Quasi-Periodic Water Waves

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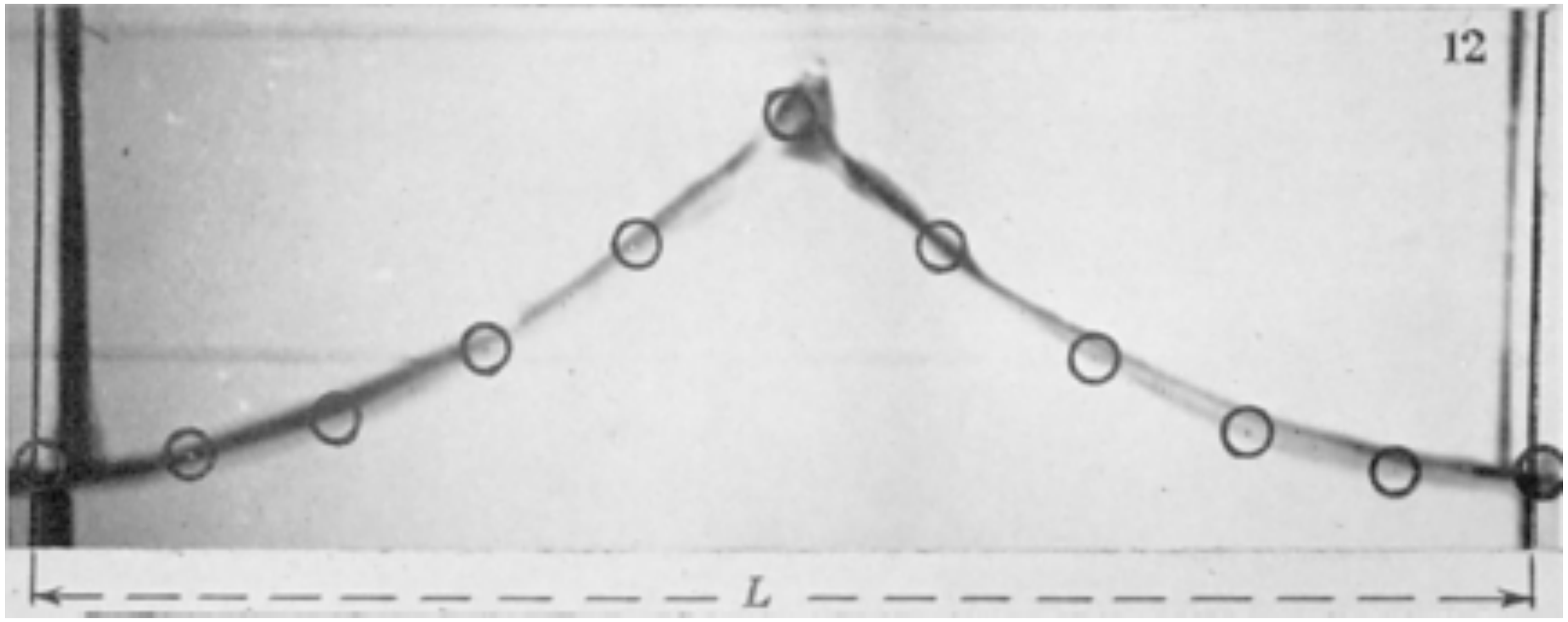

