Conformal flows on spheres

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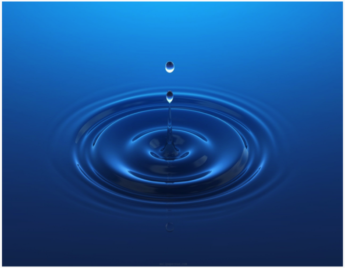
Banff, 4 July 2019



System settles down to equilibrium via dissipation of energy by dispersion

Waves keep interacting for all times, generating out-of-equilibrium dynamics

Understanding of long-time behavior of nonlinear waves in spatially confined systems is challenging. Key questions:

- How the energy injected into the system gets distributed over the degrees of freedom during the evolution?
- Can the energy flow to arbitrarily small spatial scales (*wave turbulence*)?

Examples of spatially confined systems

- Nonlinear string

$$\phi_{tt} - \phi_{xx} + \phi^3 = 0, \qquad \phi(t,0) = \phi(t,\pi) = 0$$

- Cubic Klein-Gordon equation on the d-dimensional sphere

$$\phi_{tt} - \Delta_{\mathbb{S}^d}\phi + m^2\phi + \phi^3 = 0$$

- Gross-Pitaevskii equation with isotropic harmonic potential

$$i\partial_t\psi = -\Delta\psi + |x|^2\psi + |\psi|^2\psi$$

- Vacuum Einstein equation with negative cosmological constant λ

$$R_{\mu\nu} = \lambda g_{\mu\nu}$$

General strategy

- For a spatially confined system, the associated linearized system has a purely discrete spectrum of frequencies
- Expanding solutions in the basis of linear eigenstates one transforms the original PDE into an infinite-dimensional dynamical system with discrete degrees of freedom ('modes').
- The nonlinearity generates new frequencies that may lead to resonances between the modes. The resonances dominate the transfer of energy.
- Dropping all nonresonant terms from the Hamiltonian one obtains a simplified infinite-dimensional dynamical system, called the resonant system, which accurately approximates the dynamics of small amplitude solutions of the original PDE on long time scales
- Strategy: try to understand the dynamics of the resonant system and then export this knowledge back to the original PDE.

Conformally invariant wave equations

- Consider a $(d+1)$-dimensional manifold $\mathscr{M}$ with Lorentzian metric g
- On $(\mathscr{M}, g)$ we put a real scalar field ϕ satisfying

$$\left(\Box_g - \frac{d-1}{4d} R(g)\right)\phi - \gamma|\phi|^{\frac{4}{d-1}}\phi = 0,$$

  where $\Box_g = g^{\mu\nu}\nabla_\mu\nabla_\nu$ and $R(g)$ is the Ricci scalar
- This equation is invariant under conformal transformations

$$g \mapsto \Omega^2 g, \qquad \phi \mapsto \Omega^{\frac{1-d}{2}}\phi$$

- We restrict to analytic nonlinearities: cubic ($d=3$) and quintic ($d=2$)
- For small amplitude solutions the sign of γ is irrelevant. We set $\gamma = 1$

Conformally invariant cubic wave equation on $\mathbb{S}^3$

- Let $\mathscr{M} = \mathbb{R} \times \mathbb{S}^3$ (Einstein cylinder) with the metric

$$g = -dt^2 + dx^2 + \sin^2 x\, d\omega^2, \quad (t, x, \omega) \in \mathbb{R} \times [0, \pi] \times \mathbb{S}^2$$

  This spacetime has constant scalar curvature $R(g) = 6$

- The conformally invariant cubic wave equation

$$\phi_{tt} - \Delta_{\mathbb{S}^3} \phi + \phi + \phi^3 = 0$$

- Remark: eigenvalues of $-\Delta_{\mathbb{S}^{d-1}} + m^2$ are $\omega_n^2 = n(n+d-2) + m^2$
- Almost global existence for small smooth initial data for almost all values of m^2 (Bambusi-Delort-Grébert-Szeftel, 2005)
- We assume that $\phi = \phi(t, x)$. Then $u(t, x) = \sin(x)\phi(t, x)$ satisfies

$$u_{tt} - u_{xx} + \frac{u^3}{\sin^2 x} = 0, \qquad u(t, 0) = u(t, \pi) = 0$$

- Linear eigenstates: $e_n(x) = \sqrt{\frac{2}{\pi}} \sin(\omega_n x)$ with $\omega_n = n + 1$ $(n = 0, 1, 2, \ldots)$

