Threshold for blowup in supercritical wave equations

Birgit Schörkhuber (Karlsruhe Institute of Technology) [1]

Joint work with

Irfan Glogić (University of Vienna)

and Maciej Maliborski (University of Vienna)

Workshop on
Dynamics in Geometric Dispersive Equations and the Effects of Trapping, Scattering and Weak Turbulence
Banff, 04.02.2020

[1] Funded by DFG - Project-ID 258734477 - SFB 1173

Wave maps into the sphere

- Wave maps: $u : \mathbb{R}^{1,n} \to \mathbb{S}^n \subset \mathbb{R}^{n+1}$

$$S(u) = \int_{\mathbb{R}^{1,n}} \partial^\mu u \cdot \partial_\mu u$$

Critical points satisfy

$$(\partial_t^2 - \Delta_x) u(t,x) = u(t,x) \left(|\nabla u(t,x)|^2 - |\partial_t u(t,x)|^2 \right)$$

- Wave maps: $u : \mathbb{R}^{1,n} \to \mathbb{S}^n \subset \mathbb{R}^{n+1}$

$$S(u) = \int_{\mathbb{R}^{1,n}} \partial^\mu u \cdot \partial_\mu u$$

Critical points satisfy

$$(\partial_t^2 - \Delta_x) u(t,x) = u(t,x) \left(|\nabla u(t,x)|^2 - |\partial_t u(t,x)|^2 \right)$$

- Co-rotational maps:

$$u(t, r\omega) = \begin{pmatrix} \sin\psi(t,r)\omega \\ \cos\psi(t,r) \end{pmatrix}$$

$$\left(\partial_t^2 - \partial_r^2 - \frac{n-1}{r}\partial_r \right) \psi(t,r) + \frac{(n-1)\sin(2\psi(t,r))}{2r^2} = 0$$

Scaling $\psi_\lambda(t,r) = \psi(t/\lambda, r/\lambda)$, $\lambda > 0$

$$E(\psi)(t) = \int_0^\infty \left(|\partial_t \psi(t,r)|^2 + |\partial_r \psi(t,r)|^2 + \frac{(n-1)\sin^2(\psi(t,r)}{r^2} \right) r^{n-1} dr$$

$E(\psi_\lambda) = \lambda^{n-2} E(\psi) \Rightarrow$ energy supercritical in $n \geq 3$

- Self-similar blowup: (Shatah '88)

$$\psi(t,r) = U\left(\tfrac{r}{T-t}\right), \quad T > 0$$

Ground state profile (Turok-Spergel '90, Biernat-Bizoń '15)

$$U_0(\rho) = 2\arctan\left(\tfrac{\rho}{\sqrt{n-2}}\right), \quad T > 0$$

Numerical experiments: Generic blowup profile described by U_0 (Biernat-Chmaj-Tabor '00)

- Self-similar blowup: (Shatah '88)

$$\psi(t,r) = U\left(\tfrac{r}{T-t}\right), \quad T > 0$$

Ground state profile (Turok-Spergel '90, Biernat-Bizoń '15)

$$U_0(\rho) = 2\arctan\left(\tfrac{\rho}{\sqrt{n-2}}\right), \quad T > 0$$

Numerical experiments: Generic blowup profile described by U_0 (Biernat-Chmaj-Tabor '00)

- Stable blowup behavior: Nonlinear asymptotic stability of the ground state under small co-rotational perturbations

$n = 3$, local (Donninger-S.-Aichelburg '12, Donninger '11, Costin-Donninger-Xia '16)

odd $n \geq 5$, local (Costin-Donninger-Glogić '17, Chatzikaleas-Donninger-Glogić '17)

global, $n = 3$ (Biernat-Donninger-S. '20)

- $3 \leq n \leq 6$: Infinitely many self-similar solutions
  Existence of smooth profiles $\{U_k\}_{k \in \mathbb{N}_0}$ (Bizoń '99, Biernat-Bizoń-Maliborski '17)

Self-similar blowup solutions - Threshold phenomena

- $3 \leq n \leq 6$: Infinitely many self-similar solutions
  Existence of smooth profiles $\{U_k\}_{k \in \mathbb{N}_0}$ (Bizoń '99, Biernat-Bizoń-Maliborski '17)
- Threshold for blowup:
  Numerical experiments $3 \leq n \leq 6$: Self-similar profile U_1 intermediate attractor appears close to threshold for blowup (Biernat-Chmaj-Tabor '00, Biernat-Bizoń-Maliborski '17)