Support: NSF, DMS-1716560 DOE, DE-AC02-05CH11231

Collaborator

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Collaborator

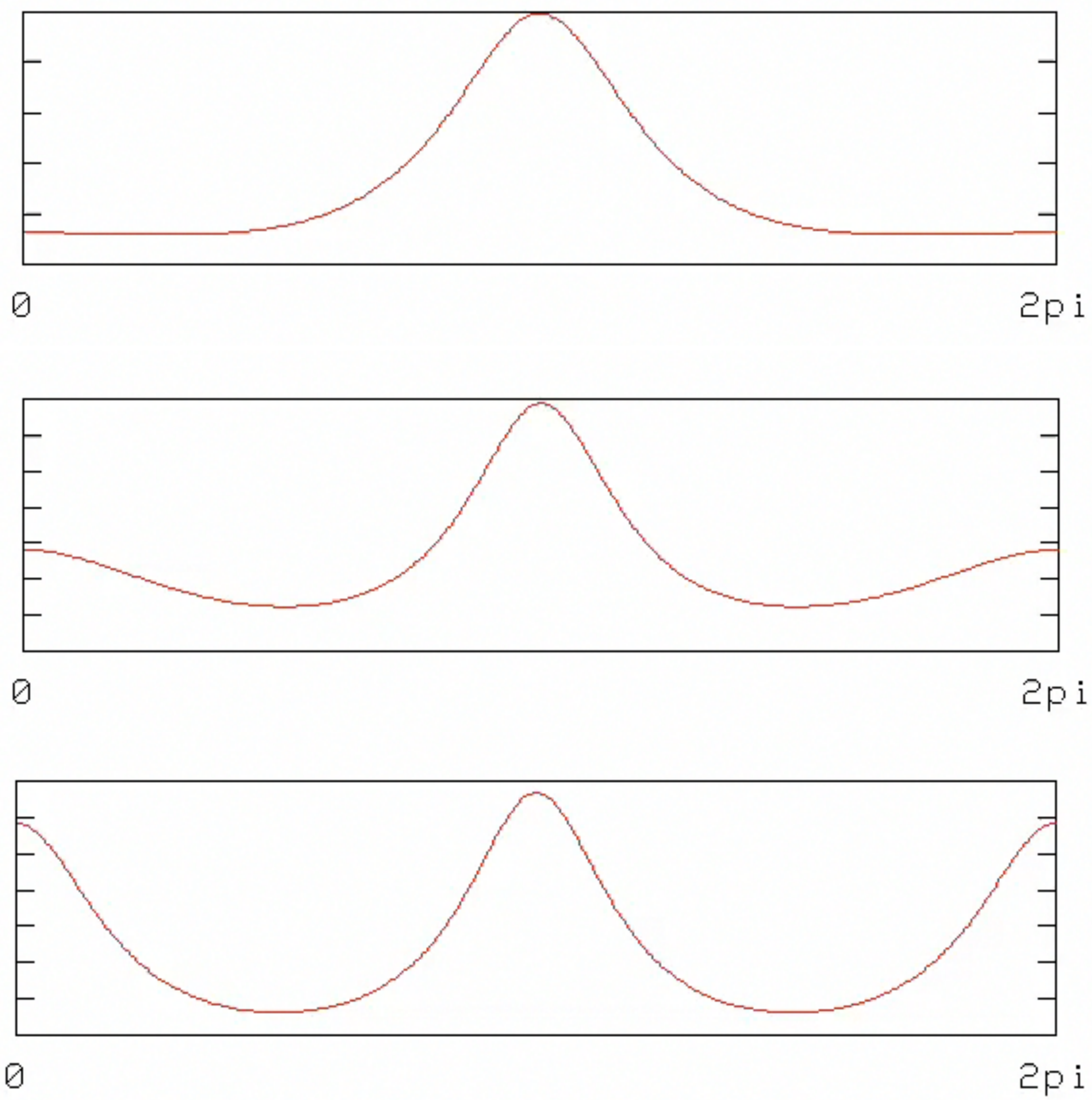
SPATIALLY QUASI-PERIODIC GRAVITY-CAPILLARY WATER WAVES OF INFINITE DEPTH

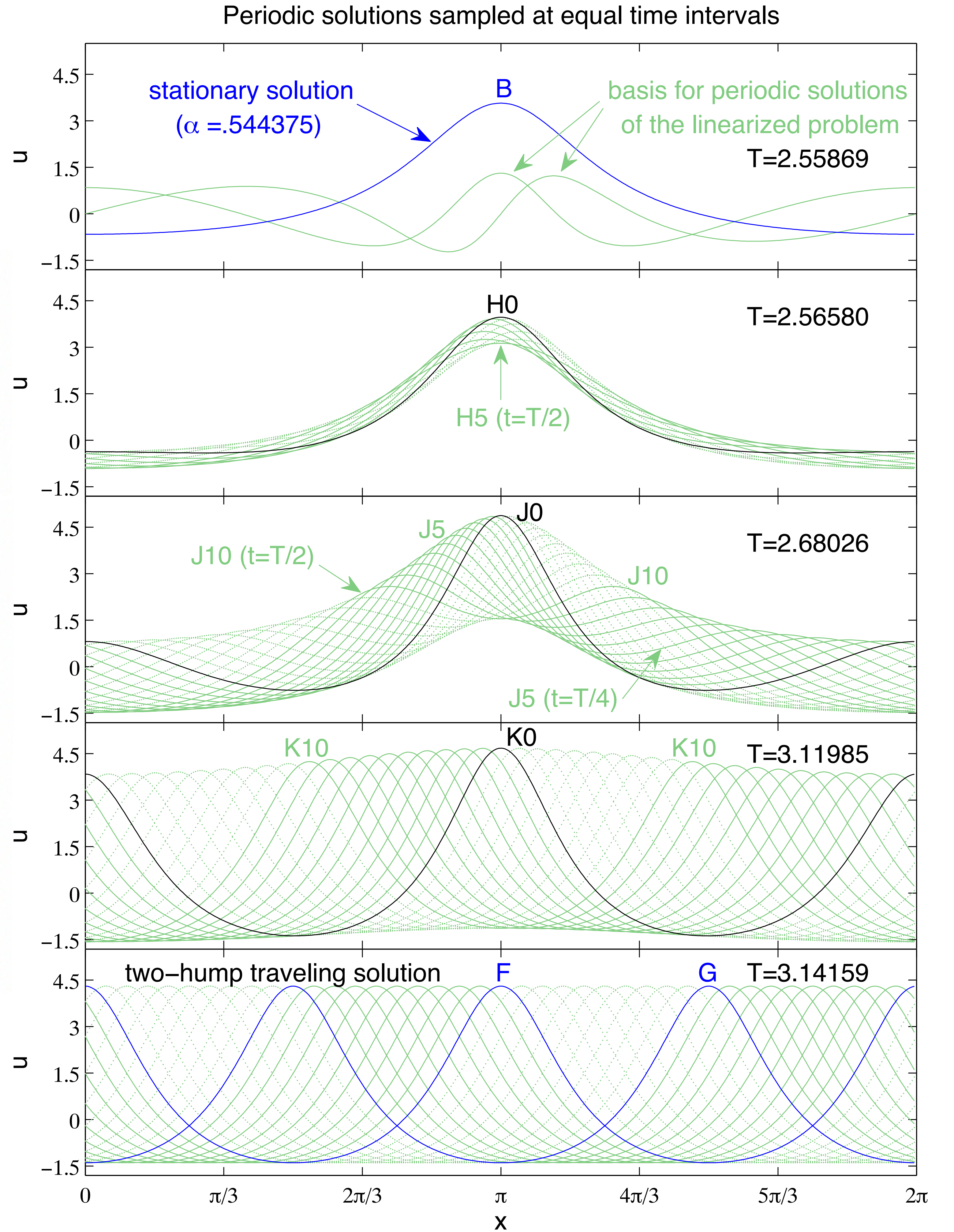
JON WILKENING AND XINYU ZHAO

Abstract. We formulate the two-dimensional gravity-capillary water wave equations in a spatially quasi-periodic setting and present a numerical study of traveling waves and more general solutions of the initial value problem. The former are a generalization of the classical Wilton ripple problem. We adopt a conformal mapping formulation and employ a quasi-periodic version of the Hilbert transform to determine the normal velocity of the free surface. We compute traveling waves in a nonlinear least-squares framework using a variant of the Levenberg-Marquardt method. We propose four methods for timestepping the initial value problem, two explicit Runge-Kutta (ERK) methods and two exponential time-differencing (ETD) schemes. The latter approach makes use of the small-scale decomposition to eliminate stiffness due to surface tension. We investigate various properties of quasi-periodic traveling waves, including Fourier resonances and the dependence of wave speed and surface tension on the amplitude parameters that describe a two-parameter family of waves. We also present an example of a periodic wave profile containing vertical tangent lines that is set in motion with a quasi-periodic velocity potential that causes some of the waves to overturn and others to flatten out as time evolves.

