Asymptotic of multiplicities and of character distributions for large tensor products of representations of simple Lie algebras

Olga Postnova
PDMI RAS (Saint Petersburg)
joint work
with N.Reshetikhin, V. Serganova (UC Berkeley)

March 12, 2019

Notations

- Let $\mathfrak{g}$ be a simple finite dimensional Lie algebra.
- $\mathfrak{h}$ Cartan subalgebra
- $\alpha_a \in \mathfrak{h}^*$, $a = 1, \ldots, r = rank(\mathfrak{g})$ are simple roots
- The Killing form $(.,.)$ on $\mathfrak{g}$ defines scalar products on $\mathfrak{h}$ and therefore on $\mathfrak{h}^*$. Symmetrized Cartan matrix $B_{ab} = d_a C_{ab} = (\alpha_a, \alpha_b)$. In the basis of simple roots $(x, y) = \sum_{a,b} x_a B_{ab} y_b$.
- Linear isomorphisms $\mathfrak{g} \simeq \mathfrak{g}^*$ and $\mathfrak{h} \simeq \mathfrak{h}^*$ are fixed by the Killing form.

Multiplicities

- $V_1, \ldots, V_m$ be finite dimensional representations, $N_1, \ldots, N_m$ be positive integers.
- $$\otimes_{i=1}^m V_i^{\otimes N_i} \simeq \oplus_\lambda V_\lambda^{\oplus m_\lambda(N)}$$

  Here $m_\lambda(N)$ is the multiplicity of V_λ in the tensor product. Formulae for multiplicities. Asymptotic for large N_i?

Character probability distribution

- Let W–finite dimensional $\mathfrak{g}$-module

$$W \simeq \oplus_{\lambda \in D_N} V_\lambda^{\oplus m_\lambda}$$

Here D_N is the set of irreducible components of V.

- Let $t \in \mathfrak{h}_\mathbb{R}$, V be a finite dimensional $\mathfrak{g}$–module and $\chi_V(e^t) = tr_V(\pi(e^t))$ be the character of V evaluated on e^t.

  Character probability measure on D_N:

$$Prob(\lambda) = \frac{m_\lambda \chi_{V_\lambda}(e^t)}{\chi_W(e^t)}$$

- Plancherel measure corresponds to $t = 0$

$$Prob(\lambda) = \frac{m_\lambda \dim(V_\lambda)}{\dim(W)}$$

Problem 1: Find the asymptotic of $m_\lambda(N)$ is the limit when $N_i = \tau_i/\epsilon, \lambda = \xi/\epsilon$, $\epsilon \to 0$ and $\tau_i > 0$ and $\xi \in \mathfrak{h}^*_{>0}$ are fixed.

- When $m = 1$, and ξ is inside (not on a wall) of the positive Weyl chamber, the asymptotic was computed in

  T.Tate, S. Zelditch, *Lattice path combinatorics and asymptotics of multiplicities of weights in tensor powers*, J. Funct. Anal. 217 (2004), no. 2, 402–447. arXiv:math/0305251.

- For general m the proof is very similar, we will outline it.

  O. Postnova, N. Reshetikhin, *On multiplicities of irreducibles in large tensor product of representations of simple Lie algebras.*

- Particular case sl_{n+1}, powers of $\mathbb{C}^{n+1}$, immediately follows from the hook formula for dimensions of irreducible representations of S_N. and from the Stirling formula.

Example: The multiplicity function in sl_{n+1} case

The multiplicity function $m_\lambda^{(N)}$ is determined by the hook length formula:

$$m_\lambda^{(N)} = N! \frac{\prod_{i<j} (l_i - l_j - i + j)}{\prod_{i=1}^{n+1} (l_i + n + 1 - i)!}.$$

The Stirling formula:

$$N! = \sqrt{2\pi N} e^{N \ln N - N} \left(1 + O\left(\frac{1}{N}\right)\right),$$