Time averaging

- Expanding $u(t,x) = \varepsilon \sum_{n=0}^{\infty} c_n(t) e_n(x)$ we get

$$\frac{d^2 c_n}{dt^2} + \omega_n^2 c_n = \varepsilon^2 \sum_{j,k,l \geq 0} S_{njkl}\, c_j c_k c_l, \quad S_{jkln} = \int_0^{\pi} \frac{dx}{\sin^2 x} e_n(x) e_j(x) e_k(x) e_l(x)$$

- Using variation of constants

$$c_n = \alpha_n e^{i\omega_n t} + \bar{\alpha}_n e^{-i\omega_n t}, \qquad \frac{dc_n}{dt} = i\omega_n \left(\alpha_n e^{i\omega_n t} - \bar{\alpha}_n e^{-i\omega_n t}\right)$$

  we factor out fast oscillations

$$2i\omega_n \frac{d\alpha_n}{dt} = \varepsilon^2 \sum_{j,k,l \geq 0} S_{njkl}\, c_j c_k c_l\, e^{-i\omega_n t},$$

- Each term in the sum has a factor $e^{-i\Omega_{njkl} t}$, where

$$\Omega_{njkl} = \omega_n \pm \omega_j \pm \omega_k \pm \omega_l$$

  The terms with $\Omega_{njkl} = 0$ correspond to resonant interactions

- Let $\tau = \varepsilon^2 t$. For $\varepsilon \to 0$ the non-resonant terms $\propto e^{-i\Omega_{njkl}\tau/\varepsilon^2}$ are highly oscillatory and therefore negligible.

Cubic conformal flow

- Keeping only the resonant terms we obtain the cubic conformal flow (B-Craps-Evnin-Hunik-Luyten-Maliborski, 2016)

$$i\,(n+1)\frac{d\alpha_n}{d\tau} = \sum_{\substack{j,k,l\geq 0\\ n+j=k+l}} S_{njkl}\,\bar{\alpha}_j\alpha_k\alpha_l\,,$$

  where $S_{njkl} = \min\{n,j,k,l\}+1$

- This system provides an accurate approximation to the conformally invariant cubic wave equation on the timescale $\sim \varepsilon^{-2}$

- This is a Hamiltonian system

$$i\,(n+1)\frac{d\alpha_n}{d\tau} = \frac{\partial H}{\partial\bar{\alpha}_n}$$

  with

$$H = \frac{1}{2}\sum_{\substack{n,j,k,l\geq 0\\ n+j=k+l}} S_{njkl}\bar{\alpha}_n\bar{\alpha}_j\alpha_k\alpha_l$$

Quintic conformal flow

- Conformally invariant quintic wave equation on $\mathbb{S}^2$

$$\phi_{tt} - \Delta_{\mathbb{S}^2}\phi + \frac{1}{4}\phi + \phi^5 = 0$$

- Assuming that $\phi = u(t,x)$, where $x = \cos\vartheta$, we get

$$u_{tt} - \partial_x\left((1-x^2)u_x\right) + \frac{1}{4}u + u^5 = 0$$

- Linear eigenstates: $e_n(x) = P_n(x)$ with $\omega_n = n + \frac{1}{2}$ $(n = 0, 1, 2, \ldots)$
- Time-averaging gives the quintic conformal flow (for slow time $\tau = \varepsilon^4 t$) (Biasi-B-Evnin, 2019)

$$i\frac{d\alpha_n}{d\tau} = \sum_{n+j+k=l+m+i} S_{njklmi}\bar{\alpha}_j\bar{\alpha}_k\alpha_l\alpha_m\alpha_i$$

$$S_{njklmi} = \int_{-1}^{1} P_n(x)P_j(x)P_k(x)P_l(x)P_m(x)P_i(x)\,dx$$

Other Hamiltonian systems of the form

$$i\,\frac{d\alpha_n}{d\tau} = \sum_{\substack{j,k,l\geq 0\\ n+j=k+l}} S_{njkl}\,\bar{\alpha}_j\alpha_k\alpha_l$$