Benjamin-Ono

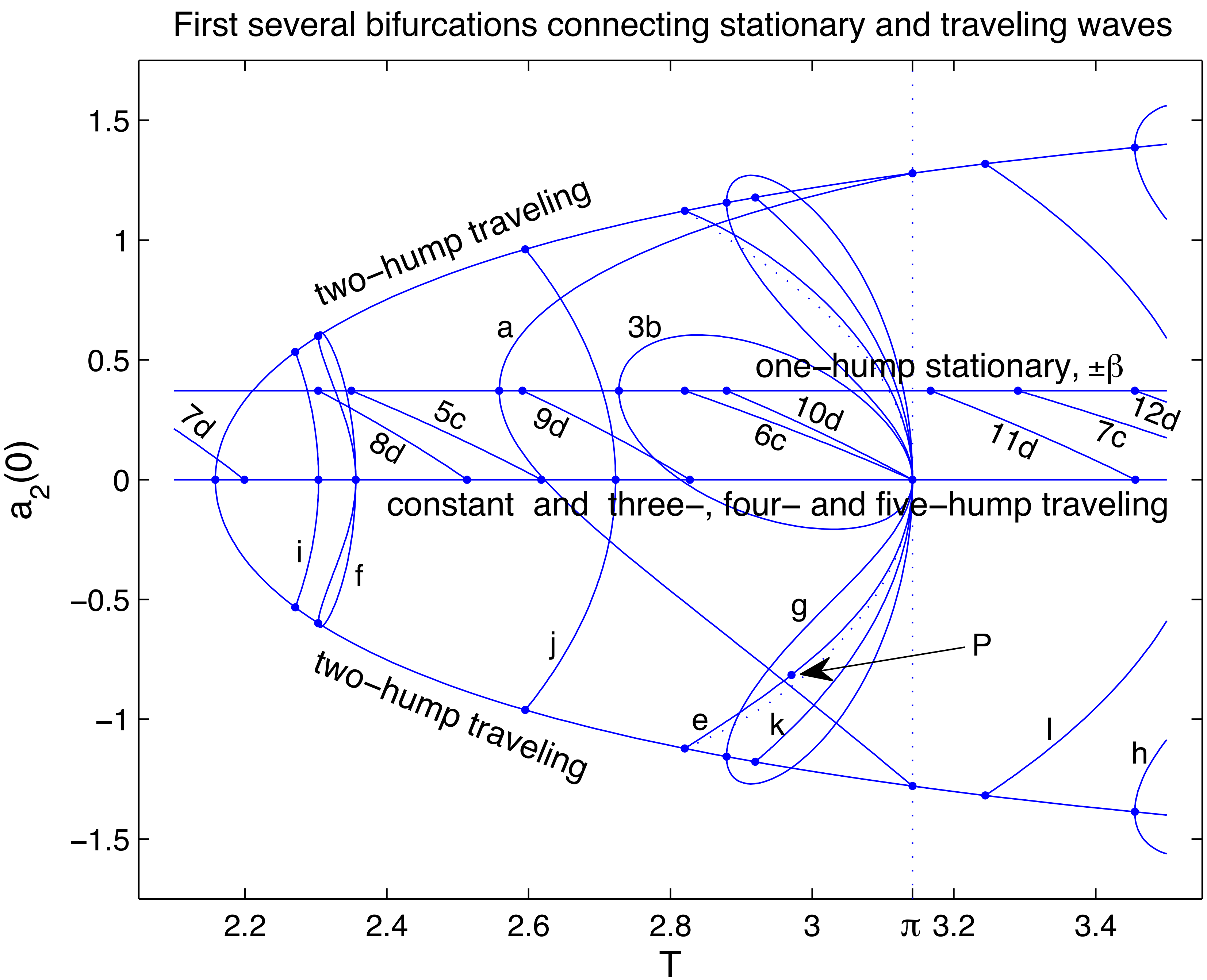
$$u_t + uu_x - Hu_{xx} = 0$$





Transition from one to two humps

paths connecting pairs of traveling waves



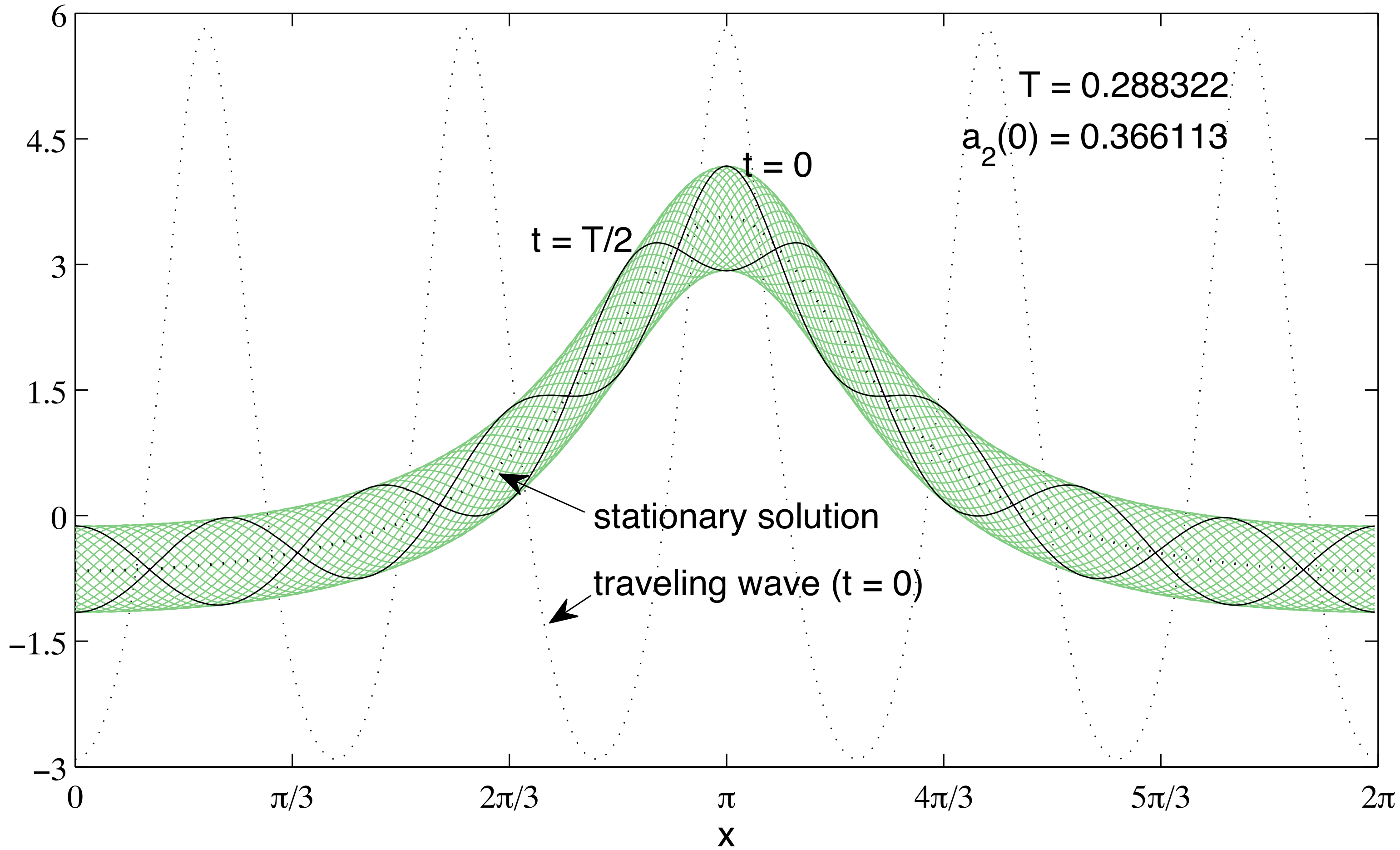
$$\alpha_0 = 0.544375$$

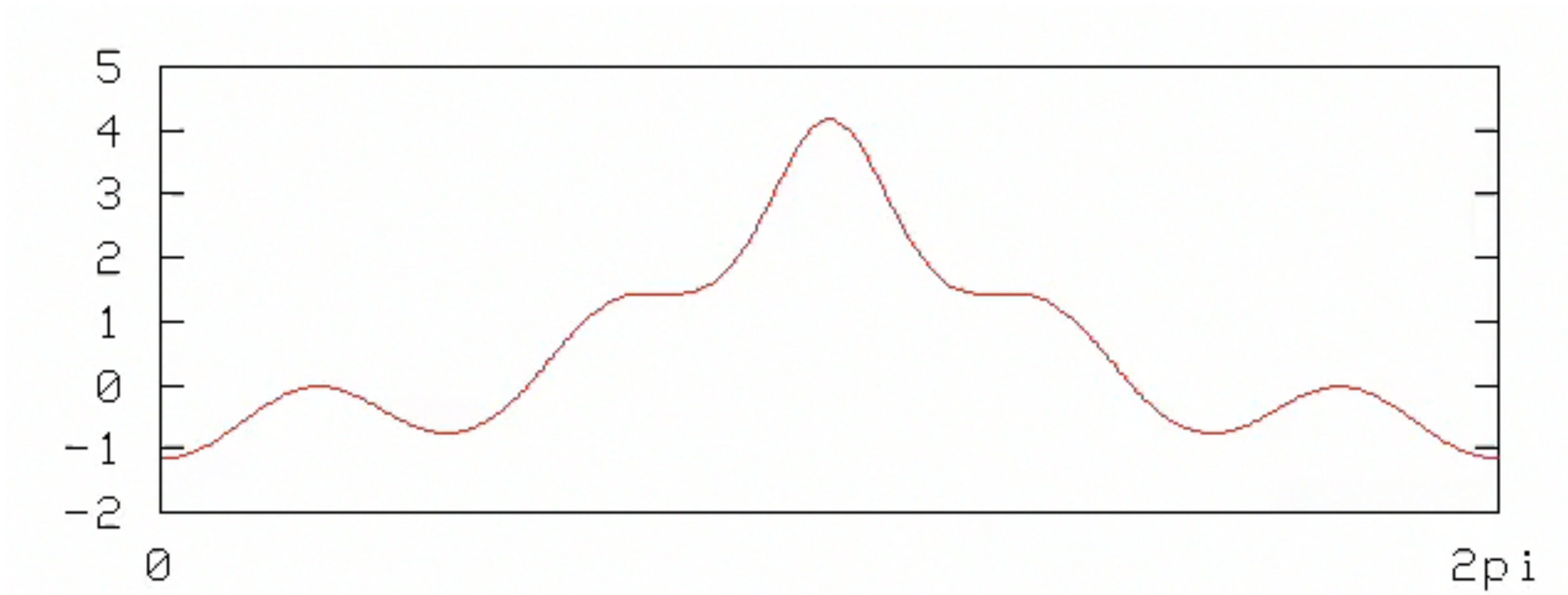
$$(N', \nu', n', m') \longleftrightarrow (N, \nu, n, m)$$

