$, $b_i/n \to \beta_i$ for $i = 1, \ldots, k$



$|\lambda^{(n)}| \sim \gamma n$

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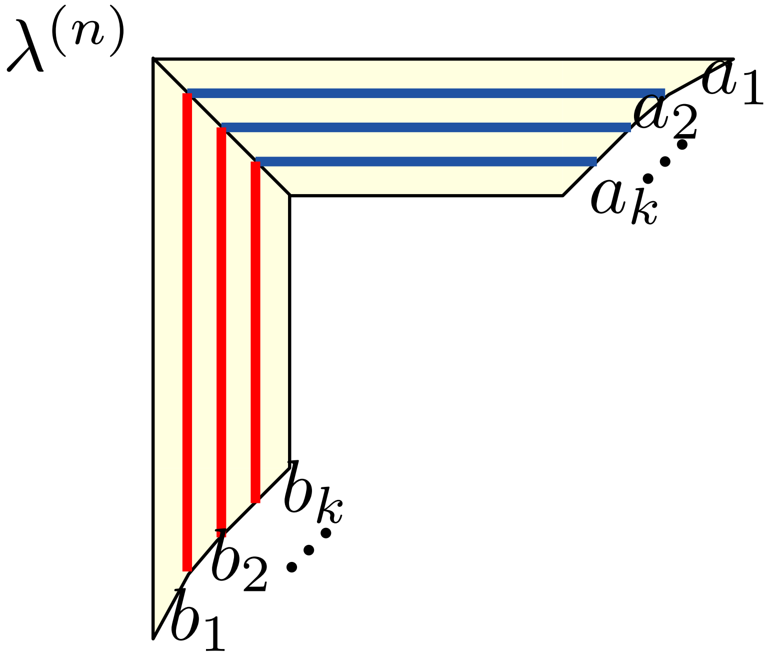

$$|\lambda^{(n)}| \sim \gamma n$$

Theorem (M-Panova-Pak 2017)

If $\theta_n = \lambda^{(n)}/\mu^{(n)}$ is a sequence with $\lambda^{(n)}, \mu^{(n)}$ having TVK limit $(\alpha, \beta), (\pi, \phi)$ then

$$\log f^{\theta_n} = cn + o(n).$$

Asymptotics for $f^{\lambda/\mu}$: sub-polynomial depth shapes

Let $g(n) = n^{o(1)}$ be a sub-polynomial function. Sequence $\{\theta_n\} = \{\lambda^{(n)}/\mu^{(n)}\}$ have **sub-polynomial shape** if $\max(\lambda_1^{(n)}, \lambda_1^{(n)'}) = \Theta(n/g(n))$ and $\max_{u\in\theta_n} h(u) = \Theta(g(n))$.

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Theorem (M-Panova-Pak 2017)
If $\theta_n = \lambda^{(n)}/\mu^{(n)}$ is a sequence of sub-polynomial shapes then

$$\log f^{\theta_n} = n \log n + \Theta(n \log g(n)).$$

Asymptotics for $f^{\lambda/\mu}$: balanced shapes

λ of size n has **balanced shape** if $\lambda_1, \ell(\lambda) \leq s\sqrt{n}$.

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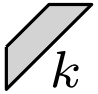
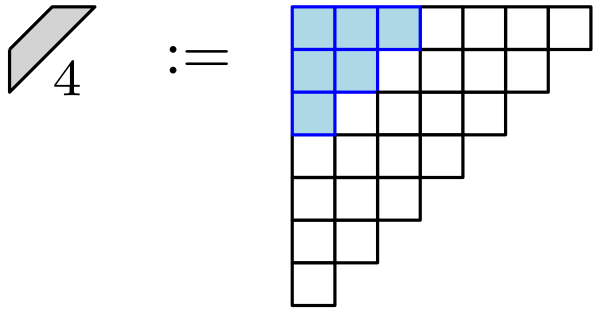
Theorem (Pak 2017, M-Panova-Pak 2017)

For $\{\theta_n\} = \{\lambda^{(n)}/\mu^{(n)}\}$ as above, then there exist constants c_1, c_2

$$c_1 \leq \frac{1}{n}\left(\log f^{\theta_n} - \frac{1}{2} n \log n\right) \leq c_2.$$

Asymptotics for $f^{\lambda/\mu}$: benchmark balanced shapes

Let  be shape $(2k-1, 2k-2, \ldots, 1)/(k-1, k-2, \ldots, 1)$



$n = k(3k-1)/2$

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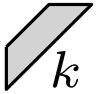
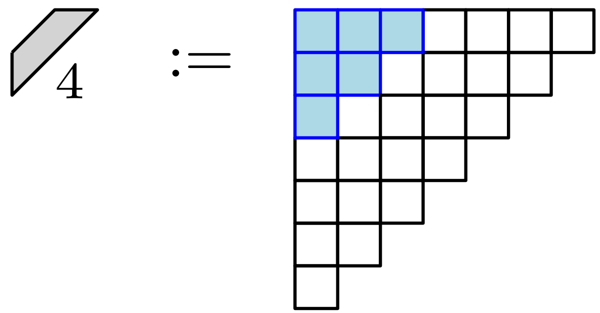
Let  be shape $(2k-1, 2k-2, \ldots, 1)/(k-1, k-2, \ldots, 1)$



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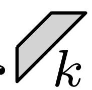
Example (M., Pak, Panova 16)

$$-0.3237 \le \frac{1}{n}\left(\log f^{\,\text{[shape]}_k} - \frac{1}{2} n \log n\right) \le -0.0621$$



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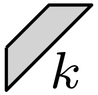
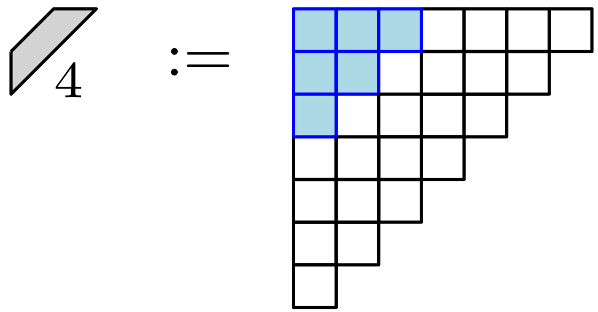
Let  be shape $(2k-1, 2k-2, \ldots, 1)/(k-1, k-2, \ldots, 1)$



$n = k(3k-1)/2$

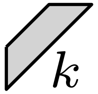
Example (M., Pak, Panova 16)

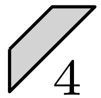
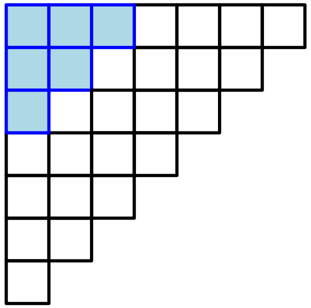
$$-0.3237 \leq \frac{1}{n}\left(\log f^{\diagup_k} - \frac{1}{2} n \log n\right) \leq -0.0621$$

Conjecture:

$$\frac{1}{n}\left(\log f^{\diagup_k} - \frac{1}{2} n \log n\right) \to c$$

Asymptotics for $f^{\lambda/\mu}$: benchmark balanced shapes

Let  be shape $(2k-1, 2k-2, \ldots, 1)/(k-1, k-2, \ldots, 1)$

 $:=$ 

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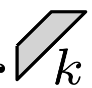

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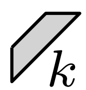

Jay Pantone estimated $c \approx -0.1842$

Asymptotics for $f^{\lambda/\mu}$: balanced shapes

Theorem (Pak 2017, M-Panova-Pak 2017)

Let $\{\theta_n\} = \{\lambda^{(n)}/\mu^{(n)}\}$ be sequence of shapes where $\lambda^{(n)} \to \psi, \mu^{(n)} \to \phi$ scaling by $1/\sqrt{n}$ both directions, then

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Conjecture

For $\{\theta_n\} = \{\lambda^{(n)}/\mu^{(n)}\}$ as above then

$$\tfrac{1}{n}\left(\log f^{\theta_n} - \tfrac{1}{2} n \log n\right) \to c(\psi/\phi).$$

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~~Conjecture~~ Theorem (M-Pak-Tassy 2018)

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- use (weighted) variational principle of Kenyon 09

Outline

$$f^\lambda = \frac{|\lambda|!}{\prod_{u\in\lambda} h(u)}$$

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- relation to lozenge tilings ✓
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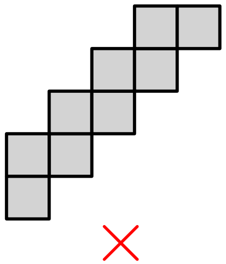
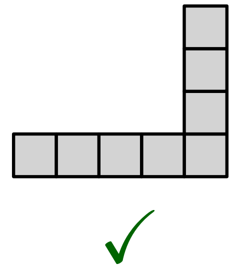
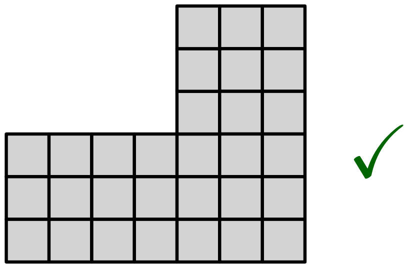
Shapes with product formulas for $f^{\lambda/\mu}$







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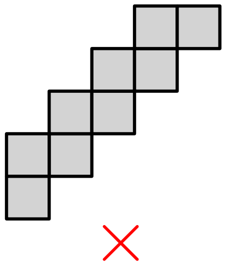

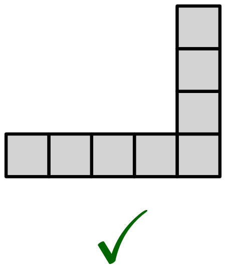

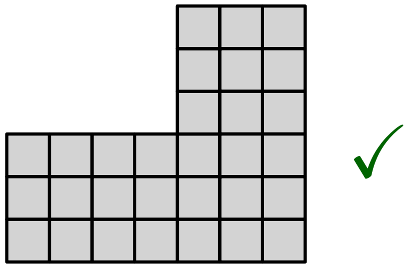

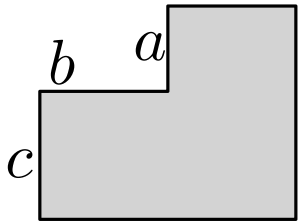

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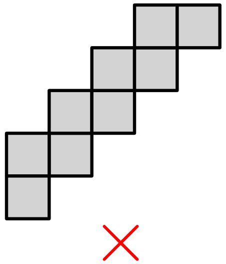

