Detection of conductivity inclusions in a semilinear elliptic problem arising from cardiac electrophysiology

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In collaboration with

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Outline of the presentation

1. Inverse problem and motivation
2. Phase-field approach and Reconstruction algorithm
3. Numerical results

Inverse problem and motivation

Problem formulation

Direct problem

For a fixed inclusion $\omega \subset \Omega$, introduce $K_\omega(x) = K_{out} + (K_{in} - K_{out})\chi_\omega$ with $K_{in} << K_{out}$ and define y as the solution of

$$\begin{cases} -div(K_\omega \nabla y) + \chi_{\Omega\setminus\omega} y^3 = f & \text{in } \Omega \\ K_{out}\partial_\nu y = 0 & \text{on } \partial\Omega \end{cases}$$

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Inverse problem

Given a boundary measurement y_{meas} on $\partial\Omega$, find the inclusion ω such that the solution y of the direct problem satisfies $y|_{\partial\Omega} = y_{meas}$.

Motivation

The direct problem is a simplified version of the **monodomain** model for the electrical activity of the heart:

- y: transmembrane potential;
- K: conductivity coefficient;
- **non-linear** constitutive law for ionic current: $I_{ion}(y) = y^3$;
- f is an external source of current.

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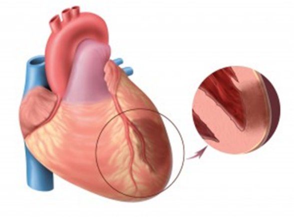

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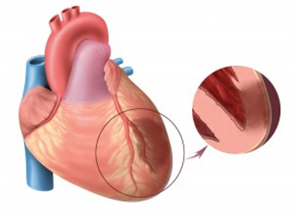

Long-term purpose

Identify the presence of ischemic regions from non invasive electrical measurements.

Phase-field approach and Reconstruction algorithm

Arbitrary inclusions

Assume ω of arbitrary shape.

$$\omega \subset \Omega \text{ is a } \textit{finite-perimeter set}, \text{ i.e. } u = \chi_\omega \in BV(\Omega)$$

Rewrite the problem in terms of u

Forward problem

$$\int_\Omega a(u)\nabla y \nabla \varphi + \int_\Omega b(u) y^3 \varphi = \int_\Omega f\varphi,$$

being $a(u) = 1 - (1-k)u$ and $b(u) = 1 - u \quad (k << 1)$.

Define the **solution map** $S : X_{0,1} \to H^1(\Omega)$, $S(u) = y$, where

$$X_{0,1} = \{v \in BV : v \in \{0,1\}, v = 0 \ a.e. \in \Omega^{d_0}\}$$

where $\Omega^{d_0} = \{x \in \Omega : d(x, \partial\Omega) \leq d_0\}$

Constrained minimization problem

Inverse problem

$$Find \quad u \in X_{0,1} \quad s.t. \quad S(u)|_{\partial\Omega} = y_{meas}$$

Goal: minimize the mismatch with the data

$$\min_{u \in X_{0,1}} J(u),$$

$$J(u) = \frac{1}{2} \int_{\partial\Omega} (S(u) - y_{meas})^2$$

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The problem is ill-posed!

Tikhonov regularization

Tikhonov regularization of the functional via the Total Variation term:

$$\min_{u \in X_{0,1}} J_{reg}(u), \quad J_{reg}(u) = \frac{1}{2} \int_{\partial\Omega} (S(u) - y_{meas})^2 + \alpha TV(u),$$

where

$$TV(u) = \sup \left\{ \int_{\Omega} u \mathrm{div}(\phi); \quad \phi \in C_0^1(\Omega; \mathbb{R}^2), \ \|\phi\|_\infty \leq 1 \right\}.$$