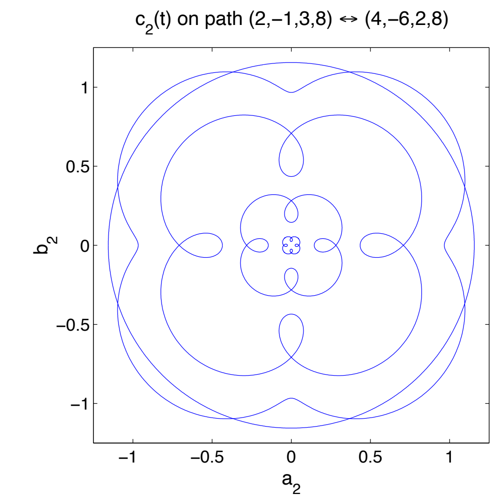
$a:$	$(1,0,1,1)$	$\longleftrightarrow$	$(2,-1,1,1)$,
$b:$	$(1,0,2,1)$	$\longleftrightarrow$	$(3,-1,2,1)$,
$c:$	$(1,0,3,1)$	$\longleftrightarrow$	$(4,-1,3,1)$,
$d:$	$(1,0,4,1)$	$\longleftrightarrow$	$(5,-1,4,1)$,
$e:$	$(2,-1,2,3)$	$\longleftrightarrow$	$(3,-3,1,3)$,
$f:$	$(2,-1,3,6)$	$\longleftrightarrow$	$(4,-5,2,6)$,
$g:$	$(2,-1,3,8)$	$\longleftrightarrow$	$(4,-6,2,8)$,
$h:$	$(2,-1,3,10)$	$\longleftrightarrow$	$(4,-7,2,10)$,
$i:$	$(2,-1,4,11)$	$\longleftrightarrow$	$(5,-8,3,11)$,
$j:$	$(2,-1,4,13)$	$\longleftrightarrow$	$(5,-9,3,13)$,
$k:$	$(2,-1,4,15)$	$\longleftrightarrow$	$(5,-10,3,15)$,
$l:$	$(2,-1,4,17)$	$\longleftrightarrow$	$(5,-11,3,17)$.

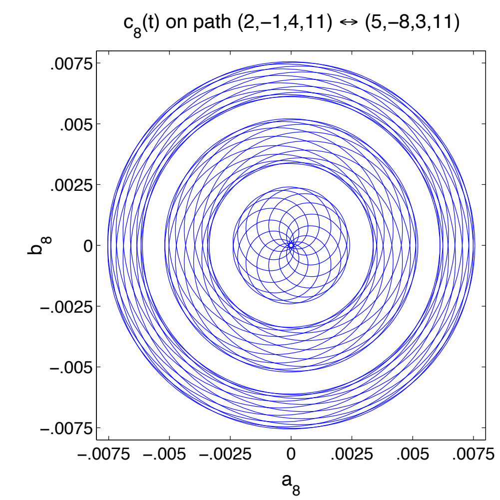
A solution on the path $(1, 0, 4, 1) \longleftrightarrow (5, -1, 4, 1)$

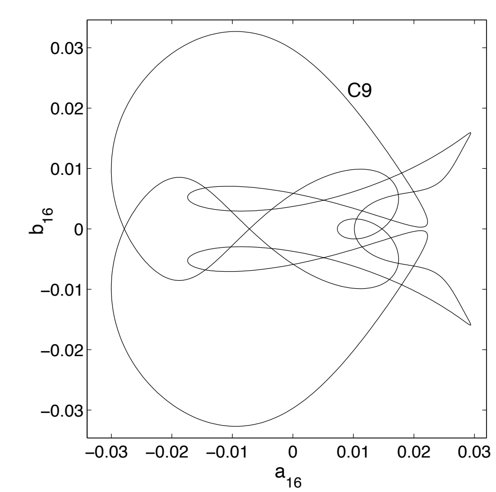
$$\alpha_0 = 0.544375$$











Quasi-periodic solutions

$(\star)$ def: $u(x,t) = U(\vec{\kappa}\,x + \vec{\omega}\,t + \vec{\theta})$ $\begin{cases} U \in C(\mathbb{T}^n) \\ \vec{\kappa},\ \vec{\omega},\ \vec{\theta} \in \mathbb{R}^n \end{cases}$

Example: Benjamin-Ono (Satsuma/Ishimori 79, Dobrokhotov/Krichever 91)

Pick $\vec{\theta} \in \mathbb{R}^n$ and $C < a_1 < b_1 < \cdots < a_n < b_n$

define $\begin{cases} c_m = |c_m| e^{i\theta_m}, \qquad |c_m|^2 = -\frac{(b_m - C)\prod_{j \neq m}(a_m - a_j)(b_m - b_j)}{(a_m - C)\prod_j (b_m - a_j)(a_m - b_j)} \\ M_{jm}(y) = |c_m| e^{iy_m}\delta_{jm} - \frac{1}{b_j - a_m} \qquad (n \times n \text{ matrix}) \end{cases}$

Then $(\star)$ is a solution of $u_t + 2uu_x - Hu_{xx} = 0$ with

$$U(y) = C + \textstyle\sum_m (a_m - b_m) - 2\,\mathrm{Im}\,\partial_x \ln\det M(y),$$

$$\kappa_m = -(b_m - a_m), \qquad \omega_m = b_m^2 - a_m^2$$

Are there analogues for the water wave?