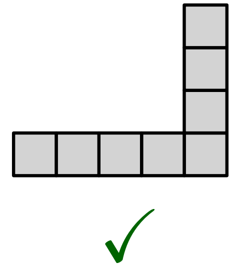

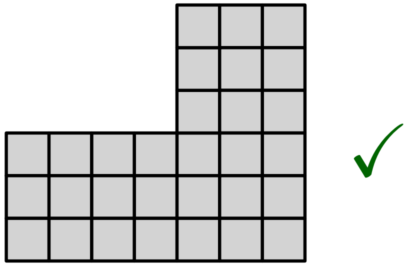

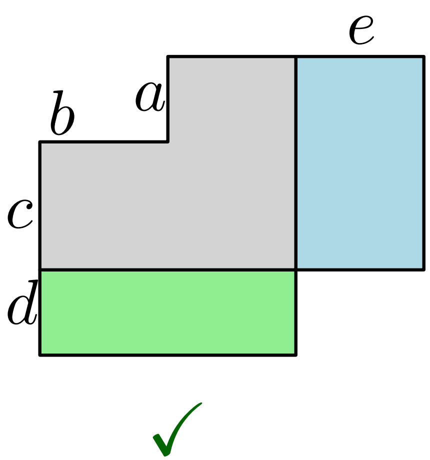

Oh-Kim 2014

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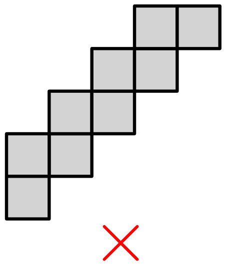

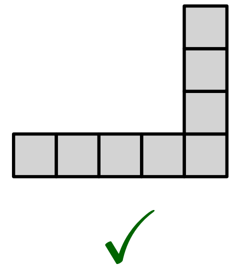

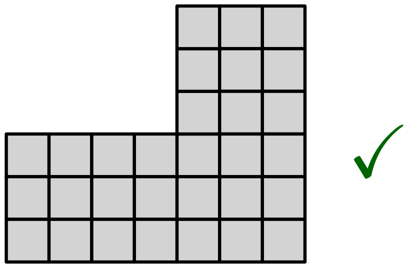

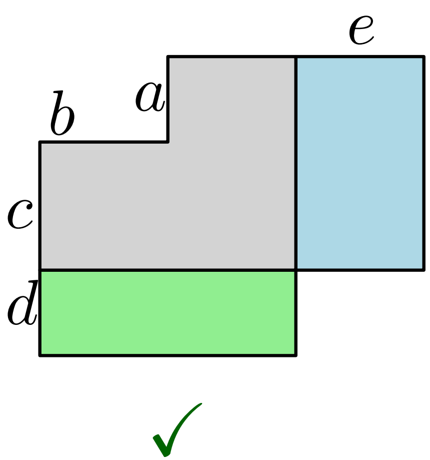

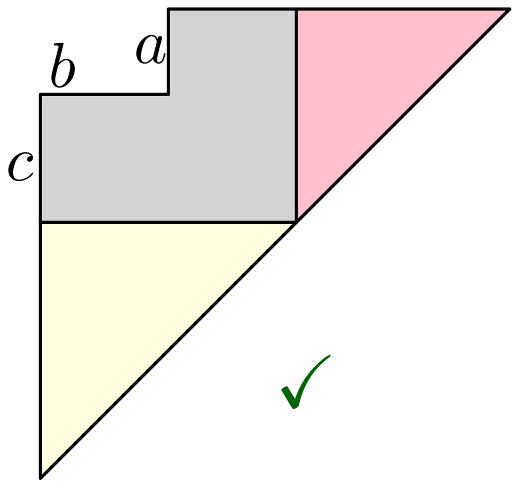

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DeWitt 2012

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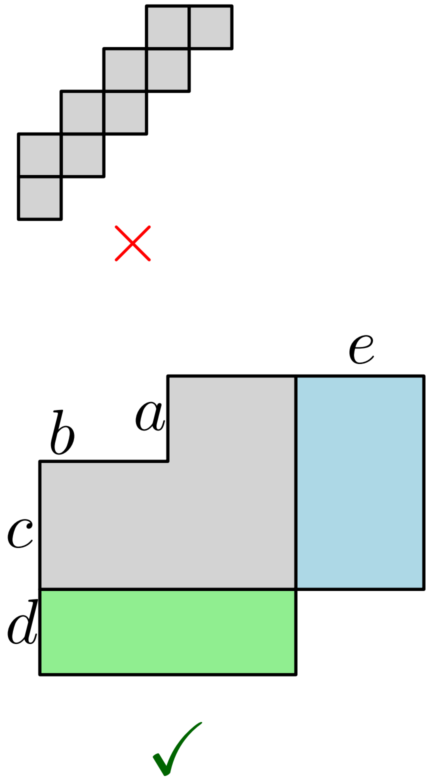

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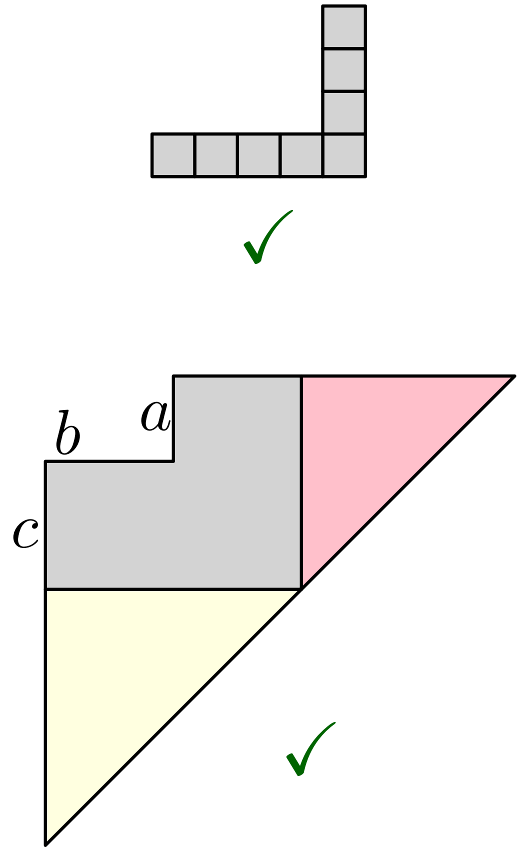

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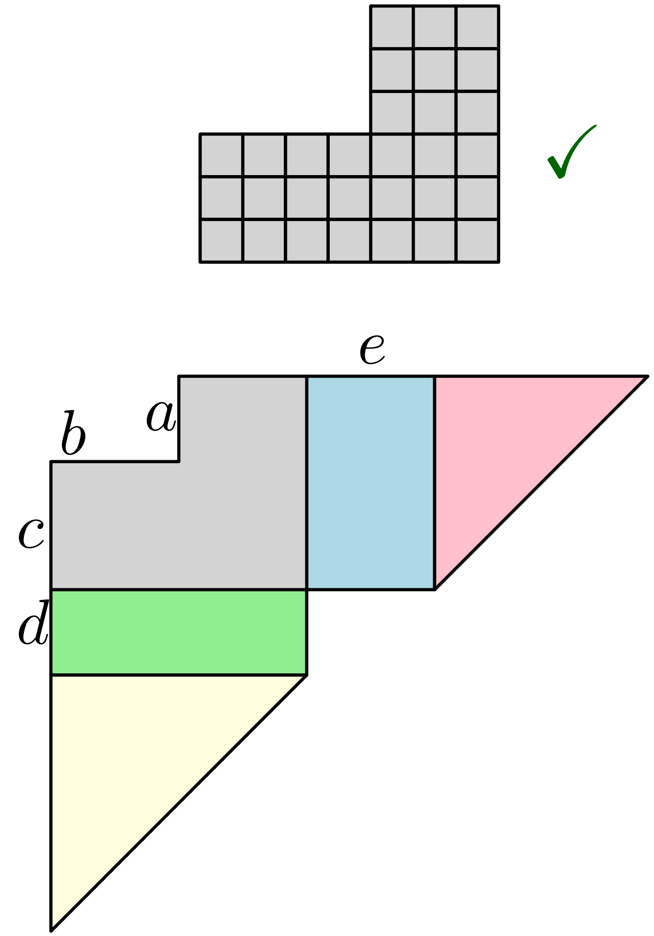

M-Pak-Panova 2017

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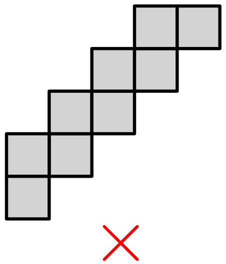
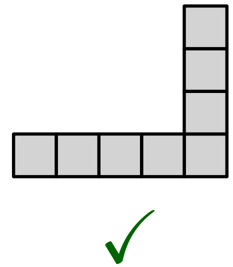
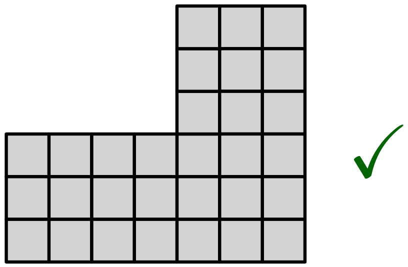