- $3 \leq n \leq 6$: Infinitely many self-similar solutions
  Existence of smooth profiles $\{U_k\}_{k\in\mathbb{N}_0}$ (Bizoń '99, Biernat-Bizoń-Maliborski '17)
- Threshold for blowup:
  Numerical experiments $3 \leq n \leq 6$: Self-similar profile U_1 intermediate attractor appears close to threshold for blowup (Biernat-Chmaj-Tabor '00, Biernat-Bizoń-Maliborski '17)
- Remark on $n \geq 7$: non-self-similar blowup Type II blowup solutions (Ghoul-Ibrahim-Nguyen '18)

$$\psi(t,r) \sim Q\left(\frac{r}{\lambda(t)}\right)$$

Self-similar blowup solutions - Threshold phenomena

- $3 \le n \le 6$: Infinitely many self-similar solutions
  Existence of smooth profiles $\{U_k\}_{k\in\mathbb{N}_0}$ (Bizoń '99, Biernat-Bizoń-Maliborski '17)
- Threshold for blowup:
  Numerical experiments $3 \le n \le 6$: Self-similar profile U_1 intermediate attractor appears close to threshold for blowup (Biernat-Chmaj-Tabor '00, Biernat-Bizoń-Maliborski '17)
- Remark on $n \ge 7$: non-self-similar blowup Type II blowup solutions (Ghoul-Ibrahim-Nguyen '18)

$$\psi(t,r) \sim Q\left(\frac{r}{\lambda(t)}\right)$$

- Toy model for co-rotational wave maps: For $\psi = ru$

$$\left(\partial_t^2 - \partial_r^2 - \frac{n+1}{r}\partial_r\right) u(t,r) = u(t,r)^3 F(ru(t,r))$$

with F smooth, bounded and non-negative.

$\Rightarrow$ Toy model: focusing cubic wave equation

The focusing non-linear wave equation

- Focusing cubic wave equation in $d \geq 5$

$$\left(\partial_t^2 - \partial_r^2 - \frac{d-1}{r}\partial_r\right) u(t,r) = u(t,r)^3$$

- Scale invariance:

$$u_\lambda(t,r) = \lambda^{-1} u(t/\lambda, r/\lambda), \quad \lambda > 0$$

- Self-similar blowup solutions:

$$u(t,x) = (T-t)^{-1} U\left(\tfrac{r}{T-t}\right), \quad T > 0$$

- Stable blowup behavior: ODE blowup

$$U_0(\rho) = \sqrt{2}$$

  $d \geq 5$ odd: stable blowup in backward lightcone (Donninger-S. '17)

- Non-trivial self-similar blowup:
  $d < 13$: Numerical experiments, $\{U_k\}_{k \in \mathbb{N}_0}$ (Kycia '11)
  $d \geq 13$: Non-self-similar blowup solutions (Collot '13)

Explicit self-similar solution for $d \geq 5$

$$u_T^*(t,r) = (T-t)^{-1} U^* \left(\tfrac{r}{T-t}\right), \quad U^*(\rho) = \frac{2\sqrt{2(d-1)(d-4)}}{d-4+3\rho^2}$$

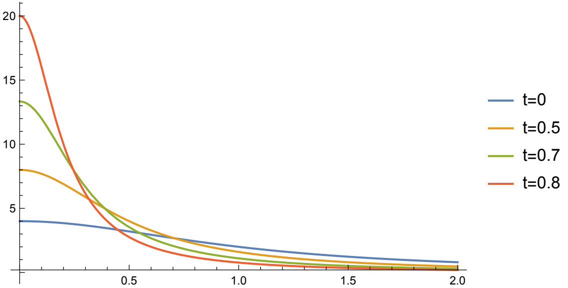


Figure: Blowup solution $u_1^*(t,r) = (1-t)^{-1} U^*(\frac{r}{1-t})$ for $d = 7$

Theorem (Glogić-S.)

$d = 7$: u_T^* is asymptotically stable under small (non-radial) peturbation satisfying a co-dimension one condition.

Theorem (Glogić-S.)

$d = 7$: u_T^* is asymptotically stable under small (non-radial) peturbation satisfying a co-dimension one condition.

Numerical experiments (Maliborski-Glogić-S.)