Part 0: Introduction

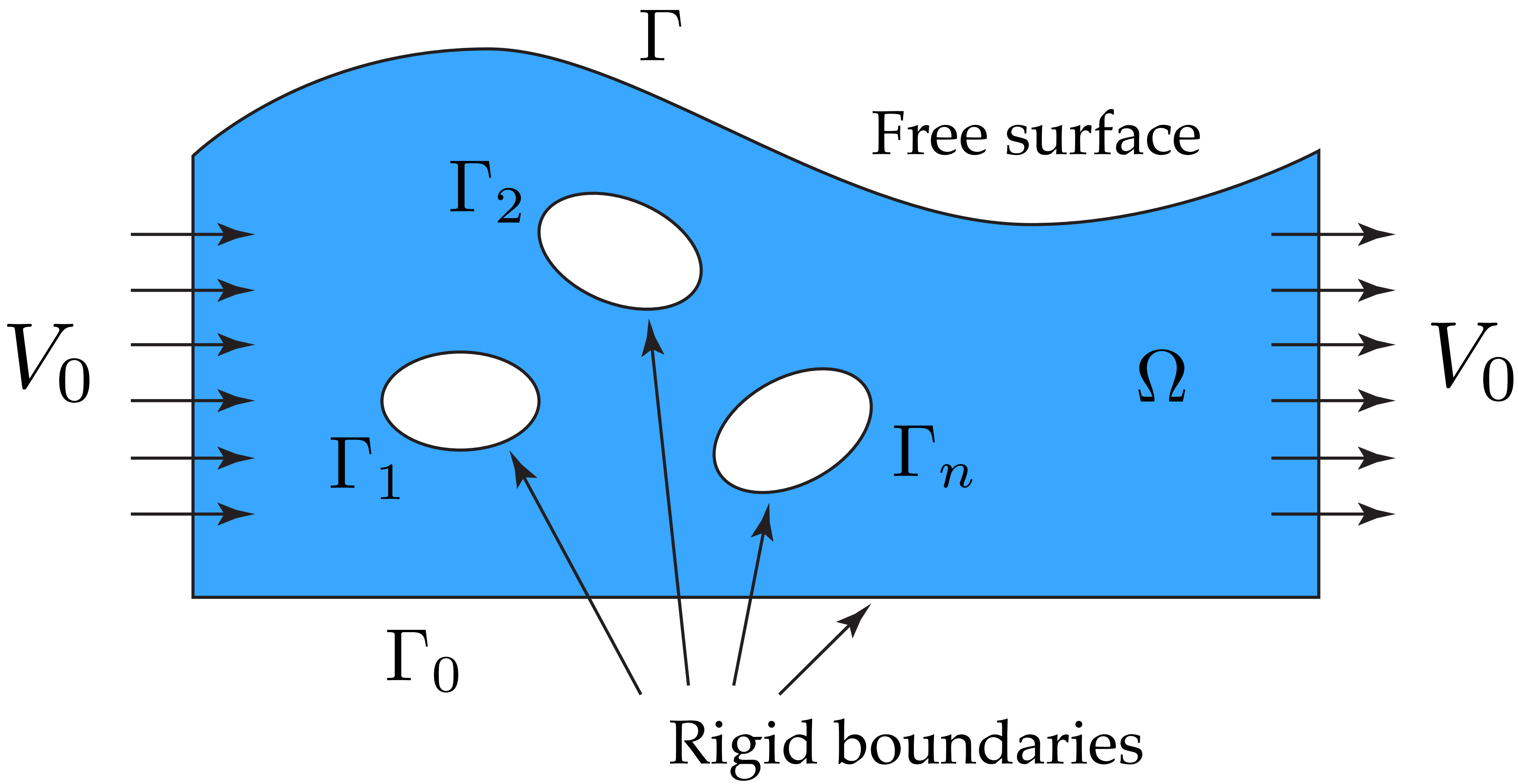
Part 1: spatially quasi-periodic water waves

- Equations of motion
- Initial value problem
- Traveling waves (two quasi-periods)

Part 2: temporally Quasi-periodic solutions (if time permits)

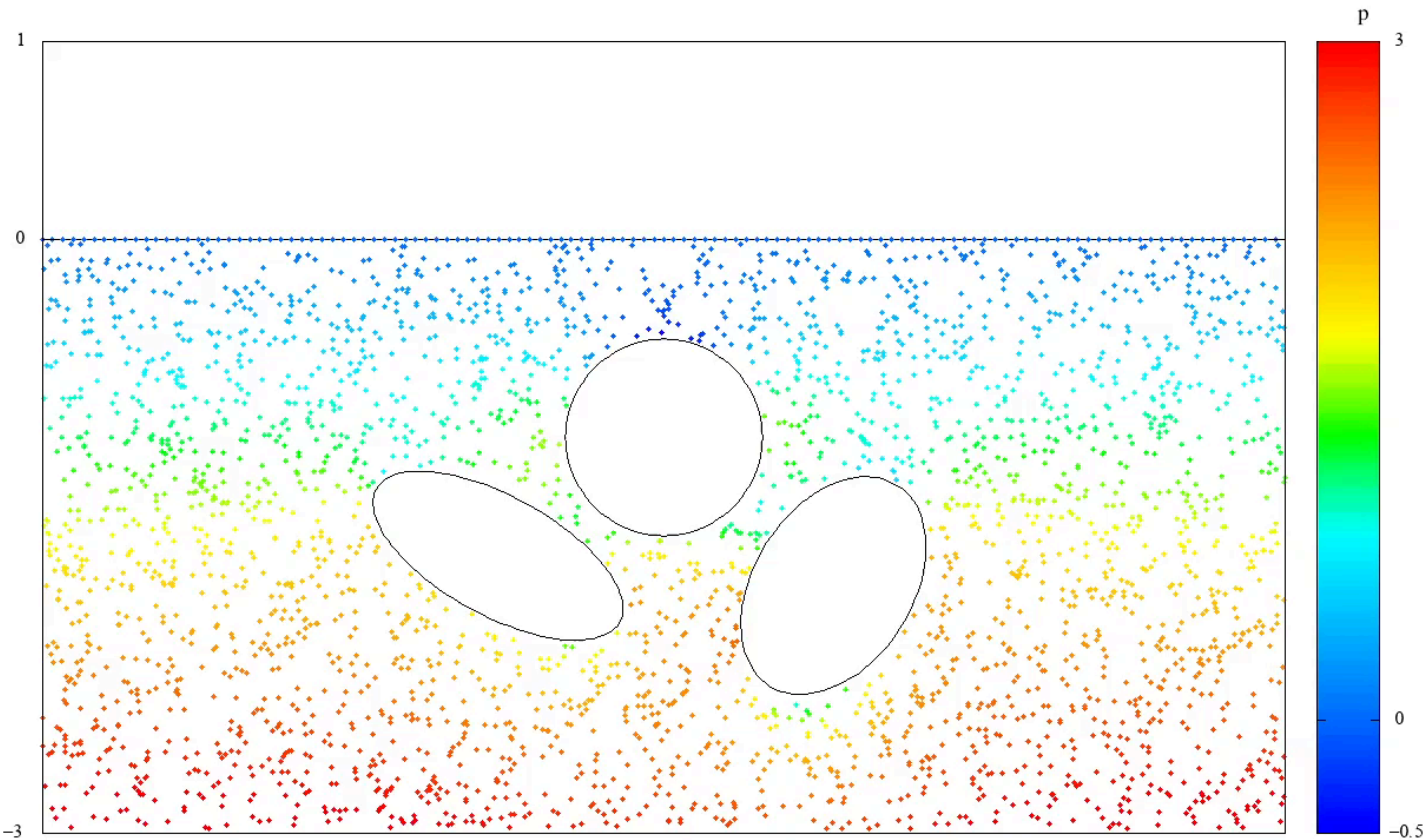
- two quasi-periods (methods and examples)
- three quasi-periods (method and example)

Setup



$\zeta_j(\alpha, t)$ parametrizes Γ_j

Cauchy integral code (method 3 below)



Ambrose, Camassa, Marzuola, McLaughlin, Robinson, Wilkening
Numerical Algorithms for Water Waves with Background Flow over Obstacles and Topology, (coming soon!)