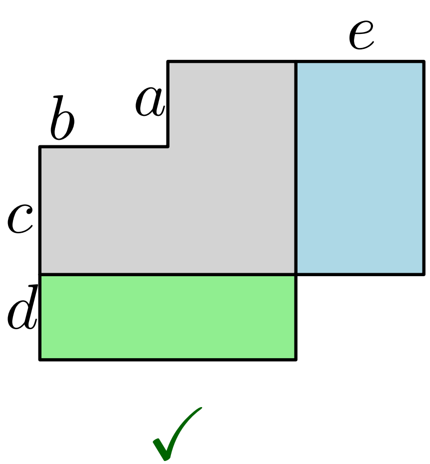

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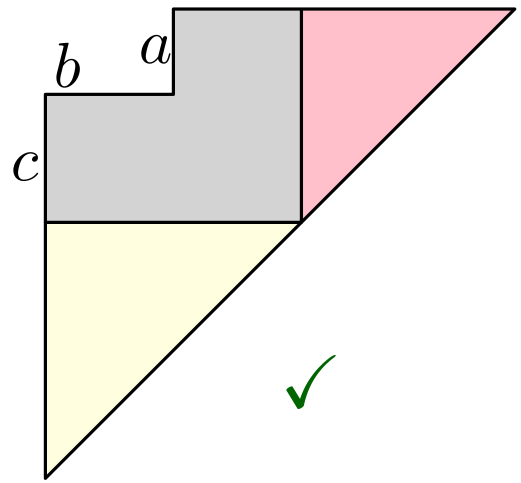

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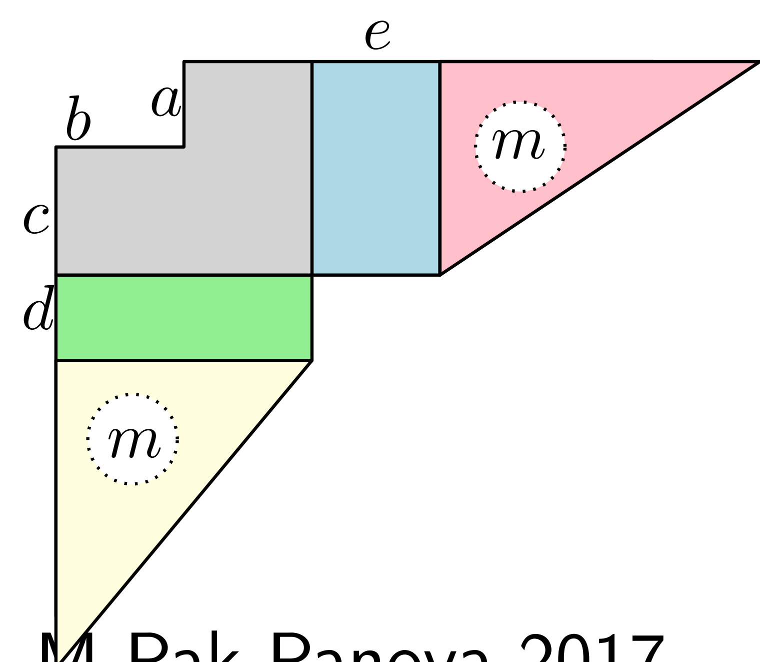

M-Pak-Panova 2017

Theorem (M., Pak, Panova 17)

$$f^{\lambda/(b^a)} = n! \prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c} \frac{i+j+k-1}{i+j+k-2} \prod_{(i,j)\in\lambda/(0^a,b^a)} \frac{1}{\lambda_i+\lambda'_j-i-j+1}$$

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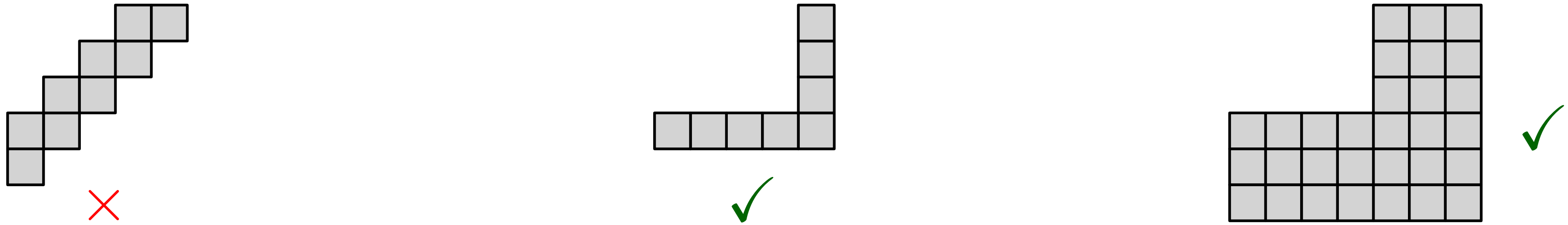

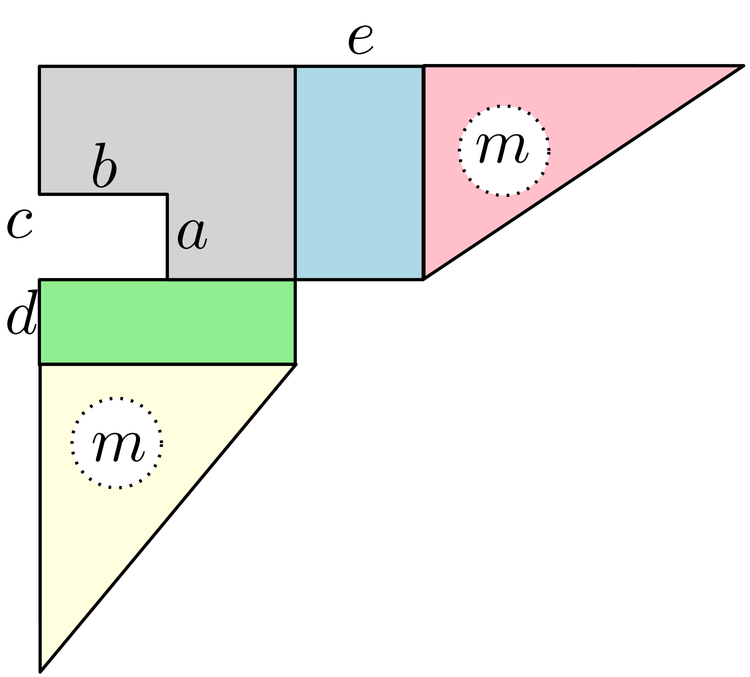

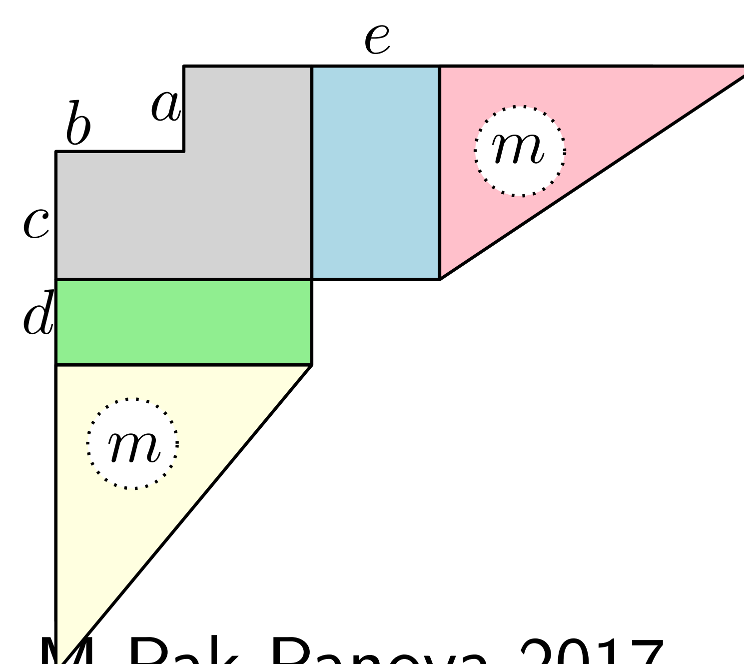

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about Naruse's proof

Naruse's proof: recurrence for skew SYT

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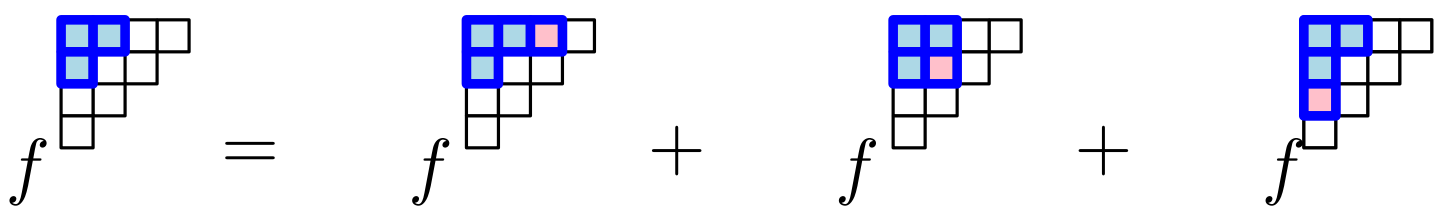

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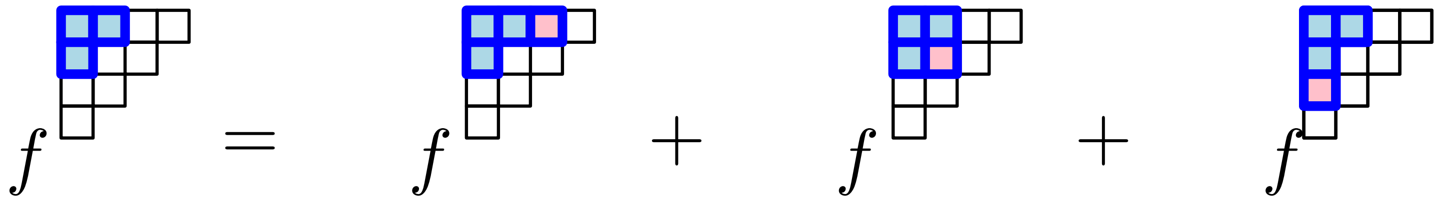

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(bijective approach by Konvanlinka 17,18)

Naruse's proof: where excited diagrams come from

σ_λ is the **equivariant** Schubert class of the Schubert variety $X_\lambda \subseteq \mathrm{Gr}(d, \mathbb{C}^n)$

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  No known rules for the general (equivariant) Schubert structure constants $c^u_{w,v}$ for permutations u, v, w.