- Generic data with "small" amplitude $a > 0 \Rightarrow$ dispersion
- Generic data with "large" amplitude $a > 0 \Rightarrow$ finite-time blowup
- Fine-tune to threshold $a \sim a^*$: u_T^* intermediate attractor in evolution

Theorem (Glogić-S.)

$d = 7$: u_T^* is asymptotically stable under small (non-radial) peturbation satisfying a co-dimension one condition.

Numerical experiments (Maliborski-Glogić-S.)

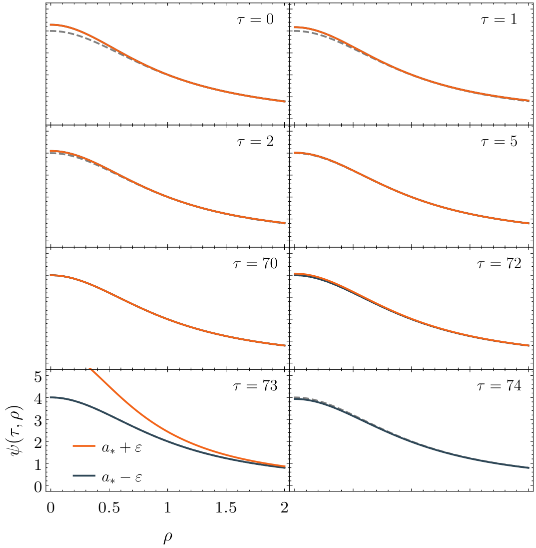
- Generic data with "small" amplitude $a > 0 \Rightarrow$ dispersion
- Generic data with "large" amplitude $a > 0 \Rightarrow$ finite-time blowup
- Fine-tune to threshold $a \sim a^*$: u_T^* intermediate attractor in evolution

Conjecture

u_T^* describes a threshold for singularity formation

Threshold behavior

$d = 7$: Evolution for near critical data in self-similar variables (τ, ρ)



Co-dimension 1 stable blowup (radial case)

- Study small perturbations of blowup data: Fix $T = 1$ and study evolution for

$$u(0,\cdot) = u_1^*(0,\cdot) + f, \quad \partial_t u(0,\cdot) = \partial_t u_1^*(0,\cdot) + g$$

- Restriction to backward lightcone

$$\mathcal{C}_T = \{(t,r) : 0 \leq r \leq T - t, \quad t \in [0,T)\}$$

- Similarity coordinates

$$\rho = \frac{r}{T-t}, \quad \tau = -\log(T-t) + \log T$$

Set $u(t,r) = (T-t)^{-1} v(-\log(T-t) + \log T, \frac{r}{T-t})$

- Tranformation of blowup solution: $u_T^*(t,r) \mapsto$ static solution $U^*(\rho)$
- Ansatz: $v(\tau,\rho) = U^*(\rho) + \varphi(\tau,\rho)$

$$\left(\partial_\tau^2 + 3\partial_\tau + 2\rho\partial_\rho\partial_\tau - \Delta_\rho + \rho^2\partial_\rho^2 + 4\rho\partial_\rho + 2 - V(\rho)\right)\varphi(\tau,\rho) = N(\varphi(\tau,\rho))$$

$V(\rho) = 3U^*(\rho)^2$ and $N(\varphi) = (U^* + \varphi)^3 - 3{U^*}^2\varphi$

Co-dimension 1 stable blowup (radial case)

- Abstract evolution equation for perturbation:

$$\partial_\tau \Phi(\tau) = (\mathbf{L}_0 + \mathbf{L}')\Phi(\tau) + \mathbf{N}(\Phi(\tau)), \quad \tau > 0$$

Transformed initial data: $\Phi(0) = \mathbf{U}((f,g),T)$

- Function space:

$$\mathcal{H} := H^k_{\text{rad}} \times H^{k-1}_{\text{rad}}(\mathbb{B}^d), \quad k = \frac{d}{2} - \frac{1}{2} > s_c = \frac{d}{2} - 1$$

Free wave evolution

$$\|\mathbf{S}_0(\tau)\mathbf{u}\|_{\mathcal{H}} \lesssim e^{-\frac{1}{2}\tau}\|\mathbf{u}\|_{\mathcal{H}} \quad \forall \tau \geq 0$$

Linearized evolution: $\mathbf{L}$ generates semigroup $\{\mathbf{S}(\tau) : \tau \geq 0\}$ on $\mathcal{H}$