Relaxation

Phase-field relaxation (cf. [Deckelnick, Elliott, Styles '16]): Let

$$\mathcal{K} = \{v \in H^1(\Omega) : \ 0 \leq v \leq 1 \ a.e. \ in \ \Omega, \ v = 0 \ a.e. \ in \ \Omega^{d_0}\}$$

and, for every $\varepsilon > 0$ ($\varepsilon \simeq$ thickness of diffuse interface separating two sets on which the conductivity coefficient is constant), introduce the relaxed optimization problem:

$$\arg\min_{u \in \mathcal{K}} J_\varepsilon(u);$$

$$J_\varepsilon(u) = \frac{1}{2}\|S(u) - y_{meas}\|^2_{L^2(\partial\Omega)} + \alpha \int_\Omega \left(\varepsilon|\nabla u|^2 + \frac{1}{\varepsilon}u(1-u)\right),$$

where the **solution map** $S : \mathcal{K} \to H^1(\Omega)$, $S(u) = y$, and y solves

Forward problem

$$\int_\Omega a(u)\nabla y \nabla \varphi + \int_\Omega b(u) y^3 \varphi = \int_\Omega f \varphi,$$

being $a(u) = 1 - (1-k)u$ and $b(u) = 1 - u$.

Relaxation

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Existence

$\forall \varepsilon > 0$ there exists a minimizer of J_ε in $\mathcal{K}$.

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Stability

Fix $\alpha, \varepsilon > 0$. Let $\{y^k\} \subset L^2(\partial\Omega)$ such that $y^k \xrightarrow{L^2(\partial\Omega)} y_{meas}$ and let u_ε^k be a solution with data y^k. Then, up to a subsequence, $u_\varepsilon^k \xrightarrow{H^1} u_\varepsilon$, where u_ε is a solution with data y_{meas}.

Optimality conditions

Optimality conditions of the phase-field problem

A minimizer u_ε of J_ε satisfies the variational inequality:

$$J'_\varepsilon(u_\varepsilon)[v - u_\varepsilon] \geq 0 \qquad \forall v \in \mathcal{K};$$

$$J'_\varepsilon(u)[\vartheta] = \int_\Omega (1-k)\vartheta \nabla S(u) \cdot \nabla p + \int_\Omega \vartheta S(u)^3 p + 2\alpha\varepsilon \int_\Omega \nabla u \cdot \nabla \vartheta + \frac{\alpha}{\varepsilon} \int_\Omega (1-2u)\vartheta,$$

where p is the solution of the *adjoint problem*:

$$\int_\Omega a(u)\nabla p \cdot \nabla \psi + \int_\Omega 3b(u)S(u)^2 p\psi = \int_{\partial\Omega} (S(u) - y_{meas})\psi \qquad \forall \psi \in H^1(\Omega).$$

Reconstruction Algorithm

Introduce the Parabolic Obstacle Problem (POP):

Find $u(\cdot,t) \in \mathcal{K}$, $t \geq 0$ s.t. $u(\cdot,0) = u_0$ and

$$\int_\Omega \partial_t u(\cdot,t)(v - u(\cdot,t)) + J'_\varepsilon(u(\cdot,t))[v - u(\cdot,t)] \geq 0 \qquad \forall v \in \mathcal{K}$$

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Formally: take $v = u(\cdot, t - \Delta t)$, divide by Δt and let $\Delta t \to 0$:

$$\|u_t\|^2 + J'_\varepsilon(u)[u_t] \leq 0, \quad \text{i.e.} \quad \frac{d}{dt} J'_\varepsilon(u(\cdot, t)) \leq 0.$$

$\rightsquigarrow$ Cost functional decreases along the evolution

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Goal: discretize POP to obtain **discrete** Reconstruction Algorithm

Discrete direct problem

Let $\mathcal{T}_h$ be a shape regular triangulation of Ω and define $V_h \subset H^1(\Omega)$:

$$V_h = \{v_h \in C(\bar{\Omega}), v_h|_K \in \mathbb{P}_1(K) \ \forall K \in \mathcal{T}_h\}; \qquad \mathcal{K}_h = V_h \cap \mathcal{K}.$$

For every fixed $h > 0$, we define the (well-posed) discrete solution map $S_h : \mathcal{K} \to V_h$, where $S_h(u)$ solves

$$\int_\Omega a(u) \nabla S_h(u) \nabla v_h + \int_\Omega b(u) S_h(u)^3 v_h = \int_\Omega f v_h \quad \forall v_h \in V_h.$$

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Convergence I

Let $f \in L^2(\Omega)$. Then, for every $u \in \mathcal{K}$, $S_h(u) \to S(u)$ strongly in $H^1(\Omega)$.