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Theorem (Ikeda-Naruse 09, Kreiman 05)

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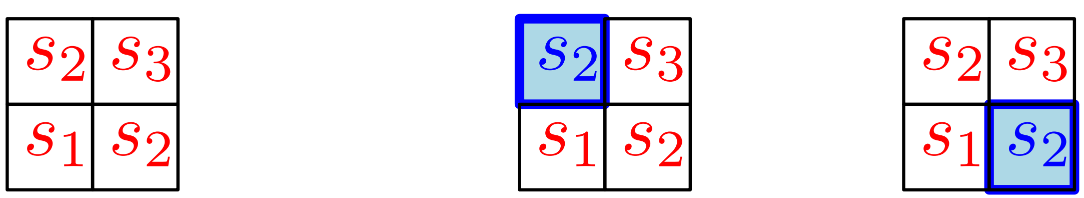

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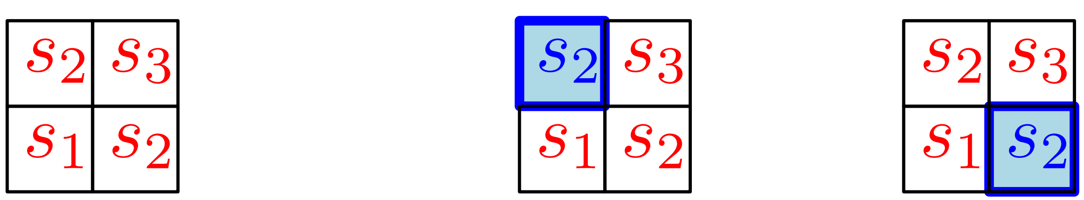

$$c^{22}_{1,22} = c^{v}_{w,v} = (y_1 - y_4) + (y_2 - y_3)$$

Naruse's proof: where SYT come from

$$f^{\lambda/\mu}/n! = \frac{1}{n} \sum_{\nu=\mu+\square\subseteq\lambda} f^{\lambda/\nu}/(n-1)! \qquad (\star)$$

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Then iterating and evaluating $y_p = p$ gives:

Key lemma

$$(-1)^{|\lambda/\mu|} \frac{c^{\lambda}_{\mu,\lambda}}{c^{\lambda}_{\lambda,\lambda}} \Bigg|_{y_p=p} = \frac{f^{\lambda/\mu}}{|\lambda/\mu|!}$$

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$$f^\lambda = \frac{|\lambda|!}{\prod_{u\in\lambda} h(u)}$$

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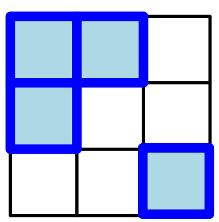

Ikeda–Naruse 09

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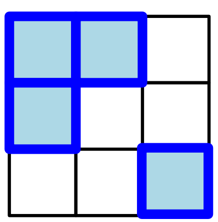

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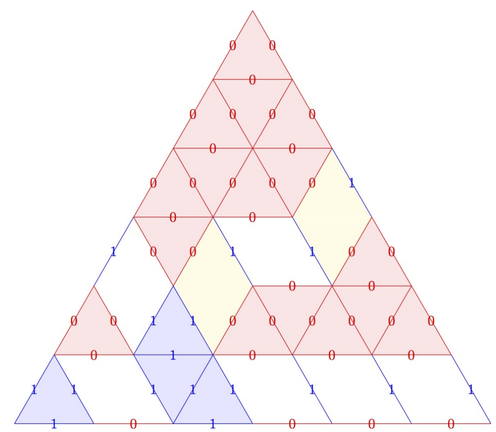

Knutson–Tao 03

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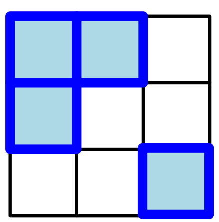

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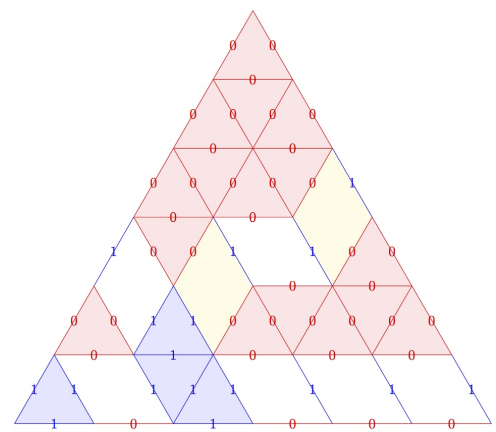

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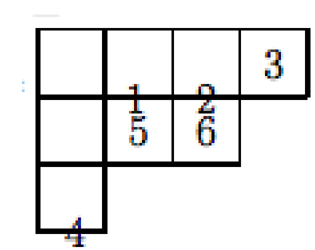


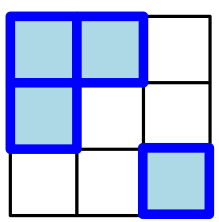
Thomas–Yong 12

A formula for $f^{\lambda/\mu}$ for every rule for $c^{\lambda}_{\lambda,\mu}$

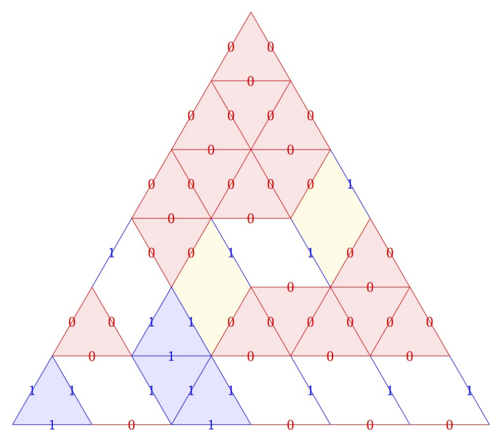
Key lemma

$$(-1)^{|\lambda/\mu|}\frac{c^{\lambda}_{\mu,\lambda}}{c^{\lambda}_{\lambda,\lambda}}\Bigg|_{y_p=p} = \frac{f^{\lambda/\mu}}{|\lambda/\mu|!}$$

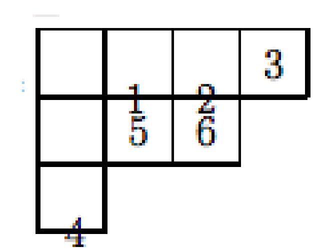
Any rule for $c^{\lambda}_{\mu,\nu}$ will give a formula for $f^{\lambda/\mu}$.



Ikeda–Naruse 09



Knutson–Tao 03



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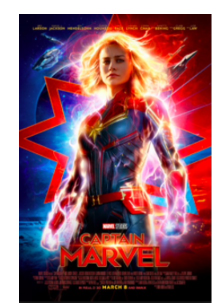

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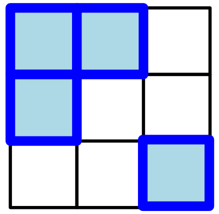

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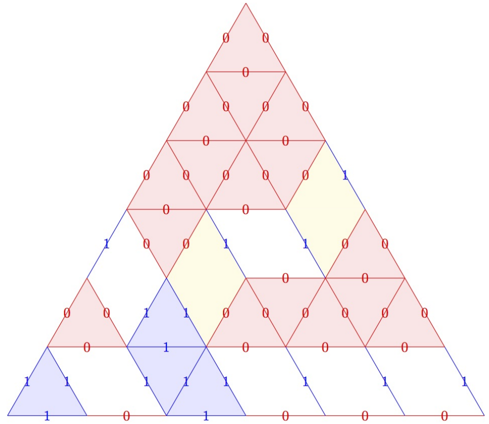

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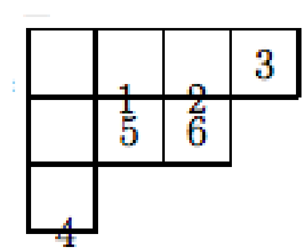


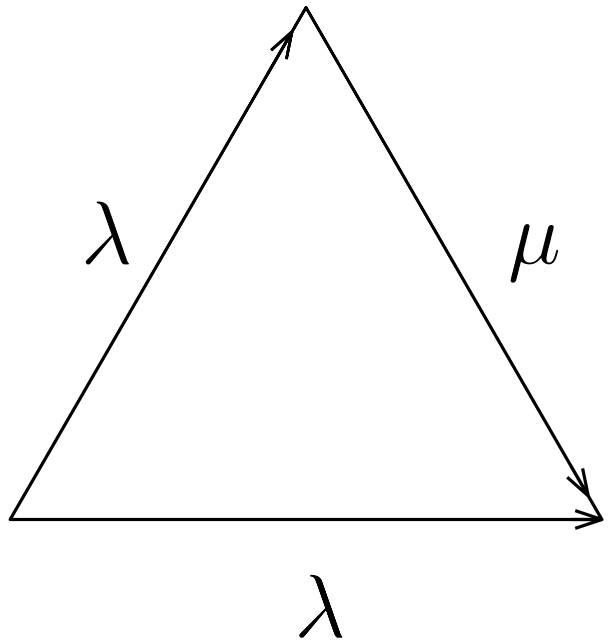
Thomas–Yong 12

Questions: Is the Okounkov–Olshanski formula in this universe?

Is the Littlewood–Richardson formula in this universe?