- Spectral problem: Unstable spectrum of $\mathbf{L}$ given by finitely many isolated eigenvalues $\Rightarrow$ reduces to ODE problem
- Spectral ODE: $\lambda \in \mathbb{C}$, $\text{Re}\lambda \geq 0$ that allow for *smooth* solutions of

$$(1-\rho^2)f''(\rho) + \left[\frac{d-1}{\rho} - 2(\lambda+2)\rho\right] f'(\rho) - [(\lambda+1)(\lambda+2) - V(\rho)]\, f(\rho) = 0,$$

for $\rho \in [0,1]$, where $V(\rho) = 3U^*(\rho)^2$.

Co-dimension 1 stable blowup (radial case)

- Abstract evolution equation for perturbation:

$$\partial_\tau \Phi(\tau) = (\mathbf{L}_0 + \mathbf{L}')\Phi(\tau) + \mathbf{N}(\Phi(\tau)), \quad \tau > 0$$

  Transformed initial data: $\Phi(0) = \mathbf{U}((f,g),T)$

- Function space:

$$\mathcal{H} := H^k_{\text{rad}} \times H^{k-1}_{\text{rad}}(\mathbb{B}^d), \quad k = \tfrac{d}{2} - \tfrac{1}{2} > s_c = \tfrac{d}{2} - 1$$

  Free wave evolution

$$\|\mathbf{S}_0(\tau)\mathbf{u}\|_{\mathcal{H}} \lesssim e^{-\frac{1}{2}\tau}\|\mathbf{u}\|_{\mathcal{H}} \quad \forall \tau \geq 0$$

  Linearized evolution: $\mathbf{L}$ generates semigroup $\{\mathbf{S}(\tau) : \tau \geq 0\}$ on $\mathcal{H}$

- Spectral problem: Unstable spectrum of $\mathbf{L}$ given by finitely many isolated eigenvalues $\Rightarrow$ reduces to ODE problem

- Spectral ODE: $\lambda \in \mathbb{C}$, $\text{Re}\lambda \geq 0$ that allow for *smooth* solutions of

$$(1-\rho^2)f''(\rho) + \left[\tfrac{d-1}{\rho} - 2(\lambda+2)\rho\right] f'(\rho) - [(\lambda+1)(\lambda+2) - V(\rho)]\, f(\rho) = 0,$$

  for $\rho \in [0,1]$, where $V(\rho) = 3U^*(\rho)^2$.

Co-dimension 1 stable blowup (radial case)

- Abstract evolution equation for perturbation:

$$\partial_\tau \Phi(\tau) = (\mathbf{L}_0 + \mathbf{L}')\Phi(\tau) + \mathbf{N}(\Phi(\tau)), \quad \tau > 0$$

Transformed initial data: $\Phi(0) = \mathbf{U}((f,g),T)$

- Function space:

$$\mathcal{H} := H^k_{\text{rad}} \times H^{k-1}_{\text{rad}}(\mathbb{B}^d), \quad k = \frac{d}{2} - \frac{1}{2} > s_c = \frac{d}{2} - 1$$

Free wave evolution

$$\|\mathbf{S}_0(\tau)\mathbf{u}\|_{\mathcal{H}} \lesssim e^{-\frac{1}{2}\tau}\|\mathbf{u}\|_{\mathcal{H}} \quad \forall \tau \geq 0$$

Linearized evolution: $\mathbf{L}$ generates semigroup $\{\mathbf{S}(\tau) : \tau \geq 0\}$ on $\mathcal{H}$

- Spectral problem: Unstable spectrum of $\mathbf{L}$ given by finitely many isolated eigenvalues $\Rightarrow$ reduces to ODE problem

- Spectral ODE: $\lambda \in \mathbb{C}$, $\text{Re}\lambda \geq 0$ that allow for *smooth* solutions of

$$(1-\rho^2)f''(\rho) + \left[\frac{d-1}{\rho} - 2(\lambda+2)\rho\right] f'(\rho) - \left[(\lambda+1)(\lambda+2) - V(\rho)\right] f(\rho) = 0,$$

for $\rho \in [0,1]$, where $V(\rho) = 3U^*(\rho)^2$.