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Convergence II

Let $\{h_k\}, \{u_k\}$ be two sequences such that $h_k \to 0$, $u_k \in \mathcal{K}_{h_k}$ and $u_k \xrightarrow{L^1} u \in \mathcal{K}$. Then $S_{h_k}(u_k) \xrightarrow{H^1} S(u)$.

Discrete optimization problem

Define the discrete cost functional, $J_{\varepsilon,h} : \mathcal{K}_h \to \mathbb{R}$

$$J_{\varepsilon,h}(u_h) = \frac{1}{2}\|S_h(u_h) - y_{meas,h}\|^2_{L^2(\partial\Omega)} + \alpha \int_\Omega \left(\varepsilon|\nabla u_h|^2 + \frac{1}{\varepsilon}u_h(1-u_h)\right)$$

$y_{meas,h}$ is the $L^2(\Omega)$-projection of the boundary datum y_{meas} in the space of the traces of V_h functions.

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Existence of minimizers

For each $h > 0$, there exists $u_h \in \mathcal{K}_h$ such that

$$u_h = argmin_{v_h \in \mathcal{K}_h} J_{\varepsilon,h}(v_h).$$

Every sequence $\{u_{h_k}\}$ s.t. $\lim_{k\to\infty} h_k = 0$ admits a subsequence that converges in $H^1(\Omega)$ to a minimum of the cost functional J_ε.

Discrete optimality condition

$$\boxed{u_h \in \mathcal{K}_h : \quad J'_{\varepsilon,h}(u_h)[v_h - u_h] \geq 0 \quad \forall v_h \in \mathcal{K}_h}$$

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with p_h finite element solution of discrete adjoint problem

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Convergence to continuous optimality condition

Let $\{h_k\}$ s.t. $h_k \to 0$ and u_k corresp. solution of the discrete variational inequality. Then there exists a subsequence of $\{u_k\}$ that converges a.e. and in $H^1(\Omega)$ to a solution u of the continuous optimality condition.

Discrete Reconstruction Algorithm

Continuous parabolic obstacle problem (POP):

$$\begin{cases} \int_\Omega \partial_t u(\cdot,t)(v - u(\cdot,t)) + J'_\varepsilon(u(\cdot,t))[v - u(\cdot,t)] \geq 0 & \forall v \in \mathcal{K}, \quad t \in (0,+\infty) \\ u(\cdot,0) = u_0 & \text{an initial guess in } \mathcal{K} \end{cases}$$

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Time discretization via a **semi-implicit** scheme:

$$\begin{cases} u_h^0 = u_0 \in \mathcal{K}_h \qquad \textit{(a prescribed initial datum)} \\ \boxed{u_h^{n+1} \in \mathcal{K}_h} : \tau_n^{-1} \int_\Omega (u_h^{n+1} - u_h^n)(v_h - u_h^{n+1}) + \int_\Omega (1-k)\nabla S_h(u_h^n) \cdot \nabla p_h^n (v_h - u_h^{n+1}) \\ \qquad + \int_\Omega S_h(u_h^n)^3 p_h^n (v_h - u_h^{n+1}) + 2\alpha\varepsilon \int_\Omega \nabla \mathbf{u_h^{n+1}} \cdot \nabla(v_h - u_h^{n+1}) \\ \qquad + \alpha \frac{1}{\varepsilon} \int_\Omega (1 - 2u_h^n)(v_h - u_h^{n+1}) \geq 0 \quad \forall v_h \in \mathcal{K}_h, \; n = 0, 1, \ldots \end{cases}$$