Knutson–Tao rule for $f^{\lambda/\mu}$

board



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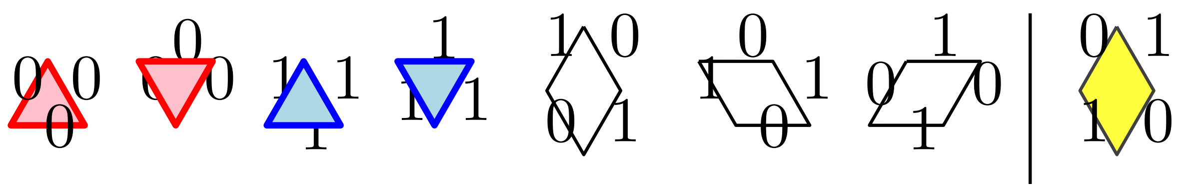

Knutson–Tao rule for $f^{\lambda/\mu}$

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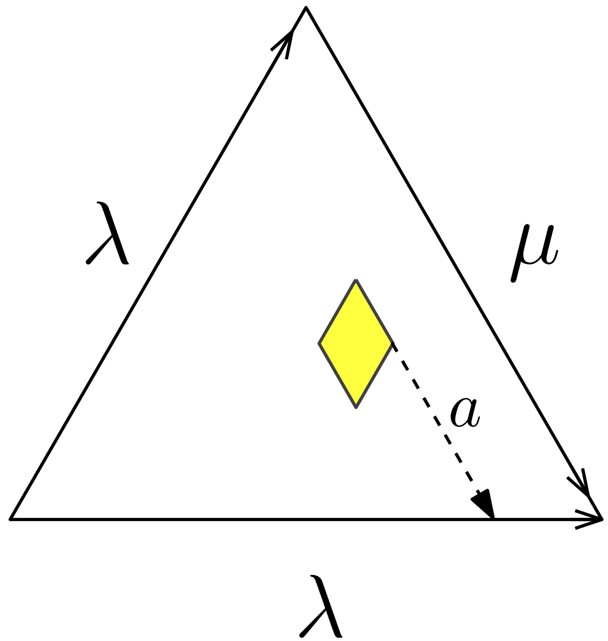

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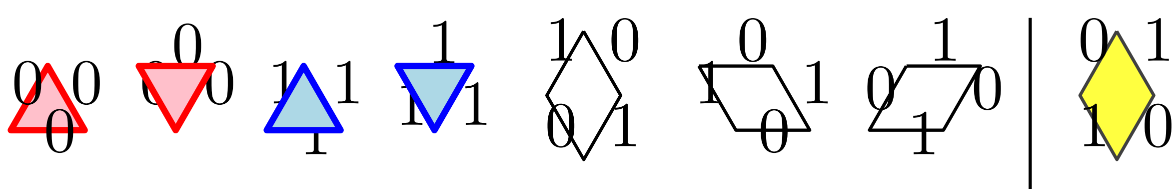

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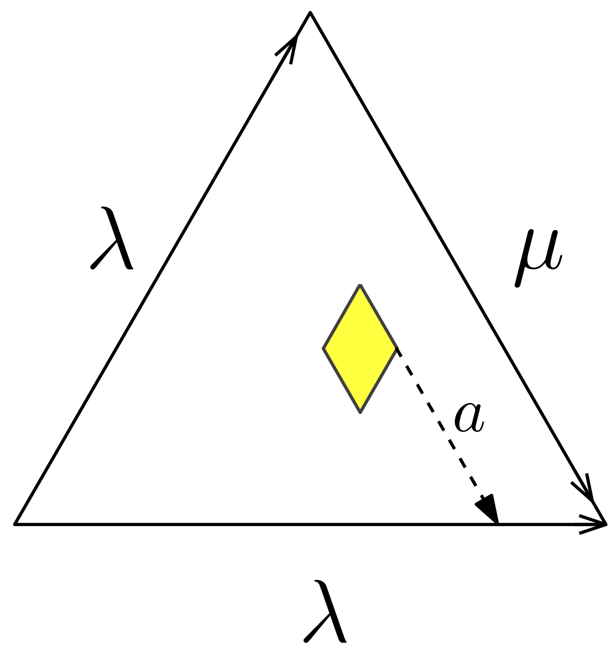

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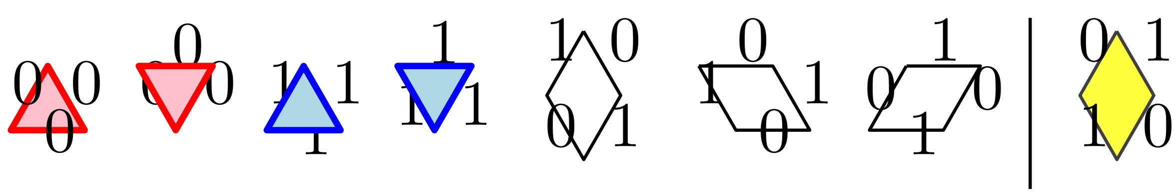

Knutson–Tao 2003

$$f^{\lambda/\mu} = \frac{|\lambda/\mu|!}{\prod_{u\in\lambda} h(u)} \sum_{\mathsf{p}\in {}^{\lambda}\underset{\lambda}{\Delta}{}^{\mu}} \prod_{\lozenge\in\mathsf{p}} a(\lozenge),$$

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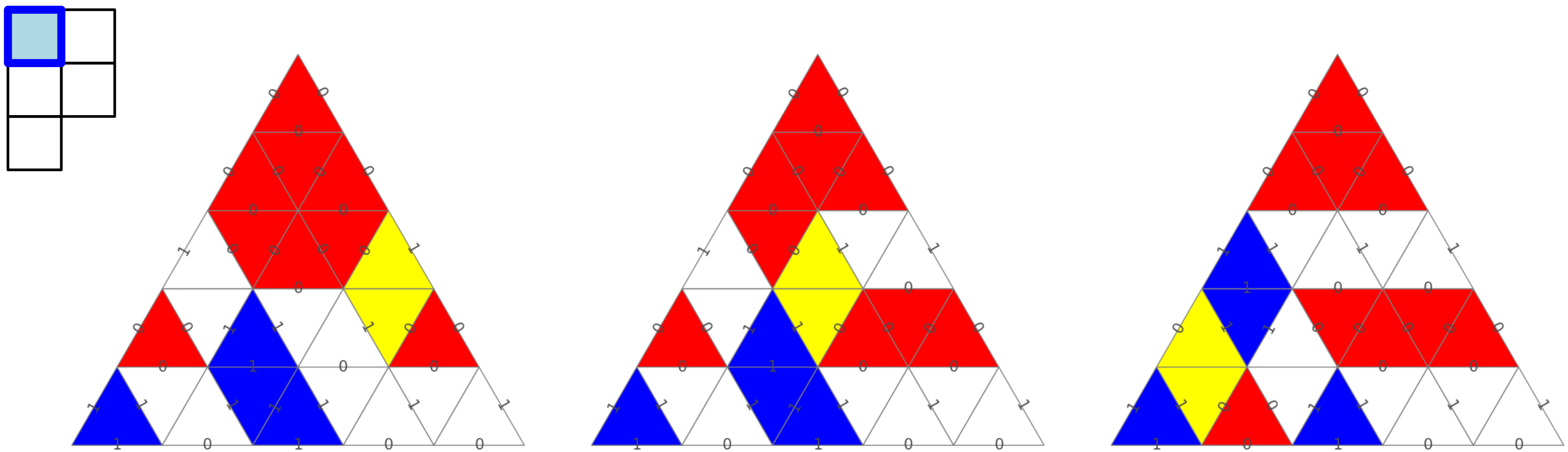
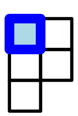
Example



f 

$$= \frac{4!}{2 \cdot 3 \cdot 4} \cdot (2 \quad + \quad 2 \quad + \quad 1)$$

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Recall: Okounkov–Olshanski formula

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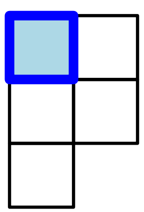
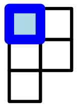


f 

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Equivalence of two rules

Theorem (Morales-Zhu 19+)

The Okounkov–Olshanski and the Knutson–Tao formulas for $f^{\lambda/\mu}$ are term-by-term equal.

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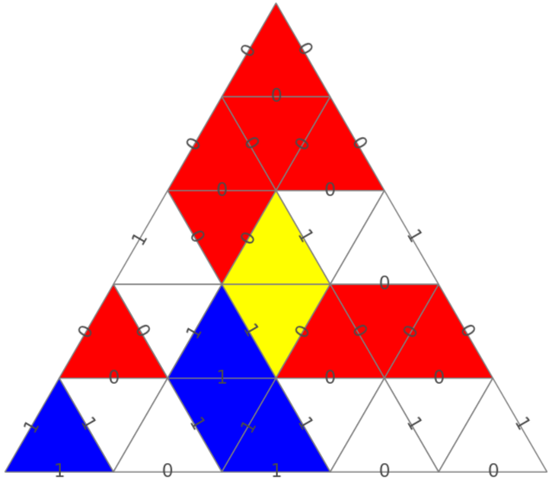

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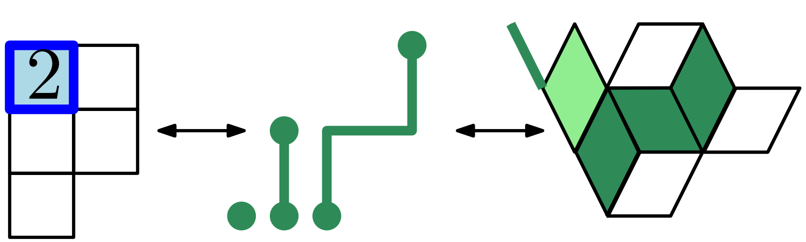

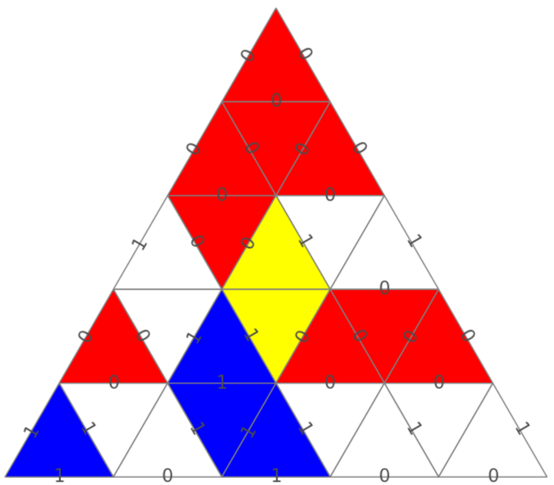

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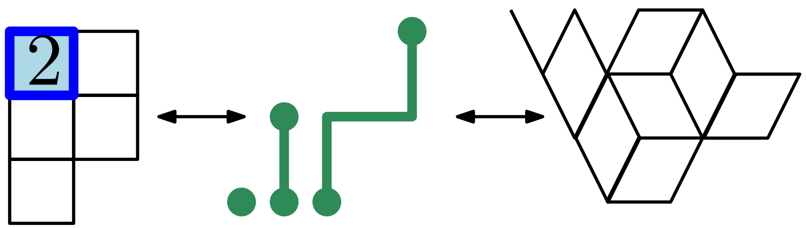