- Cubic Szegő equation $S_{njkl} = 1$ (Gérard-Grellier, 2010)
- Lowest Landau Level (LLL) equation: resonant system for the maximally rotating Bose-Einstein condensate (Germain-Hani-Thomann, 2015)

$$S_{njkl} = \frac{(n+j)!}{2^{n+j}\sqrt{n!j!k!l!}}$$

- Resonant system for radial scalar perturbations of AdS$_{d+1}$ spacetime (Balasubramanian et al., Craps-Evnin-Vanhoof, 2014)
- Schrödinger-Newton-Hooke (SNH) system: resonant system for a non-relativistic self-gravitating condensate (B-Evnin-Ficek, 2018)

Aside on anti-de Sitter space

- Anti-de Sitter (AdS) metric in d spatial dimensions

$$g = \frac{l^2}{\cos^2 x}\left(-dt^2 + dx^2 + \sin^2 x\, d\omega^2_{S^{d-1}}\right)$$

- AdS metric is the unique globally regular static solution of the vacuum Einstein equations $R_{\alpha\beta} = \lambda g_{\alpha\beta}$ with $\lambda = -d/l^2$.
- AdS space is the ground state among spacetimes with negative λ (much as Minkowski space is the ground state among spacetimes with $\lambda = 0$)
- Conjecture: AdS space is unstable under arbitrarily small perturbations
- Mechanism of instability: transfer of energy from low to high frequencies (due to the fully resonant spectrum of linearized perturbations)
- For $d \geq 3$ the resonant flow becomes singular in finite time (for some initial data). For $d = 2$ the dynamics appears weakly turbulent.

Back to the cubic conformal flow

$$i\,(n+1)\frac{d\alpha_n}{d\tau} = \sum_{\substack{j,k,l\geq 0\\ n+j=k+l}} (\min\{n,j,k,l\}+1)\,\bar{\alpha}_j\alpha_k\alpha_l$$

- Symmetries

$$\begin{aligned}
\text{Scaling:} \qquad & \alpha_n(\tau) \to \lambda\,\alpha_n(\lambda^2\tau)\\
\text{Global phase rotation:} \qquad & \alpha_n(\tau) \to e^{i\theta}\alpha_n(\tau)\\
\text{Local phase rotation:} \qquad & \alpha_n(\tau) \to e^{in\theta}\alpha_n(\tau)
\end{aligned}$$

- Conserved quantities due to the phase rotation symmetries

$$Q = \sum_{n=0}^{\infty}(n+1)|\alpha_n|^2, \qquad E = \sum_{n=0}^{\infty}(n+1)^2|\alpha_n|^2$$

- Additional (complex) conserved quantity

$$Z = \sum_{n=0}^{\infty}(n+1)(n+2)\bar{\alpha}_n\alpha_{n+1}$$

Energy transfer

- Key question: how the energy of initial data gets distributed over the modes during evolution? Does energy flow from low to high modes?
- Given a sequence $\alpha = (\alpha_0, \alpha_1, \ldots)$, we define the norm

$$\|\alpha\|_s^2 = \sum_{n=0}^{\infty} (n+1)^{2s} |\alpha_n|^2$$

- Do there exist weakly turbulent solutions, i.e. global solutions $\alpha(\tau)$ s.t.

$$\limsup_{\tau \to \infty} \|\alpha(\tau)\|_s = +\infty \quad \text{for some } s > 1$$

- Such solutions exist for the cubic Szegő equation (Gérard-Grellier, 2017)
- Ultraviolet asymptotics of the interaction coefficients for the cubic conformal flow suggests that the transfer of energy to high frequencies is less efficient than for the cubic Szegő equation