The asymptotic for multiplicities for large N and l_i:

$$m_\lambda^{(N)} = \frac{\sqrt{2\pi N} e^{N \ln N - N} \prod_{i<j} (l_i - l_j)}{(\sqrt{2\pi})^{n+1} \prod_{i=1}^{n+1} l_i^{n+1-i+1/2} e^{l_i \ln l_i - l_i}} \left(1 + O\left(\frac{1}{N}\right)\right).$$

Assume $N = \tau/\epsilon$, $l_i = \sigma_i/\epsilon$ and $\sum_{i=1}^{n+1} \sigma_i = \tau$ and when $\epsilon \to 0$:

$$m_\lambda^{(N)} = \left(\frac{\epsilon}{2\pi}\right)^{\frac{n}{2}} \tau^{\frac{1}{2}} \prod_{i<j} (\sigma_i - \sigma_j) \prod_{i=1}^{n+1} \sigma_i^{-n+i-3/2} e^{\frac{1}{\epsilon} S(\tau,\sigma)} (1 + O(\epsilon)),$$

where $S(\tau, \sigma) = \tau \ln \tau - \sum_{i=1}^{n+1} \sigma_i \ln \sigma_i$.

Problem 2: Find the asymptotic of the character probability measure in the limit $\epsilon \to 0$ (when t is fixed)

As we will see the asymptotical distribution depends on the stabilizer of t in the Weyl group.

- When t is inside the positive Weyl chamber, the stabilizer is trivial, the asymptotic distribution is Gaussian.

  O. Postnova, N. Reshetikhin *On multiplicities of irreducibles in large tensor product of representations of simple Lie algebras.*

- When $t = 0$ (Plancherel), the stabilizer is W, the asymptotic distribution is proportional to the product of Gaussian distribution and polynomial.

  $\mathfrak{g} = sl_{n+1}$: S. Kerov, *On asymptotic distribution of symmetry types of high rank tensors*, Zapiski Nauchnykh Seminarov POMI, **155**, 1986.

$$p(a) = \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \frac{(n+1)^{\frac{(n+1)^2}{2}}}{1! \cdot 2! \cdot \ldots n!} \prod_{i<j} (a_i - a_j)^2 \, e^{-\frac{1}{2}\frac{n+1}{\tau}\sum_{i+1}^{n+1} a_i^2}$$

where $a_k = \frac{l_k - \frac{N}{n+1}}{\sqrt{N}}, k = 1 \ldots n+1$, l_k - lengths of rows of Young diagram.

Dimensions of sl_{n+1}-modules are given by the Weyl formula

$$\dim(V_\lambda) = \frac{\prod_{i\leqslant j}(l_i - l_j)}{\prod_{k=1}^n k!} \simeq \epsilon^{\frac{(n+1)^2-n-1}{2}} \frac{\prod_{i\leqslant j}(\sigma_i - \sigma_j)}{\prod_{k=1}^n k!}.$$

The pointwise asymptotic of $\mathrm{Prob}(\lambda)$

$$p_\lambda \simeq \left(\frac{\epsilon}{2\pi}\right)^{\frac{n^2+2n}{2}} \frac{\prod_{i<j}(\sigma_i - \sigma_j)^2}{\prod_{k=1}^n k!} \prod_{i=1}^{n+1} \sigma_i^{-n+i-3/2} e^{\frac{1}{\epsilon}(S(\tau,\sigma)-\tau\ln(n+1))}.$$

$S(\tau,\sigma) = \tau\ln\tau - \sum_{i=1}^{n+1}\sigma_i\ln\sigma_i$ has the critical point $\sigma_i = \frac{\tau}{n+1}$.