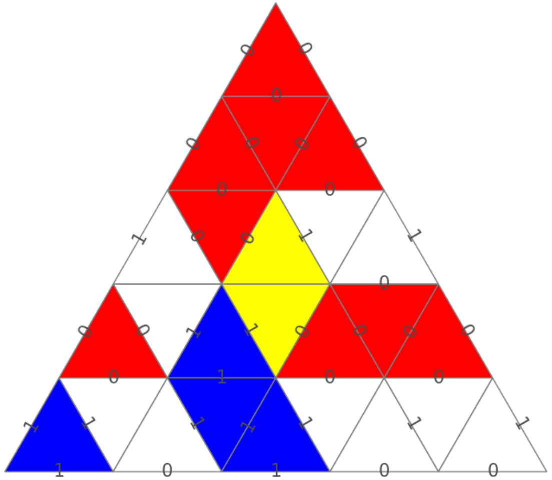

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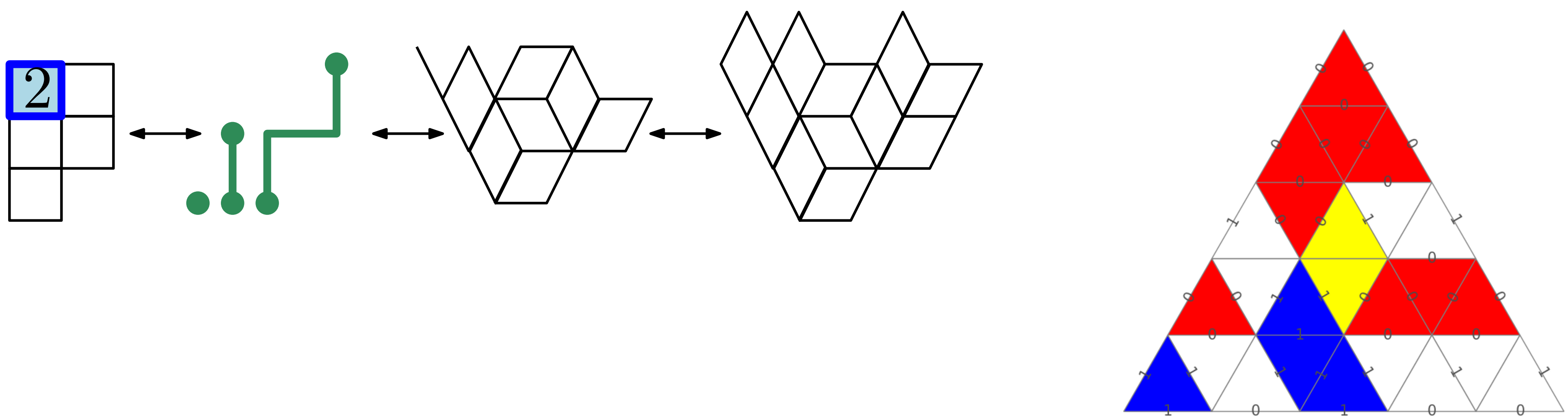

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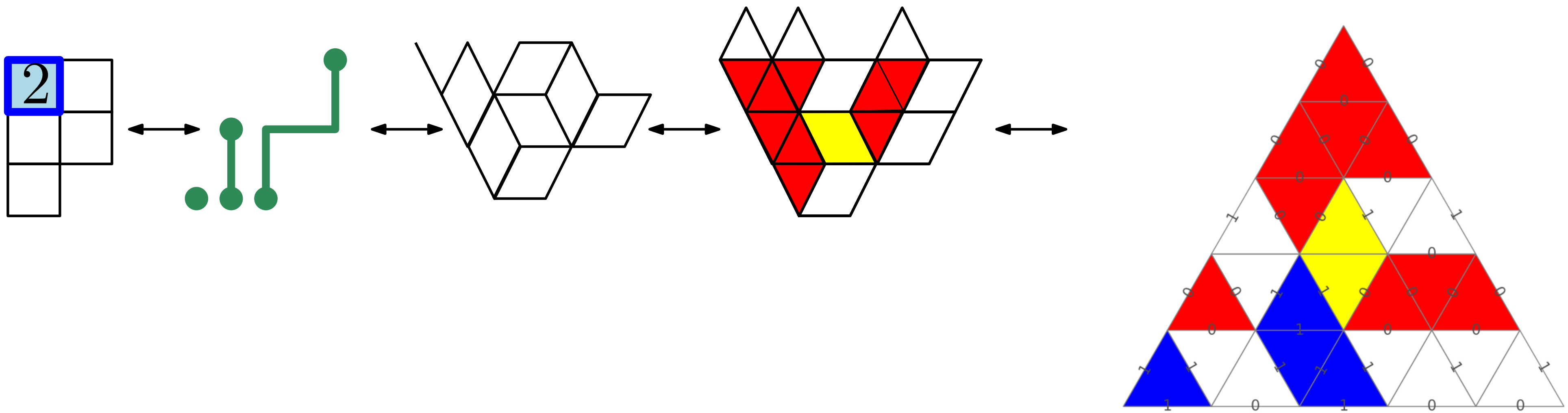

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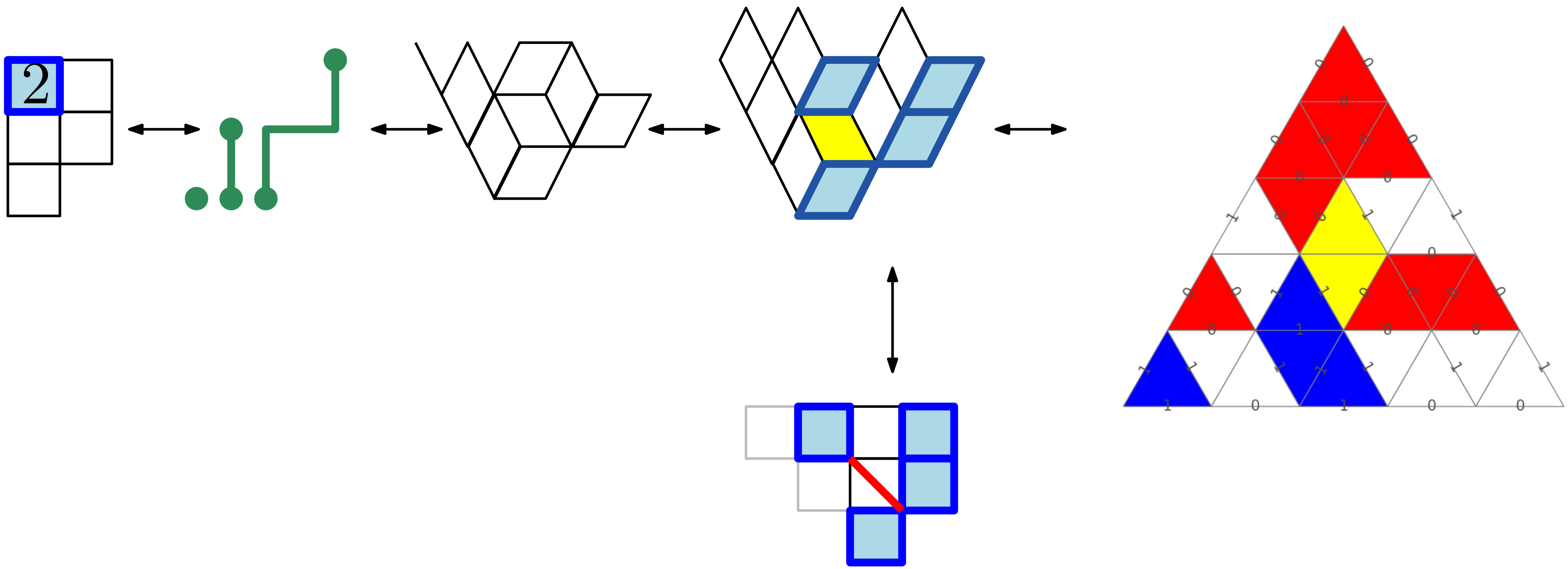

Number of terms of the Okoukov–Olshanski formula

Corollary (Morales-Zhu 19+)

$$\#(\text{nonzero terms shape } \lambda/\mu) \quad = \det\left[\binom{\lambda'_i}{\mu'_j+i-j}\right]_{i,j=1}^{\ell(\mu')}$$

Excited diagram reformulation of Okounkov–Olshanski

Corollary

$$f^{\lambda/\mu} = \frac{n!}{\prod_{u\in\lambda} h(u)} \sum_{D\in\mathcal{E}^{\nwarrow}(\lambda/\mu)} \prod_{u\in \mathrm{diag}(D)} \mathrm{arm}(u)$$

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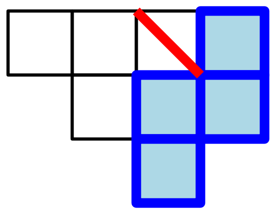

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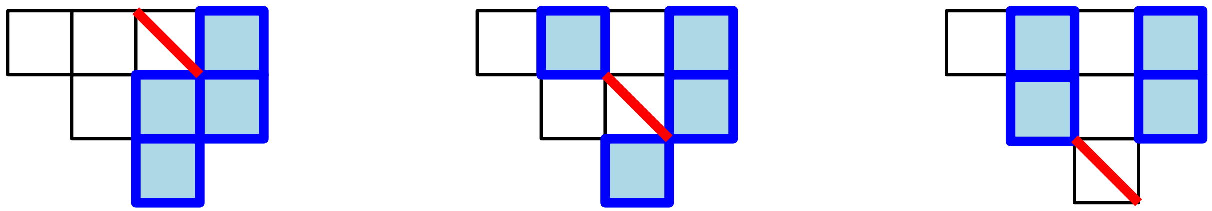