Finite-dimensional invariant manifolds

- For one-mode initial data $\alpha_n(0) = \delta_{nN}$ the solution is $\alpha_n(\tau) = \delta_{nN} e^{-i\tau}$
- Three-dimensional invariant manifold

$$\alpha_n = (b + an)p^n$$

with complex-valued functions $a(\tau), b(\tau), p(\tau)$

- The dynamics of the invariant manifold is described by a reduced Hamiltonian system

$$\frac{da}{d\tau} = f_1(a,b,p), \quad \frac{db}{d\tau} = f_2(a,b,p), \quad \frac{dp}{d\tau} = f_3(a,b,p)$$

- The reduced system is completely integrable thanks to the three conserved quantities Q, E, and H (that are in involution)

- The reduced system (here $y = \frac{|p|^2}{1-|p|^2}$)

$$\frac{i\dot{p}}{(1+y)^2} = \frac{p}{6}\left(2y|a|^2 + \bar{b}a\right)$$
$$\frac{i\dot{a}}{(1+y)^2} = \frac{a}{6}\left(5|b|^2 + (18y^2+4y)|a|^2 + (6y-1)\bar{b}a + 10y\bar{a}b\right)$$
$$\frac{i\dot{b}}{(1+y)^2} = b\left(|b|^2 + (6y^2+2y)|a|^2 + 2yb\bar{a}\right) + a\left(2y|b|^2 + (4y+2)^2|a|^2 + y^2\bar{b}a\right)$$

- This can be solved exactly, in particular

$$y(\tau) = B + A\cos(\Omega\tau)$$

  where the constants A, B, Ω are determined by initial data

- The turning points $y_\pm = B \pm A$ provide lower and upper bounds for the inverse and direct cascades of energy

- Here y_+ is uniformly bounded from above (in contrast to the cubic Szegő)

Complex-plane representation

- In terms of the generating function $u(\tau,z) = \sum_{n=0}^{\infty} \alpha_n(\tau) z^n$, the cubic conformal flow is equivalent to

$$i\partial_\tau \partial_z (zu) = \frac{1}{2\pi i} \oint_{|\zeta|=1} \frac{d\zeta}{\zeta} \overline{u(\tau,\zeta)} \left(\frac{\zeta u(\tau,\zeta) - z u(\tau,z)}{\zeta - z} \right)^2$$

- This formulation is convenient in some calculations
- On the three-dimensional invariant subspace

$$u(t,z) = \frac{b(t)}{1-p(t)z} + \frac{a(t)p(t)z}{(1-p(t)z)^2}$$

the radius of analyticity (the distance of the closest pole to the unit circle) is uniformly bounded from below

Stationary states

- Solutions of the form $\alpha_n(t) = A_n e^{-i(\lambda - n\omega)t}$ are called *stationary states*.
- Stationary states are the critical points of the functional

$$K = \frac{1}{2}H - \lambda Q + \omega(E - Q)$$

- The conformal flow has a variety of stationary states. Examples:
  - Single-mode states $u(t,z) = c\, z^N e^{-i|c|^2 t}$
  - "Ground state"

$$u(t,z) = \frac{c}{1-pz} \exp\left(-\frac{i|c|^2 t}{(1-|p|^2)^2}\right), \qquad |p| < 1$$

    This state saturates the inequality $H \leq Q^2$ which plays a key role in the proof of its orbital stability (B-Hunik-Pelinovsky, 2018)
  - Blaschke products

$$u(t,z) = c \prod_{k=1}^{N} \frac{\bar{p}_k - z}{1 - p_k z}\, e^{-i|c|^2 t}, \qquad |p_k| < 1$$

Some open problems

- Is the cubic conformal flow integrable?
- Are there weakly turbulent solutions?
- Are there higher dimensional invariant subspaces?
- Classification of stationary states
- What are the implications for the original PDEs?
    - Time-periodic solutions (Bambusi-Paleari)
    - Soliton resolution?