Discrete Reconstruction Algorithm

Time discretization via a **semi-implicit** scheme:

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\end{cases}
$$

Discrete Reconstruction algorithm

- Set $n = 0$ and $u_h^0 = u_0$, the initial guess for the inclusion;
- while $\left\| u_h^n - u_h^{n-1} \right\|_{L^\infty(\Omega)} > tol$
  1. compute $S(u_h^n)$ solving the discrete direct problem;
  2. compute p_h^n solving the discrete adjoint problem;
  3. update u_h^{n+1} according to the discrete POP (e.g. via Primal-Dual Active Set algorithm);
  4. update $n = n + 1$;

Properties of the discrete reconstruction algorithm

Discrete Energy dicrease

For each $n > 0$, there exists a positive constant $\mathcal{B}_n$ such that, if $\tau_n \leq \mathcal{B}_n$ it holds:

$$\left\| u_h^{n+1} - u_h^n \right\|_{L^2}^2 + J_{\varepsilon,h}(u_h^{n+1}) \leq J_{\varepsilon,h}(u_h^n) \qquad n > 0.$$

$\mathcal{B}_n = \mathcal{B}_n(\Omega, h, k, \|p_h^n\|_{H^1}, \|y_h^n\|_{H^1}, \left\|y_h^{n+1}\right\|_{H^1})$

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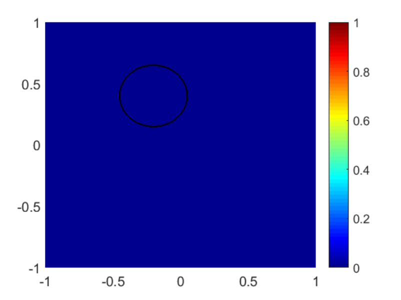
Convergence to discrete optimality conditions

There exist timesteps $\{\tau_n\}$ s.t. the sequence $\{u_h^n\}$ has a converging subsequence to u_h satisfying the discrete optimality condition.

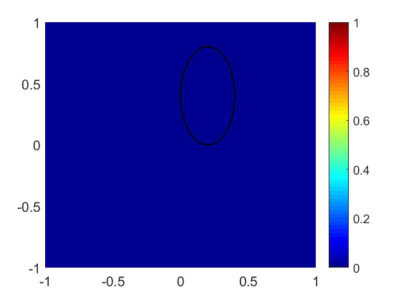
Numerical results

Numerical results

$\varepsilon = 1/(8\pi)$, $h = 0.04$, $\tau = 0.01/\varepsilon$, $\alpha = 10^{-3}$, $k = 10^{-2}$