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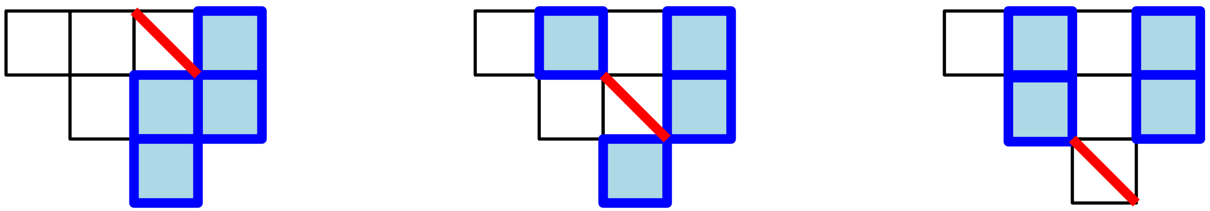
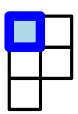


f  $= \frac{4!}{2\cdot 3\cdot 4}\cdot(2 \quad + \quad 2 \quad + \quad 1) = 5$

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$$f^\lambda = \frac{|\lambda|!}{\prod_{u\in\lambda} h(u)}$$

Naruse's formula for $f^{\lambda/\mu}$

Applications

- relation to lozenge tilings
- bounds and asymptotics for $f^{\lambda/\mu}$
- family of skew shapes with product formulas

about Naruse's proof

relations among formulas for $f^{\lambda/\mu}$

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Thank you - Gracias

Some references

- **Kostant polynomials and the cohomology ring for G/B**, S. Billey, Duke Math. J. 96 (1999), 205–224.
- **Billey's formula in combinatorics, geometry, and topology**, J. S. Tymoczko, `arXiv:1309.0254`
- **Schubert calculus and hook formula**, H. Naruse, slides Séminaire Lotharingien de Combinatoire 73, 2014
- **Skew hook formula for d-complete posets**, H. Naruse, S. Okada, `arXiv:1802.09748`
- **Hook formulas for skew shapes I, II, III**, M., I. Pak, G. Panova, `arxiv:1512:08348`, `arxiv:1610.04744`, `arxiv:1707.00931`
- **Asymptotics for the number of standard Young tableaux of skew shape**, M., I. Pak, G. Panova, `arxiv:1610.07561`

Positive formulas for $f^{\lambda/\mu}$

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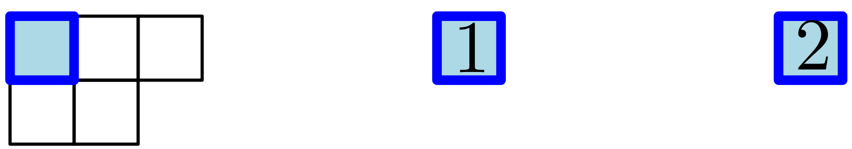
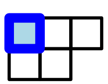


f  $= \frac{4!}{2\cdot 3\cdot 4} \cdot ((3+0)+(2+0))$

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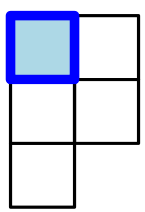
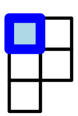


f 

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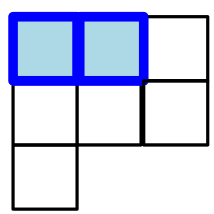

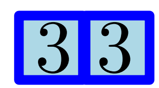

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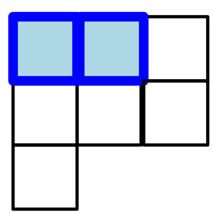

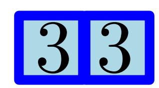

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Study of Okounkov–Olshanski formula

(M-Zhu 2018+, MIT PRIMES)

I. q-analogue of Naruse's skew shifted formula

Theorem (Naruse 2014)

$\mu \subset \lambda$ strict partitions,

$$g^{\lambda/\mu} = |\lambda/\mu|! \sum_{D \in \mathcal{E}'(\lambda/\mu)} \prod_{(i,j) \in \lambda \setminus D} \frac{1}{h'(i,j)},$$

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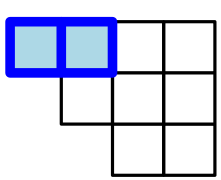
$(4,3,2)/(2)$



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Hook formulas for increasing tableaux

$$\beta^{|\lambda|} \sum_{T \in SIT(\lambda)} \prod_{k=0}^{m-1} \frac{1}{\left(\prod_{i=1}^{d} \frac{1+\beta(\nu_i^{(k)}+d-i+1)}{1+\beta(\lambda_i+d-i+1)}\right) - 1} =$$

$$= \frac{\prod_{i=1}^{\ell(\lambda)} (-\beta(\lambda_i + d - i + 1) - 1)^{\lambda_i}}{\prod_{(i,j)\in[\lambda]} h(i,j)}.$$

where $SIT(\lambda)$ are increasing tableaux

II. How to compute $|\mathcal{P}(\lambda/\mu)|$?

$|\mathcal{E}(\lambda/\mu)|$ is given by a determinant:

$$|\mathcal{E}(\lambda/\mu)| = \det\left[\binom{\mu_i - i + j + f_i - 1}{f_i - 1}\right]_{i,j=1}^{\ell(\mu)}.$$

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$$C_n = |\mathcal{E}(\delta_{n+2}/\delta_n)| = \det\left[\binom{n-i+j}{i}\right]_{i,j=1}^{n-1}.$$

$$C_3 = 5 = \det\begin{bmatrix}\binom{3}{1} & \binom{4}{1}\\ \binom{2}{2} & \binom{3}{2}\end{bmatrix}.$$

II. How to compute $|\mathcal{P}(\lambda/\mu)|$?

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Is $|\mathcal{P}(\lambda/\mu)|$ given by a determinant?

Positive formulas for $f^{\lambda/\mu}$

Okounkov-Olshanski 1998

$$f^{\lambda/\mu} = \frac{|\lambda/\mu|!}{\prod_{u\in\lambda} h(u)} \sum_{T} \prod_{(i,j)\in\mu} (\lambda_{T(i,j)} + j - i),$$

sum is over **reverse** SSYT of shape μ entries $\leq \ell := \ell(\lambda)$.

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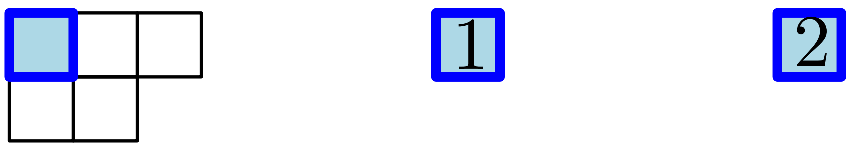
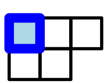
Example



f 

$$= \frac{4!}{2\cdot 3\cdot 4} \cdot ((3+0)+(2+0))$$

$$= 5.$$

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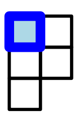
f  $= \frac{4!}{2\cdot 3\cdot 4}\cdot((2+0)+(2+0)+(1+0))$

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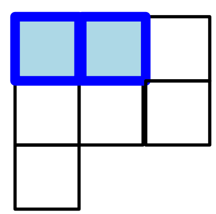

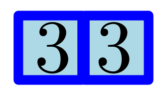

$$(1-0)(1-1)$$

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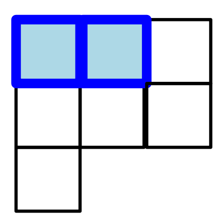

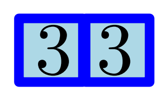

$$(1-0)(1-1)$$

Study of Okounkov–Olshanski formula

(M-Zhu 2018+, MIT PRIMES)

Naruse's proof: factorial Schur functions

The **Schur function** of μ, $\ell(\mu) \leq d$ is

$$s_\mu(x_1, \ldots, x_d) := \frac{\det \left[x_j^{\mu_i + d - i} \right]_{i,j=1}^{d}}{\prod\limits_{1 \leq i < j \leq d} (x_i - x_j)}.$$

where $(x \mid a)_r = x(x - a_1)(x - a_2) \cdots (x - a_{r-1})$

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The classical **factorial Schur function** of μ, $\ell(\mu) \leq d$ is

$$s_\mu(x_1, \ldots, x_d \,|\, 1, 2, \ldots) := \frac{\det \left[(x_j)_{\mu_i + d - i} \right]_{i,j=1}^{d}}{\prod_{1 \leq i < j \leq d} (x_i - x_j)}.$$

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$$s_{\square}(x_1, x_2 \mid a_1, a_2, \ldots) =$$

$$= \frac{\det \begin{bmatrix} (x_1 - a_1)(x_1 - a_2) & (x_2 - a_1)(x_2 - a_2) \\ 1 & 1 \end{bmatrix}}{x_1 - x_2}$$

$$= x_1 - a_2 + x_2 - a_1$$

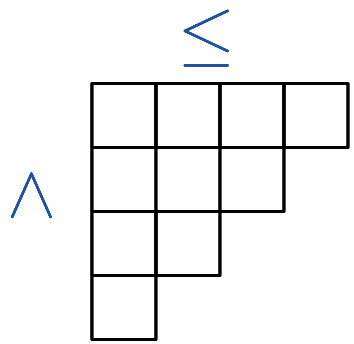
Naruse's proof: factorial Schur functions

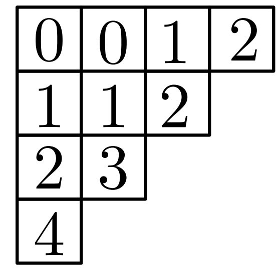
Theorem (Knutson-Tao 03, Lakshmibai-Raghavan-Sankaran 05)

$$c^{v}_{w,v} = (-1)^{\ell(w)} \cdot s^{(d)}_{\mu}(y_{v(1)}, \ldots, y_{v(d)} \mid y_1, \ldots, y_{n-1}).$$

Semistandard Young tableaux

A **semistandard tableau** is a filling the Young diagram weakly increasing in rows and strictly increasing in columns