Co-dimension 1 stable blowup (radial case)

- Abstract evolution equation for perturbation:

$$\partial_\tau \Phi(\tau) = (\mathbf{L}_0 + \mathbf{L}')\Phi(\tau) + \mathbf{N}(\Phi(\tau)), \quad \tau > 0$$

Transformed initial data: $\Phi(0) = \mathbf{U}((f,g),T)$

- Function space:

$$\mathcal{H} := H^k_{\text{rad}} \times H^{k-1}_{\text{rad}}(\mathbb{B}^d), \quad k = \tfrac{d}{2} - \tfrac{1}{2} > s_c = \tfrac{d}{2} - 1$$

Free wave evolution

$$\|\mathbf{S}_0(\tau)\mathbf{u}\|_{\mathcal{H}} \lesssim e^{-\frac{1}{2}\tau}\|\mathbf{u}\|_{\mathcal{H}} \quad \forall \tau \geq 0$$

Linearized evolution: $\mathbf{L}$ generates semigroup $\{\mathbf{S}(\tau) : \tau \geq 0\}$ on $\mathcal{H}$

- Spectral problem: Unstable spectrum of $\mathbf{L}$ given by finitely many isolated eigenvalues $\Rightarrow$ reduces to ODE problem
- Spectral ODE: $\lambda \in \mathbb{C}$, $\text{Re}\lambda \geq 0$ that allow for *smooth* solutions of

$$(1-\rho^2)f''(\rho) + \left[\tfrac{d-1}{\rho} - 2(\lambda+2)\rho\right] f'(\rho) - \left[(\lambda+1)(\lambda+2) - V(\rho)\right] f(\rho) = 0,$$

for $\rho \in [0,1]$, where $V(\rho) = 3U^*(\rho)^2$.

Co-dimension 1 stable blowup (radial case)

- Symmetry eigenvalue: Time-translation $\lambda_0 = 1$ (for all $d \geq 5$)

$$f_0(\rho) = \frac{d-4-3\rho^2}{(d-4+3\rho^2)^2},$$

- Numerical evidence for genuine instability $\lambda_1 > 0$

d	λ_1	λ_0	λ_{-1}
5	4.37213	1	-0.53721
6	3.39524	1	-0.54896
7	3.00000	1	-0.55242
8	2.782 00	1	-0.55388
9	2.642 96	1	-0.55462

$d = 7$: Explicit solution for $\lambda_1 = 3$: $f_1(\rho) = \frac{1}{(1+\rho^2)^2}$

- Spectrum of $\mathbf{L}$: in $d = 7$ we can prove that

$$\sigma(\mathbf{L}) \subset \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda \leq -\omega_0\} \cup \{1, 3\}$$

where 1 and 3 are eigenvalues with eigenfunctions (f_0, g_0) and (f_1, g_1)

Co-dimension 1 stable blowup (radial case)

- Symmetry eigenvalue: Time-translation $\lambda_0 = 1$ (for all $d \geq 5$)

$$f_0(\rho) = \frac{d-4-3\rho^2}{(d-4+3\rho^2)^2},$$

- Numerical evidence for genuine instability $\lambda_1 > 0$

d	λ_1	λ_0	λ_{-1}
5	4.37213	1	-0.53721
6	3.39524	1	-0.54896
7	3.00000	1	-0.55242
8	2.782 00	1	-0.55388
9	2.642 96	1	-0.55462

$d = 7$: Explicit solution for $\lambda_1 = 3$: $f_1(\rho) = \frac{1}{(1+\rho^2)^2}$

- Spectrum of $\mathbf{L}$: in $d = 7$ we can prove that

$$\sigma(\mathbf{L}) \subset \{\lambda \in \mathbb{C} : \text{Re}\lambda \leq -\omega_0\} \cup \{1, 3\}$$

where 1 and 3 are eigenvalues with eigenfunctions (f_0, g_0) and (f_1, g_1)

Co-dimension 1 stable blowup (radial case)

- Symmetry eigenvalue: Time-translation $\lambda_0 = 1$ (for all $d \geq 5$)

$$f_0(\rho) = \frac{d-4-3\rho^2}{(d-4+3\rho^2)^2},$$

- Numerical evidence for genuine instability $\lambda_1 > 0$

d	λ_1	λ_0	λ_{-1}
5	4.37213	1	-0.53721
6	3.39524	1	-0.54896
7	3.00000	1	-0.55242
8	2.782 00	1	-0.55388
9	2.642 96	1	-0.55462