In the vicinity of this critical point rescaling random variables σ_i as:

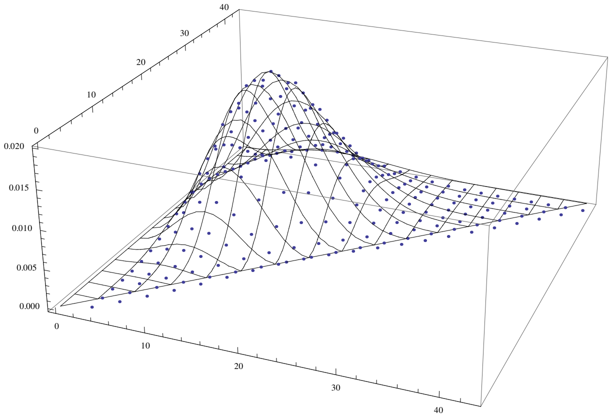
$$\sigma_i = \frac{\tau}{n+1} + \sqrt{\epsilon}a_i.$$

$$S(\tau,\sigma) = S(\tau,\tau/(n+1)) - \frac{n+1}{\tau}\sum_{i=1}^{n+1}\frac{\epsilon a_i^2}{2} + O(\epsilon^{3/2}).$$

In the vicinity of critical point the Plancherel probability distribution

$$p_\lambda^{(N)} \simeq \left(\frac{\epsilon}{2\pi}\right)^{\frac{n}{2}} \frac{1}{1!\cdot 2!\cdot \ldots n!}\tau^{\frac{-(n+1)^2+1}{2}}(n+1)^{\frac{(n+1)^2}{2}} \prod_{i<j}(a_i - a_j)^2 e^{-\frac{1}{2}\frac{n+1}{\tau}\sum_{i+1}^{n+1} a_i^2}$$

Nazarov, A.A., Postnova, O.V.,The limit shape of a probability measure on a tensor product of modules of Bn algebra,Zapiski Nauchnykh Seminarov POMI, vol.468, 82-97, 2018



$$\phi(\{x_i\}) = \frac{2^{2n} n!}{(\sqrt{2\pi})^n (2n)!(2n-2)!\dots 2!} \prod_{i<j} (x_i^2 - x_j^2)^2 \prod_{l=1}^{n} x_l^2 \exp\left(-\frac{1}{2}\sum_k x_k^2\right).$$

Weak convergence of probability measures

We say that the sequence of probability measures $(p_N)_{N\in\mathbb{N}}$ converges weakly to p ($(p_N) \Rightarrow p$) if for any bounded continuous function $f \in C(X)$

$$\lim_{N\to\infty}\int f(x)dp_N(x) = \int f(x)dp(x)$$

Criterion: Let $\mathcal{E}$ be the class of open sets in metric space X, which is closed under finite intersection, and every open set can be represented as countable or finite union of sets from $\mathcal{E}$. Let p_N, p be probability Borel measures such that $p_N(E) \longrightarrow p(E)$ for all $E \in \mathcal{E}$.Then the sequence p_N converges weakly to p.

X is the n-dimensional random vector be distributed according to p_N

$$p_N(\lambda) = \mathbf{P}\{X = \lambda\} = \mathbf{P}\{X \in U_a\} = \mathbf{P}\left\{\frac{1}{\sqrt{N}}X \in U_a(N)\right\}.$$

- Prove for $U_a(N)$ as $N \to \infty$

$$\left|p_N(\lambda)\cdot\left(\frac{\sqrt{N}}{2}\right)^n - \phi\left(\left\{\frac{1}{\sqrt{N}}a_i\right\}\right)\right| \longrightarrow 0$$

- Prove for every n-orthotope $H_n = \{c_1, d_1\} \times \{c_2, d_2\} \times \cdots \times \{c_n, d_n\}$ where all $\{c_i\} < \{d_i\}$ are fixed real numbers.

$$\lim_{N \longrightarrow \infty} \mathbf{P}\left\{c_i \leqslant \frac{1}{\sqrt{N}} X_i < d_i\right\} = \int_{H_n} \phi(\{x_i\}) dx_1 \ldots dx_n$$

- Use the criterion .

  General case
  Ph. Biane, *Miniscual weights and random walks on lattices*, Quant. Prob. Rel. Topics, v. 7 (1992), 51-65.

  T.Tate, S. Zelditch, *Lattice path combinatorics and asymptotics of multiplicities of weights in tensor powers*, J. Funct. Anal. 217 (2004), no. 2, 402–447. arXiv:math/0305251.