Equations of Motion

$$\zeta(\alpha,t) = \xi(\alpha,t) + i\eta(\alpha,t), \qquad \varphi(\alpha,t) = \phi(\zeta(\alpha,t),t)$$

$$\mathbf{u} = \nabla\phi, \qquad \zeta(\alpha,0) = \zeta_0(\alpha), \qquad \varphi(\alpha,0) = \varphi_0(\alpha), \qquad t = 0$$

$$\phi_{xx} + \phi_{yy} = 0, \qquad \text{in } \Omega$$

$$\partial_n\phi = 0, \qquad \text{on } \Gamma_j$$

$$\phi = \varphi, \qquad \text{on } \Gamma$$

$$\zeta_t \cdot \mathbf{n} = \mathbf{u} \cdot \mathbf{n} \qquad \text{on } \Gamma$$

Bernoulli

$$\nabla\left(\phi_t + \frac{1}{2}\|\mathbf{u}\|^2 + \frac{p}{\rho} + gy\right) = 0$$

$$\overbrace{\varphi_t - \nabla\phi \cdot \zeta_t}^{\phi_t} + \frac{1}{2}\phi_x^2 + \frac{1}{2}\phi_y^2 + \frac{p}{\rho} + gy - \tau\kappa = c(t), \qquad \text{on } \Gamma$$

$$\kappa = \frac{\xi_\alpha\eta_{\alpha\alpha} - \eta_\alpha\xi_{\alpha\alpha}}{s_\alpha^3}, \qquad s_\alpha = |\zeta_\alpha|$$

Equations of Motion

$$\zeta(\alpha,t) = \xi(\alpha,t) + i\eta(\alpha,t), \qquad \varphi(\alpha,t) = \phi(\zeta(\alpha,t),t)$$

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particle on surface:

$$\mathbf{n} = i\frac{\zeta_\alpha}{s_\alpha}$$

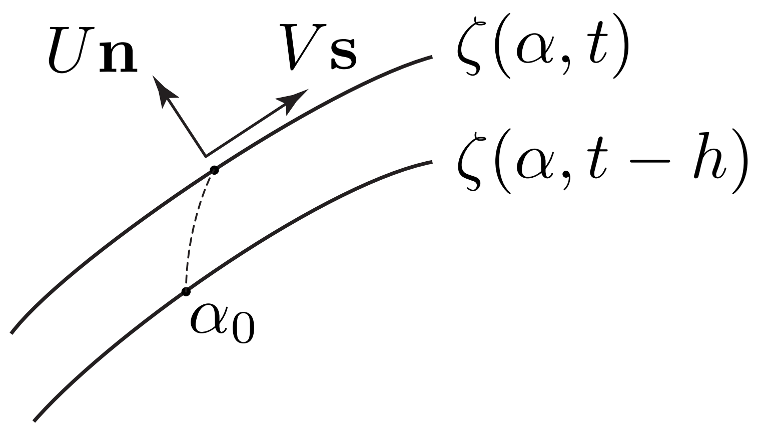
$$x = \xi(\alpha(t),t), \qquad \dot{x} = \xi_\alpha\dot{\alpha} + \xi_t = u, \qquad \eta_t\xi_\alpha - \xi_t\eta_\alpha = v\xi_\alpha - u\eta_\alpha$$

$$y = \eta(\alpha(t),t), \qquad \dot{y} = \eta_\alpha\dot{\alpha} + \eta_t = v, \qquad \zeta_t\cdot\mathbf{n} = \nabla\phi\cdot\mathbf{n}$$

Curve representations

$$U = \zeta_t \cdot \mathbf{n} = \mathbf{u} \cdot \mathbf{n} = \frac{\partial \phi}{\partial n}$$

$$\zeta_\alpha = s_\alpha e^{i\theta}, \qquad \zeta_t = (V + iU)e^{i\theta}$$



method 1: $\begin{cases} \zeta(\alpha,t) = \alpha + i\eta(\alpha,t) \\ \xi(\alpha,t) = \alpha, \quad \xi_t = 0 \end{cases}$

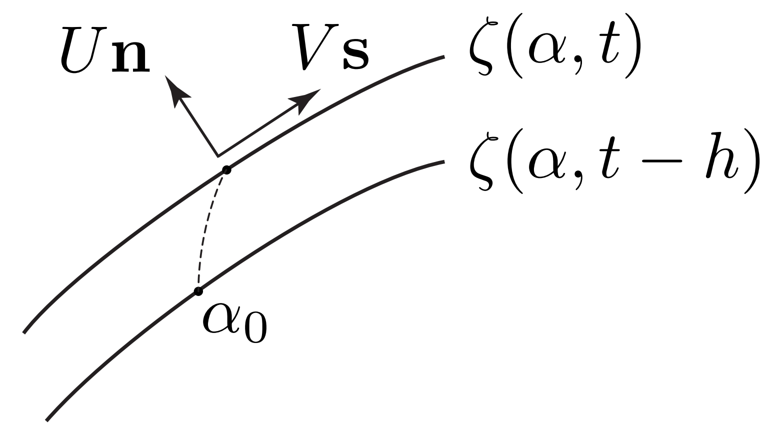
standard
graph-based
representation

$$\xi_t = \mathrm{Re}\left\{(V + iU)\frac{1 + i\eta_\alpha}{\sqrt{1+\eta_\alpha^2}}\right\} = 0 \quad \Rightarrow \quad V = \eta_\alpha U$$

$$\eta_t = \mathrm{Im}\left\{(V + iU)\frac{1 + i\eta_\alpha}{\sqrt{1+\eta_\alpha^2}}\right\} = \frac{U + \eta_\alpha^2 U}{\sqrt{1+\eta_\alpha^2}} = \sqrt{1+\eta_\alpha^2}\,\frac{\partial \phi}{\partial n} \qquad \text{(DNO)}$$

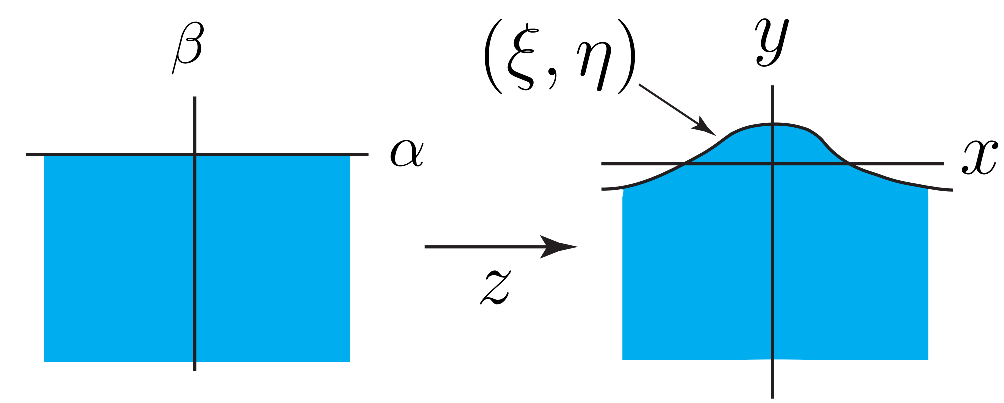
Curve representations

$$U = \zeta_t \cdot \mathbf{n} = \mathbf{u} \cdot \mathbf{n} = \frac{\partial \phi}{\partial n}$$



$$\zeta_\alpha = s_\alpha e^{i\theta}, \qquad \zeta_t = (V + iU)e^{i\theta}$$

method 2:
$$\begin{cases} \zeta(\alpha) = \xi(\alpha) + i\eta(\alpha) \\ \eta(\cdot) = -H[\xi(\alpha) - \alpha] \\ z(w) = x(w) + iy(w), \quad w = \alpha + i\beta, \\ \zeta = z|_{\beta=0}, \quad \xi = x|_{\beta=0}, \quad \eta = y|_{\beta=0} \end{cases}$$



Here we also define the complex velocity potential: $\Phi(z) = \phi(z) + i\psi(z)$