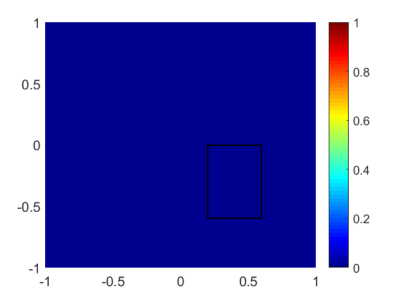


(a) Circular inclusion; 587 iterations

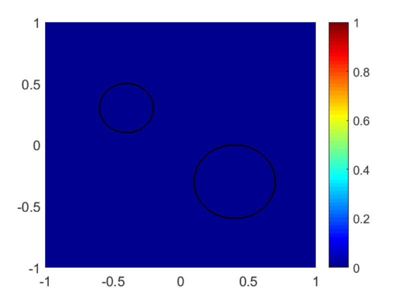


(b) Elliptical inclusion; 1497 iterations

Numerical results

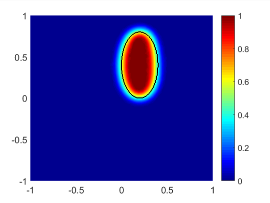


(a) Rectangular inclusion; 1272 iterations

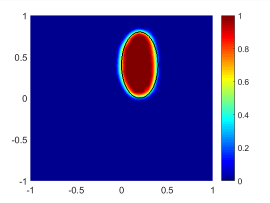


(b) Two inclusions; 4670 iterations

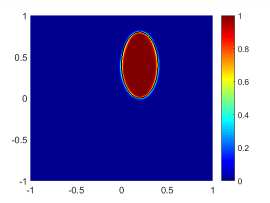
Asymptotics as $\varepsilon \to 0$



(a) $\varepsilon = \frac{1}{4\pi}$: $N_{tot} = 358$



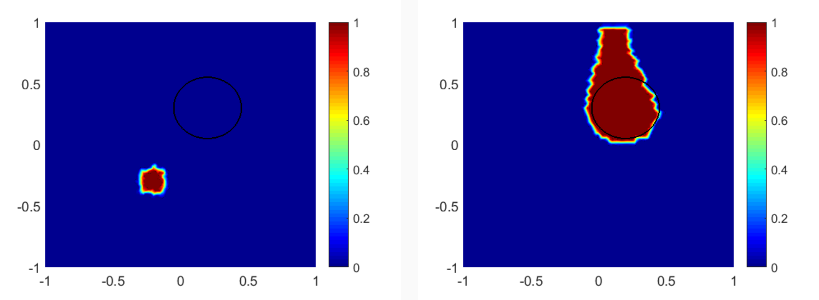
(b) $\varepsilon = \frac{1}{8\pi}$: $N_{tot} = 1500$



(c) $\varepsilon = \frac{1}{16\pi}$: $N_{tot} = 3514$

Numerical results - robustness

Initial guess

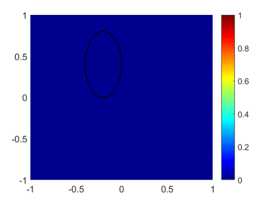


(a) Arbitrary; 661 iterations

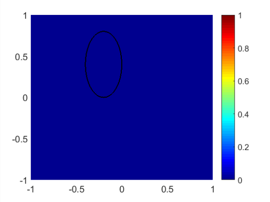
(b) Sublevel of the topological gradient of J; 489 iterations

Extra: reconstruction from noisy data

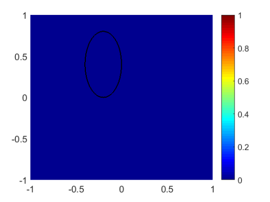
Different noise level, $\alpha = 10^{-3}$



(a) Noise level: 1%; 430 iterations



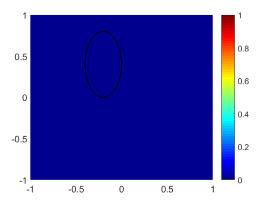
(b) Noise level: 5%; 560 iterations



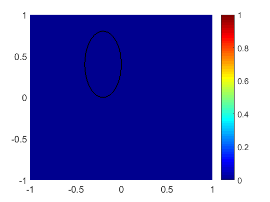
(c) Noise level: 10%; 1120 iterations

Extra: reconstruction from noisy data

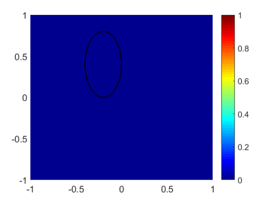
Different regularization parameters, *noiselevel* $= 10\%$



(a) $\alpha = 10^{-3}$; 1120 iterations



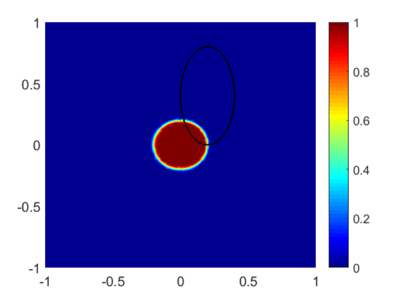
(b) $\alpha = 3 \cdot 10^{-3}$; 751 iterations