- When t is a wall of the positive Weyl chamber, the distribution is intermediate, the product measure. We describe it later.
  O. Postnova, N. Reshetikhin, V. Serganova, *The asymptotic of the character distribution on irreducible component of large tensor products.*

The asymptotic of multiplicities

Definitions of important functions:

- Define
$$f(\tau, t) = \sum_k \tau_k \ln(\chi_{\nu_k}(e^t))$$
Strictly convex in t.
- Define
$$S(\tau, \xi) = \min_y (f(\tau, y) - (y, \xi)) = f(\tau, x) - (x, \xi),$$
(y, ξ) is the Killing form: in the basis of simple roots $(y, \xi) = \sum_{ab} y_a B_{ab} \xi_b$. Here x and ξ are Legendre images of each other:
$$\frac{\partial}{\partial x_a} f(\tau, x) = \sum_b B_{ab} \xi_b$$
- Define
$$K_{ab}(\xi) = -\frac{\partial^2 S(\tau, \xi)}{\partial \xi_a \partial \xi_b}$$

Theorem

If $\xi=\epsilon\lambda$ remain finite and regular (inside the positive Weyl chamber) as $\epsilon\to 0$ the asymptotic of the multiplicity of V_λ in $\otimes_{i=1}^m V_i^{\otimes N_i}$ has the following form

$$m_\lambda(\{V_k\},\{N_k\})=\epsilon^{\frac{r}{2}}\frac{\sqrt{\det K(\xi)}}{(2\pi)^{\frac{r}{2}}}\Delta(x)e^{-(\rho,x)}e^{\frac{1}{\epsilon}S(\tau,\xi)}(1+O(\epsilon))$$

Here $x\in\mathfrak{h}$ is the Legendre image of $\xi\in\mathfrak{h}^*$, the functions S and the matrix K are as above and $\Delta(x)$ is the denominator in the Weyl formula for characters:

$$\Delta(x)=\prod_{\alpha\in\Delta_+}(e^{\frac{(x,\alpha)}{2}}-e^{-\frac{(x,\alpha)}{2}})$$

The idea of the proof:

- Let dg be the Haar measure on the simply connected compact Lie group G with the Lie algebra $\mathfrak{g}$. Then

$$m_\lambda(\{V_k\},\{N_k\})=\int_G\prod_{i=1}^m\chi_{V_i}(g)^{N_i}\overline{\chi_\lambda(g)}dg$$

From here by the steepest descent we see $m_\lambda\simeq\exp(\frac{S}{\epsilon})$

- Substitute this asymptotic into the identity

$$\prod_{i=1}^{m} \chi_{V_i}(e^x)^{N_i} = \sum_{\lambda \subset D_N} m_\lambda(\{V_k\}, \{N_k\}) \chi_\lambda(e^x)$$

where $x \in \mathfrak{h}_{\mathbb{R}}$ and replace the sum by the integral

$$e^{\frac{f(\tau,x)}{\epsilon}} = \sum_\lambda m_\lambda^N \chi_\lambda(e^x) \simeq \epsilon^{-r} \int_D e^{\frac{1}{\epsilon} S(\tau,\xi)} \mu(\tau, \xi) \chi_{\frac{\xi}{\epsilon}}(e^x) d\xi, \qquad (1)$$

From the Weyl character formula:

$$\chi_{\frac{\xi}{\epsilon}}(e^x) = \frac{e^{\frac{(x,\xi)}{\epsilon} + (\rho,x)}}{\Delta(x)} (1 + o(1))$$

- Let η be the maximum of the function $S(\tau, \xi) + (x, \xi)$, assume that it is extremum. Taking the integral over a neighborhood of η we obtain:

$$e^{\frac{f(\tau,x)}{\epsilon}} = e^{\frac{1}{\epsilon}(S(\tau,\eta)+(x,\eta))} \mu(\tau, \eta) \epsilon^{-r} \frac{e^{(\rho,x)}}{\Delta(t)} \epsilon^{\frac{r}{2}} (2\pi)^{r/2} \frac{1}{\sqrt{\det K(\eta)}} \qquad (2)$$

- From here most singular factors give

$$f(\tau, x) = max_{\xi}(S(\tau, \xi) + (x, \xi))$$

thus, S is the Legendre transfrom of f and η is the Legendre image of x.