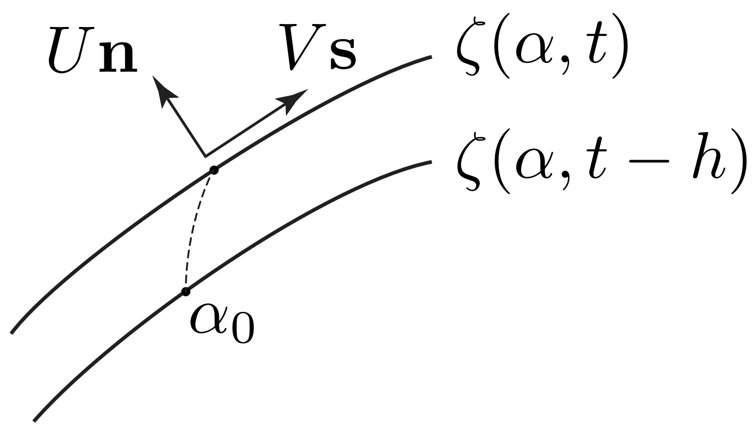
chain rule + Cauchy-Riemann:
$$\frac{\partial}{\partial \alpha}\phi(\zeta(\alpha)) = \phi_x \xi_\alpha + \phi_y \eta_\alpha = \nabla\phi \cdot (s_\alpha \hat{\mathbf{s}}) = s_\alpha \frac{\partial \phi}{\partial s}$$

$$-\frac{\partial}{\partial \alpha}\psi(\zeta(\alpha)) = -\psi_x \xi_\alpha - \psi_y \eta_\alpha = -\phi_x \eta_\alpha + \phi_y \xi_\alpha = \nabla\phi \cdot (s_\alpha \hat{\mathbf{n}}) = s_\alpha \frac{\partial \phi}{\partial n}$$

Curve representations

$$U = \zeta_t \cdot \mathbf{n} = \mathbf{u} \cdot \mathbf{n} = \frac{\partial \phi}{\partial n}$$

$$\zeta_\alpha = s_\alpha e^{i\theta}, \qquad \zeta_t = (V + iU)e^{i\theta}$$



method 2:
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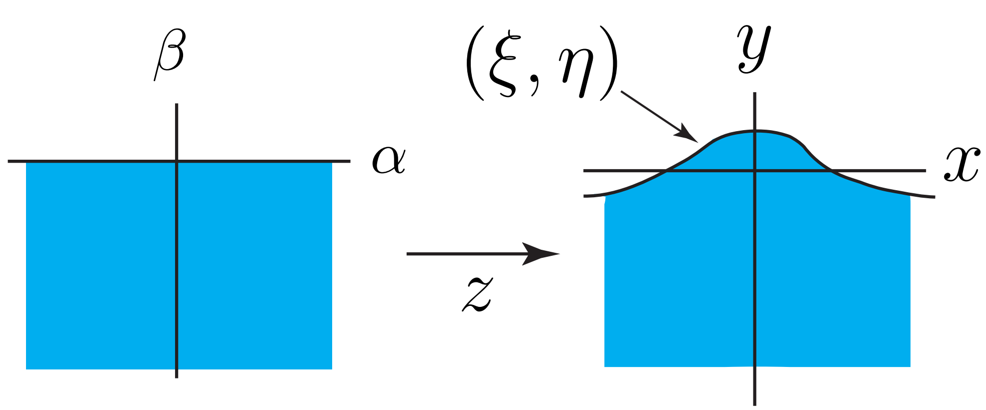
$$\xi(\alpha, t) = \alpha + x_0(t) + H[\eta](\alpha, t)$$

$$\frac{dx_0}{dt} = P_0 \left[\xi_\alpha \left(-H\left[\frac{\psi_\alpha}{J}\right] + C_1 \right) + \frac{\eta_\alpha \psi_\alpha}{J} \right]$$

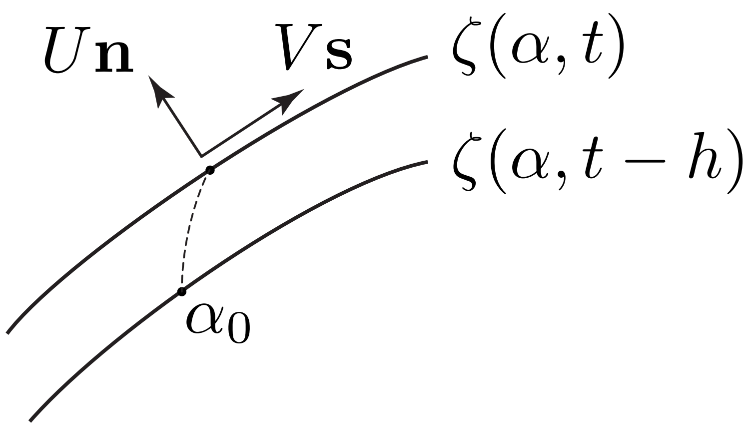


$$\left.\frac{z_t}{z_\alpha}\right|_{\beta=0} = \frac{\zeta_t}{\zeta_\alpha} = \frac{V + iU}{s_\alpha} \quad \Rightarrow \quad \frac{V}{s_\alpha} = H\left(\frac{U}{s_\alpha}\right) + C_1 = -H\left(\frac{\psi_\alpha}{s_\alpha^2}\right) + C_1$$

$$\begin{pmatrix} \xi_t \\ \eta_t \end{pmatrix} = \begin{pmatrix} \xi_\alpha & -\eta_\alpha \\ \eta_\alpha & \xi_\alpha \end{pmatrix} \begin{pmatrix} V/s_\alpha \\ U/s_\alpha \end{pmatrix} \longleftarrow \zeta_t = (\zeta_\alpha)(\zeta_t/\zeta_\alpha)$$

Curve representations

$$U = \zeta_t \cdot \mathbf{n} = \mathbf{u} \cdot \mathbf{n} = \frac{\partial \phi}{\partial n}$$



$$\zeta_\alpha = s_\alpha e^{i\theta}, \qquad \zeta_t = (V + iU)e^{i\theta}$$

$$\begin{aligned} \zeta_{\alpha t} &= s_{\alpha t} e^{i\theta} + s_\alpha \theta_t i e^{i\theta} \\ &= (V_\alpha + iU_\alpha)e^{i\theta} + (V + iU)\theta_\alpha i e^{i\theta} \end{aligned}$$

$$\begin{aligned} s_{\alpha t} &= V_\alpha - \theta_\alpha U \\ s_\alpha \theta_t &= U_\alpha + \theta_\alpha V \end{aligned}$$

method 3 (HLS):

choose V so that $s_\alpha(t) = \text{const}(t)$

$$V = \partial_\alpha^{-1} P\big[\theta_\alpha U\big] + c$$

$$s_{\alpha t} = -P_0\big[\theta_\alpha U\big]$$

$$\theta_t = \frac{U_\alpha + \theta_\alpha V}{s_\alpha}$$

$$Pf(\alpha) = f(\alpha) - P_0 f$$

$$P_0 f = \frac{1}{2\pi}\int_0^{2\pi} f(\alpha)\, d\alpha$$

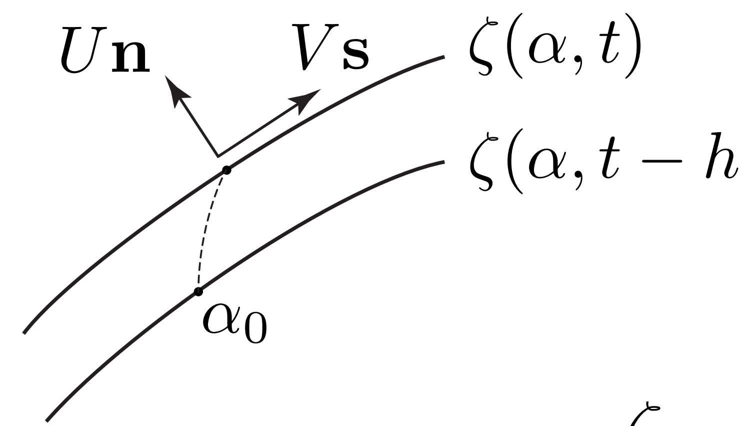
Can pick c so that e.g. $\xi(0, t) = 0$.