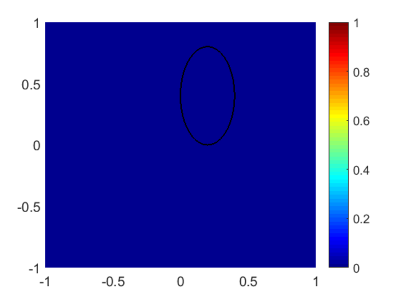
(c) $\alpha = 5 \cdot 10^{-3}$; 462 iterations

Shape Derivative approach: numerical results

Comparison with the shape gradient



(a) Shape gradient algorithm



(b) Phase field, $\varepsilon = \frac{1}{16\pi}$, mesh adaptation

Conclusions and further developments

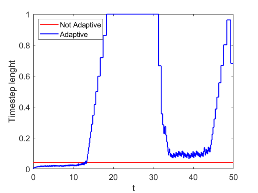
- We presented a phase field based algorithm to reconstruct inclusions in semilinear elliptic problem.
- We introduced discrete reconstruction algorithm and discussed convergence properties.
- Numerical tests show efficacy of the approach.

Conclusions and further developments

- Consider reconstrucion problem governed by *Monodomain model* $\rightsquigarrow$ system of a parabolic semilinear equation coupled with nonlinear ODE)

$$\begin{cases} \partial_t u - \nabla \cdot (M \nabla u) + f(u, w) = 0 & \text{in } \Omega \times (0, T), \\ M \partial_\nu u = 0 & \text{on } \partial\Omega \times (0, T), \\ u|_{t=0} = u_0 & \text{in } \Omega, \\ \partial_t w + g(u, w) = 0 & \text{in } \Omega \times (0, T), \\ w|_{t=0} = w_0 & \text{in } \Omega. \end{cases}$$

Challenge: reduce computational cost of the iterative reconstruction algorithm (each iteration requires solution of two parabolic eqns) $\rightsquigarrow$ a posteriori error estimates to control time and space discretization



Time step adaptivity for direct problem ($M = I$)

$\rightsquigarrow$ **cf. Luca Ratti's poster**

References

[1] E. Beretta, L. Ratti, M. Verani, *Detection of conductivity inclusions in a semilinear elliptic problem arising from cardiac electrophysiology*, Communications in Mathematical Sciences, 2018.

[2] L. Ratti, M. Verani, A posteriori error estimates for the monodomain model in cardiac electrophysiology, arxiv 1901.07468, submitted.

Constrained minimization problem

Inverse problem

$$\textit{Find} \quad u \in X_{0,1} \quad \textit{s.t.} \quad S(u)|_{\partial\Omega} = y_{meas}$$

Goal: minimize the mismatch with the data

$$\min_{u \in X_{0,1}} J(u),$$

$$J(u) = \frac{1}{2} \int_{\partial\Omega} (S(u) - y_{meas})^2$$

Continuity of the forward operator: $F : u \in X_{0,1} \to S(u)|_{\partial\Omega} \in L^2(\partial\Omega)$

If $\{u_n\} \subset X_{0,1}$ s.t. $u_n \xrightarrow{L^1} u \in X_{0,1}$, then $S(u_n)|_{\partial\Omega} \xrightarrow{L^2(\partial\Omega)} S(u)|_{\partial\Omega}$.

Issue: F is a compact operator $\Rightarrow$ The problem is ill-posed: lack of stability

Convergence

Consider $\{\varepsilon_k\}$ s.t. $\varepsilon_k \to 0$. Then, J_{ε_k} converge to J_{reg} in the sense of the $\Gamma-$convergence with respect to the L^1 norm.

As a consequence, the minimizers $\{u_{\varepsilon_k}\} \subset \mathcal{K}$ of J_{ε_k} are s.t. $u_{\varepsilon_k} \xrightarrow{L^1} u$, $u \in X_{0,1}$ minimizer of J_{reg}.