$d = 7$: Explicit solution for $\lambda_1 = 3$: $f_1(\rho) = \frac{1}{(1+\rho^2)^2}$

- Spectrum of $\mathbf{L}$: in $d = 7$ we can prove that

$$\sigma(\mathbf{L}) \subset \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda \leq -\omega_0\} \cup \{1, 3\}$$

where 1 and 3 are eigenvalues with eigenfunctions (f_0, g_0) and (f_1, g_1)

Co-dimension 1 stable blowup (radial case)

- Bounds for linearized evolution: Spectral projections $\mathbf{P}_0$, $\mathbf{P}_1$.

$$\mathbf{S}(\tau)\mathbf{P}_0\mathbf{u} = e^{\tau}\mathbf{P}_0\mathbf{u}, \quad \mathbf{S}(\tau)\mathbf{P}_1\mathbf{u} = e^{3\tau}\mathbf{P}_1\mathbf{u}$$

$$\|\mathbf{S}(\tau)[\mathbf{I}-\mathbf{P}_0-\mathbf{P}_1]\mathbf{u}\|_{\mathcal{H}} \lesssim e^{-\omega\tau}\|[\mathbf{I}-\mathbf{P}_0-\mathbf{P}_1]\mathbf{u}\|_{\mathcal{H}}$$

- Nonlinear problem:

$$\Phi(\tau) = \mathbf{S}(\tau)\mathbf{U}((f,g),T) + \int_0^{\tau}\mathbf{S}(\tau-s)\mathbf{N}(\Phi(s))ds$$

in $\mathcal{X}_\delta$ defined s.t $\|\Phi(\tau)\|_{\mathcal{H}} \leq \delta e^{-\omega\tau}$ Control of unstable behavior:

$\lambda_0 = 1$: Variation of blowup time $T > 0$

$\lambda_1 = 3$: Correction of the initial data along unstable direction $\mathbf{h}_1$

$$(f,g) + \alpha\,(f_1, g_1), \quad \alpha \in \mathbb{R}$$

Theorem (Glogić-S. (radial version))

Let $d = 7$ *and*

$$f_1(r) = (1+r^2)^{-2}, \quad g_1(r) = 4(1+r^2)^{-3}.$$

There are $\omega, \delta, c > 0$ *s.t. for all smooth, radial* (f, g) *with*

$$\|(f,g)\|_{H^4 \times H^3(\mathbb{B}^7_2)} \leq \tfrac{\delta}{c}$$

the following holds: There are $\alpha \in [-\delta, \delta]$ *and* $T \in [1-\delta, 1+\delta]$ *depending Lipschitz continuously on* (f, g) *such that for initial data*

$$u(0,\cdot) = u_1^*(0,\cdot) + f + \alpha f_1, \quad \partial_t u(0,\cdot) = \partial_t u_1^*(0,\cdot) + g + \alpha g_2$$

there is a unique solution u *in the backward light cone* $\mathcal{C}_T$ *blowing up at* $t = T$ *and converging to* u_T^* *according to*

$$(T-t)^{k-s_c} \|u(t,\cdot) - u_T^*(t,\cdot)\|_{\dot{H}^k(\mathbb{B}^7_{T-t})} \lesssim (T-t)^{\omega}$$

$$(T-t)^{k-s_c} \|\partial_t u(t,\cdot) - \partial_t u_T^*(t,\cdot)\|_{\dot{H}^{k-1}(\mathbb{B}^7_{T-t})} \lesssim (T-t)^{\omega}$$

for $k = 1, 2, 3$

- Yang-Mills equations: $A_\mu : \mathbb{R}^{1,n} \to \mathfrak{so}(n)$, $\mu = 0, \dots, d$

$$\partial_\mu F^{\mu\nu}(t,x) + [A_\mu(t,x), F^{\mu\nu}(t,x)] = 0$$

where $F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$

- Symmetry assumption: $A_\mu(t,x) = u(t,|x|)\left(\delta_\mu^k x^i - \delta_\mu^i x^k\right)$

$$\left(\partial_t^2 - \partial_r^2 - \frac{n+1}{r}\partial_r\right) u(t,r) = 3(d-2)u^2(t,r) - (d-2)r^2u^3(t,r)$$