- Next order factors give

$$\mu(\tau, \eta) = \epsilon^{r/2} \frac{\sqrt{\det K(\eta)}}{(2\pi)^{r/2}} \Delta(x) e^{-(\rho, x)}.$$

- This gives the desired formula

The asymptotic of characters

Assume that t is on a wall of the positive Weyl chamber. Denote

- $W_t \subset W$ – the stabilizer of t in the Weyl group of $\mathfrak{g}$,
- $\mathfrak{g}_t$ be the Lie subalgebra with roots which vanish on t,
- r_t – the rank of $\mathfrak{g}_t$.

The asymptotic of the character of V_λ evaluated on e^t:

$$ch_{\frac{\xi}{\epsilon}}(e^t) = \epsilon^{-r_t} e^{\frac{(\xi,t)}{\epsilon}+(\rho,t)} \prod_{\alpha\in\Delta_+^t} \frac{(\xi,\alpha)}{(\rho_t,\alpha)} \prod_{\alpha\in\Delta_+\backslash\Delta_+^t} \frac{1}{e^{\frac{(t,\alpha)}{2}} - e^{-\frac{(t,\alpha)}{2}}}$$

Here $\Delta_+^t \subset \Delta$ are positive roots of $\mathfrak{g}^t$. Let

$$\xi = \eta + \sqrt{\epsilon x} a + \sqrt{\epsilon} b,$$

where $(\alpha, a) = 0$ for $\alpha \in \Delta\backslash\Delta^t$, $(\alpha, b) = 0$ for $\alpha \in \Delta^t$, and

$$x = \sum_\nu \frac{\tau_\nu}{\dim(\mathfrak{g}^t)} \frac{\sum_\mu tr_{W_\mu}(e^t) c_2^t(\mu) \dim(V_\mu^t)}{\sum_\mu tr_{W_\mu}(e^t) \dim(V_\mu^t)}$$

Here we used the decomposition $V_\nu \simeq \oplus_\mu W_\mu^\nu \otimes V_\mu^t$ into irreps. for $\mathfrak{g}^t$.

and we assumed that factors V_i are irreducible with highest weight ν_i.

Theorem (PRS)

Let $t \in \mathfrak{h}_{\mathbb{R}}$ *as above and* $p_\lambda^{(N)}(t)$ *be the character measure. As* $\epsilon \to 0$ *it weakly converges to the probability distribution on* $\mathbb{R}_{\geqslant 0}^{r^t} \times \mathbb{R}^{r-r^t}$

$$p(a,b) = \frac{\sqrt{\det K^{(t)}}}{(2\pi)^{\frac{r-r_t}{2}}} e^{-\frac{1}{2}(b,K^{(t)}b)} \frac{\sqrt{\det B^t}}{(2\pi)^{\frac{r_t}{2}}} \prod_{\alpha\in\Delta_+^t} \frac{(a,\alpha)^2}{(\rho^t,\alpha)} e^{-\frac{1}{2}(a,a)_t}$$

Here $K^{(t)} = S^{(2)}$ *restricted to the* b*-subspace,* B^t *is the Cartan matrix of* $\mathfrak{g}^t$*.*

Further studies

- truncated tensor products
  $U_q(sl_2(C))$ - a q-deformation of universal enveloping algebra, $q = \frac{2\pi i}{r}$. Irreducible modules V_l, $(l+1)$-dimensional, $l = 0, 1, \ldots, r-2$. Problem: find the asymptotic of the multiplicities in V_l^N when $r \to \infty$, $N/r, l/r$ are finite.
- superalgebras
- multiplicities of irreducible components in large tensor products of integrable modules over affine Lie algebras