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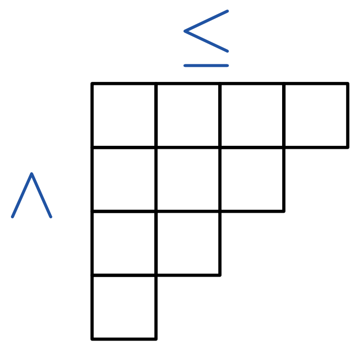

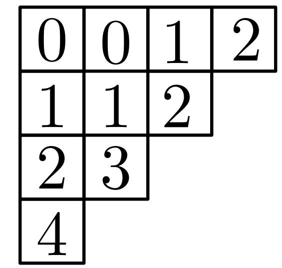

Generating function of SSYT (evaluation of Schur functions)

$$\sum_{T \in \mathrm{SSYT}(\lambda/\mu)} q^{|T|} = s_{\lambda/\mu}(1, q, q^2, \ldots)$$

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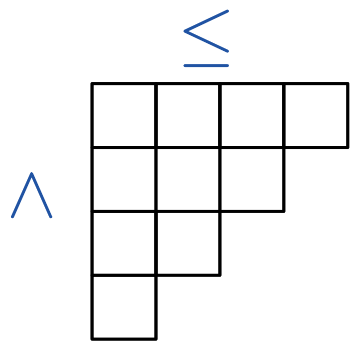

0	0	1	2
1	1	2	
2	3		
4			

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Example

$$s_{\boxplus}(1, q, q^2, \dots) = q^2 \quad + \quad q^4 \quad + \quad 2q^6 \cdots$$

$$\begin{array}{|c|c|}\hline 0 & 0 \\ \hline 1 & 1 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 0 & 1 \\ \hline 1 & 2 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 0 & 1 \\ \hline 2 & 3 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 0 & 2 \\ \hline 1 & 3 \\ \hline \end{array}$$

q-analogue hook-length formula

Theorem (Stanley 1971)

$$s_\lambda(1, q, q^2, \ldots) = q^{b(\lambda)} \prod_{(i,j)\in\lambda} \frac{1}{1 - q^{h(i,j)}}, \qquad (*)$$

where $b(\lambda) = \sum_i (i-1)\lambda_i$

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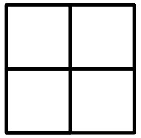

3	2
2	1

$$s_{\boxplus}(1, q, q^2, \ldots) = \frac{q^2}{(1-q^1)(1-q^2)^2(1-q^3)}$$

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By Stanley theory of P-partitions:

$$s_\lambda(1, q, q^2, \ldots) \cdot (1-q)(1-q^2)\cdots(1-q^n) = \sum_{T\in \mathrm{SYT}(\lambda)} q^{\mathrm{maj}(T)}$$

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Set $q = 1$ and get hook-length formula:

$$\Big(\prod_{(i,j)\in\lambda} \frac{1}{h(i,j)} \Big) \cdot \qquad n! \qquad = \qquad f^{\lambda/\mu}$$

Naruse's proof: recurrence for skew SYT

Theorem (Naruse 2014)

$$f^{\lambda/\mu} = |\lambda/\mu|! \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{(i,j) \in \lambda \setminus D} \frac{1}{h(i,j)},$$

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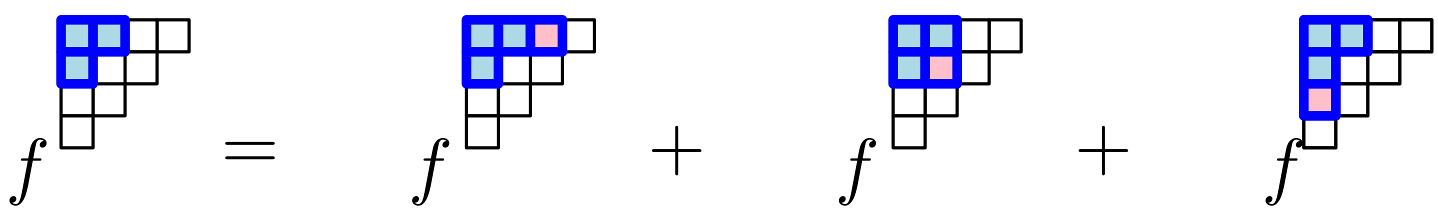

Strategy: show sum of excited diagrams satisfy same identity.

Naruse's proof: where excited diagrams come from

σ_λ is the **equivariant** Schubert class of the Schubert variety $X_\lambda \subseteq \mathrm{Gr}(d, \mathbb{C}^n)$

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- when $|\lambda| - |\mu| - |\nu| = 0$ the constants $c^\lambda_{\mu,\nu}$ are the classical Littlewood–Richardson coefficients.
- There are several rules for $c^\lambda_{\mu,\nu}$.
  No known rules for the general (equivariant) Schubert structure constants $c^u_{w,v}$ for permutations u, v, w.

Naruse's proof: excited diagrams

Theorem (Ikeda-Naruse 09, Kreiman 05)

$$c^{\lambda}_{\mu,\lambda} = \sum_{D\in\mathcal{E}(\lambda/\mu)} \prod_{(i,j)\in D} \left(y_{d+j-\lambda'_j} - y_{\lambda_i+d-i+1}\right),$$

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$s_2 s_1\ s_3 s_2$ $\qquad$ $s_2 s_1\ s_3 s_2$ $\qquad$ $s_2 s_1\ s_3 s_2$

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s_1	s_2

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s_2	s_3
s_1	s_2

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s_1	s_2

s_2	s_3
s_1	s_2

$$c^{22}_{1,22} = c^v_{w,v} = (y_1 - y_4) + (y_2 - y_3)$$

Naruse's proof: where SYT come from

Summary

$$f^{\lambda/\mu}/n! = \frac{1}{n} \sum_{\nu=\mu+\square\subseteq\lambda} f^{\lambda/\nu}/(n-1)! \qquad (\star)$$

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- evaluate $y_p = p$ and get $(\star)$

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$$(-1)^{|\lambda/\mu|} \frac{c^\lambda_{\mu,\lambda}}{c^\lambda_{\lambda,\lambda}} \Bigg|_{y_p=p} = \sum_{D\in\mathcal{E}(\lambda/\mu)} \prod_{u\in\lambda\setminus D} \frac{1}{h(u)}$$

From **associativity** of $(\sigma_\square \cdot \sigma_\mu)\sigma_\lambda = \sigma_\square(\sigma_\mu \cdot \sigma_\lambda)$, extracting coefficient of σ_λ OR by an insertion (Konvalinka 16)

$$c^\lambda_{\mu,\lambda} = \frac{1}{c^\lambda_{\square,\lambda} - c^\mu_{\square,\mu}} \sum_{\nu=\mu+\square\subseteq\lambda} c^\lambda_{\nu,\lambda}$$

- evaluate $y_p = p$ and get $(\star)$ $\square$

A formula for $f^{\lambda/\mu}$ for every rule for $c_{\lambda,\mu}^{\lambda}$

Proposition

$$(-1)^{|\lambda/\mu|} \frac{c_{\mu,\lambda}^{\lambda}}{c_{\lambda,\lambda}^{\lambda}} \Bigg|_{y_p = p} = \frac{f^{\lambda/\mu}}{|\lambda/\mu|!}$$

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Proposition

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Any rule for $c_{\mu,\nu}^{\lambda}$ will give a formula for $f^{\lambda/\mu}$.

A formula for $f^{\lambda/\mu}$ for every rule for $c^{\lambda}_{\lambda,\mu}$

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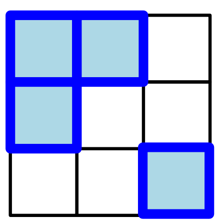

Ikeda–Naruse 09

A formula for $f^{\lambda/\mu}$ for every rule for $c^\lambda_{\lambda,\mu}$

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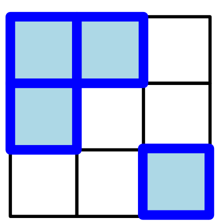
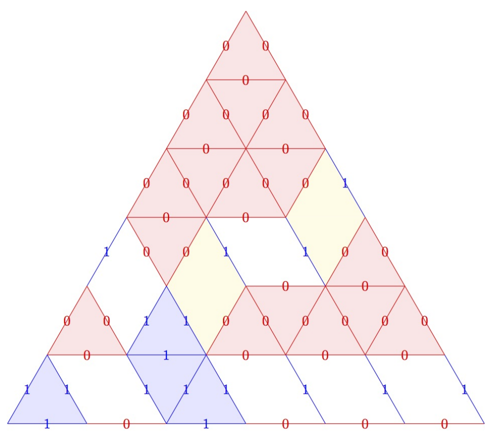


Ikeda–Naruse 09



Knutson–Tao 03

A formula for $f^{\lambda/\mu}$ for every rule for $c^{\lambda}_{\lambda,\mu}$

Proposition

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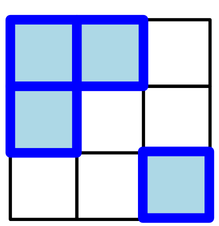
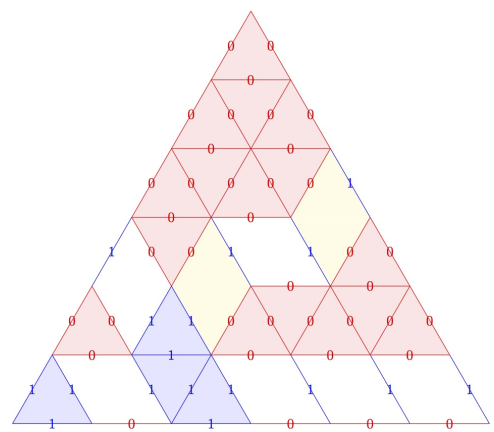
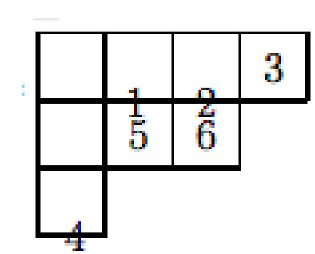


Ikeda–Naruse 09



Knutson–Tao 03



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