Energy supercritical in $n \geq 5$

- Self-similar profiles: $n = 5$, $\{U_k\}_{k\in\mathbb{N}_0}$ (Bizoń '02)
- Stable self-similar blowup $n = 5$
(Donninger '14, Costin-Donninger-Glogić-Huang '16)
- Numerical experiments: Threshold for blowup described by U_1

- Quadratic wave equation:

$$(\partial_t^2 - \Delta_x)u(t,x) = u(t,x)^2$$

Non-trivial self-similar blowup solution: (Glogić '20)

$$u_T^*(t,r) = (T-t)^{-2}U^*\left(\tfrac{r}{T-t}\right), \quad U^*(\rho) = \frac{a(d)\rho^2 + b(d)}{(\rho^2 + c(d))^2}$$

$a(d) < 0$, $b(d) > 0$, $c(d) > 0$ and $U^*(\rho) > 0$ for $\rho \in [0,1]$

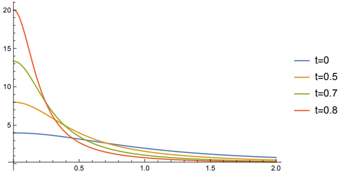


Figure: Blowup solution $u_1^*(t,r) = (1-t)^{-2}U^*(\frac{r}{1-t})$ for $d = 9$

- Co-dimension one stability: (Csobo-Glogić-S., in preparation)

Thank you for your attention!

Numerical experiments (Maliborski-Glogić-S. 2019)
based on methods developed in [Bizoń-Biernat-Maliborski 2017]

- Dynamically rescaled coordinates (y, s)

$$r = e^{-s} y, \quad \frac{dt}{ds} = e^{-s} h(s)$$

- Rescaled variables

$$e^{s} V(s, y) = u(t, r), \quad e^{2s} P(s, y) = \partial_t u(t, r).$$

- For $h(s) = 1/P(s, 0)$,

$$V(s, 0) = 1 + c e^{-s}, \quad c \in \mathbb{R}$$

and

$$P(s,0) = \begin{cases} 0, & \text{in case of dispersion,} \\ 1/f(0), & \text{in case of blowup via self-similar profile f,} \end{cases}$$

Threshold behavior

Study evolution for radial families of data depending on parameter A

- Small $A \Rightarrow$ dispersion, $P(s,0) \to 0$
- Large $A \Rightarrow$ blowup, $P(s,0) \to \frac{1}{\sqrt{2}}$
- Bisection $\Rightarrow$ fine-tune to critical A_*
- Intermediate attractor $P(s,0) \to \frac{1}{f^*(0)}$

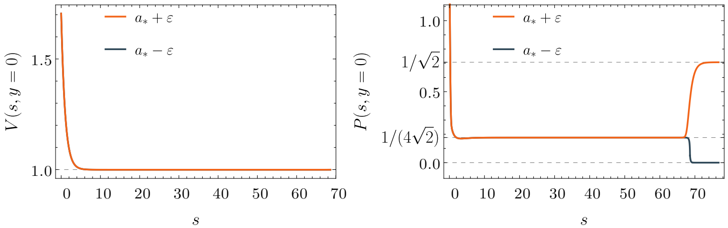
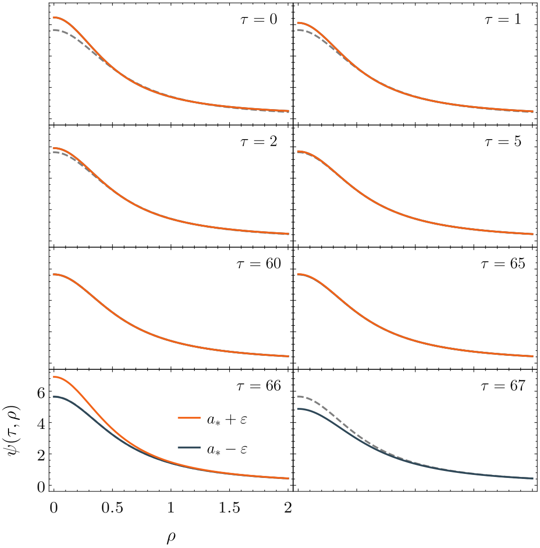


Figure: The evolution of marginally sub- (blue line) and supercritical (orange line) evolutions in $d = 5$ in computational variables

Threshold behavior

$d = 5$: Evolution for near critical data in self-similar variables (τ, ρ)



$d = 7$: Evolution for near critical data in self-similar variables (τ, ρ)