$$\zeta(\alpha, t) = \zeta(0, t) + \int_0^\alpha s_\alpha(t) e^{i\theta(\beta, t)}\, d\beta$$

Dirichlet-Neumann operator

$$U = \zeta_t \cdot \mathbf{n} = \mathbf{u} \cdot \mathbf{n} = \frac{\partial \phi}{\partial n}$$

Given φ, need to solve Laplace equation to get $\frac{\partial \phi}{\partial n}$.
The tangential derivative is just $\frac{\partial \phi}{\partial s} = \varphi_\alpha / s_\alpha$.



$$\mathbf{n} = i\frac{\zeta_\alpha}{s_\alpha}$$

$$\zeta(\alpha, t) = \xi(\alpha, t) + i\eta(\alpha, t), \qquad \varphi(\alpha, t) = \phi(\zeta(\alpha, t), t)$$

$$\mathbf{u} = \nabla\phi, \qquad \zeta(\alpha, 0) = \zeta_0(\alpha), \qquad \varphi(\alpha, 0) = \varphi_0(\alpha), \qquad t = 0$$

$$\phi_{xx} + \phi_{yy} = 0, \qquad \text{in } \Omega$$

$$\partial_n \phi = 0, \qquad \text{on } \Gamma_j$$

$$\phi = \varphi, \qquad \text{on } \Gamma$$

$$\zeta_t \cdot \mathbf{n} = \mathbf{u} \cdot \mathbf{n}, \qquad \text{on } \Gamma$$

$$\overbrace{\varphi_t - \nabla\phi \cdot \zeta_t}^{\phi_t} + \frac{1}{2}\left|\nabla\phi\right|^2 + \frac{p}{\rho} + gy - \tau\kappa = c(t), \qquad \text{on } \Gamma$$

Conformal mapping approach (method 2)

The Bernoulli equation simplifies to

$$\varphi_t = \nabla\phi \cdot \zeta_t - \frac{1}{2}|\nabla\phi|^2 - \frac{p}{\rho} - gy + \tau\kappa + c$$

$$= \underbrace{\left(\phi_x \,,\, \phi_y\right)\begin{pmatrix}\xi_\alpha & -\eta_\alpha \\ \eta_\alpha & \xi_\alpha\end{pmatrix}}_{(\varphi_\alpha \,,\, -\psi_\alpha)}\begin{pmatrix}-H\left[\psi_\alpha/s_\alpha^2\right] + C_1 \\ -\psi_\alpha/s_\alpha^2\end{pmatrix} - \frac{\varphi_\alpha^2 + \psi_\alpha^2}{2s_\alpha^2} - g\eta + \tau\kappa + c$$

$$= P\left[\frac{\psi_\alpha^2 - \varphi_\alpha^2}{2s_\alpha^2} - \varphi_\alpha H[\psi_\alpha/s_\alpha^2] + C_1\varphi_\alpha - g\eta + \tau\kappa\right]$$

Conformal mapping approach (method 2)

Summary of equations of motion:

$$\xi_\alpha = 1 + H[\eta_\alpha], \qquad \psi = -H[\varphi], \qquad J = \xi_\alpha^2 + \eta_\alpha^2, \qquad \chi = \frac{\psi_\alpha}{J},$$

$$\text{choose } C_1, \text{ e.g. as in (1)}, \qquad \text{compute } \frac{dx_0}{dt} \text{ in (2) if necessary},$$

$$\eta_t = -\eta_\alpha H[\chi] - \xi_\alpha \chi + C_1 \eta_\alpha, \qquad \kappa = \frac{\xi_\alpha \eta_{\alpha\alpha} - \eta_\alpha \xi_{\alpha\alpha}}{J^{3/2}},$$

$$\varphi_t = P\left[\frac{\psi_\alpha^2 - \varphi_\alpha^2}{2J} - \varphi_\alpha H[\chi] + C_1 \varphi_\alpha - g\eta + \tau\kappa\right].$$

$(1a)$ $C_1 = 0 :$ evolve $x_0(t)$ via (2),

$(1b)$ $C_1 = P_0\big[\xi_\alpha H[\psi_\alpha/J] - \eta_\alpha \psi_\alpha/J\big] :$ $x_0(t) = 0$,

$(1c)$ $C_1 = \big[H[\psi_\alpha/J] - \eta_\alpha \psi_\alpha/(\xi_\alpha J)\big]_{\alpha=0} :$ $\xi(0,t) = 0$.

$$(2) \quad \frac{dx_0}{dt} = P_0\left[\xi_\alpha \left(-H\left[\frac{\psi_\alpha}{J}\right] + C_1\right) + \frac{\eta_\alpha \psi_\alpha}{J}\right], \qquad \xi(\alpha,t) = \alpha + x_0(t) + H[\eta](\alpha,t)$$

Quasi-periodic, real-analytic functions

$$u(\alpha) = \tilde{u}(\boldsymbol{k}\alpha), \qquad \tilde{u}(\boldsymbol{\alpha}) = \sum_{\boldsymbol{j}\in\mathbb{Z}^d} \hat{u}_{\boldsymbol{j}} e^{i\langle \boldsymbol{j},\, \boldsymbol{\alpha}\rangle}, \qquad \alpha\in\mathbb{R}, \quad \boldsymbol{\alpha}\in\mathbb{T}^d, \quad \boldsymbol{k}\in\mathbb{R}^d$$

Bounded, analytic extension to lower half-plane satisfying $\left(\operatorname{Re} f\right)\Big|_{\beta=0} = u$

$$f(w) = \hat{u}_{\mathbf{0}} + i\hat{v}_{\mathbf{0}} + \sum_{\langle \boldsymbol{j},\boldsymbol{k}\rangle<0} 2\hat{u}_{\boldsymbol{j}} e^{i\langle \boldsymbol{j},\boldsymbol{k}\rangle w}, \qquad (w = \alpha + i\beta\,,\ \beta\le 0)$$

Extract imaginary part on real axis $\left(\operatorname{Im} f\right)\Big|_{\beta=0} = v$

$$v(\alpha) = \tilde{v}(\boldsymbol{k}\alpha), \qquad \tilde{v}(\boldsymbol{\alpha}) = \sum_{\boldsymbol{j}\in\mathbb{Z}^d} \hat{v}_{\boldsymbol{j}} e^{i\langle \boldsymbol{j},\boldsymbol{\alpha}\rangle}, \qquad \hat{v}_{\boldsymbol{j}} = i\,\mathrm{sgn}(\langle \boldsymbol{j},\boldsymbol{k}\rangle)\hat{u}_{\boldsymbol{j}}, \quad (\boldsymbol{j}\neq 0)$$

$$\boxed{v = \hat{v}_{\mathbf{0}} - H[u], \qquad u = \hat{u}_{\mathbf{0}} + H[v]}$$

Quasi-periodic Hilbert transform

$$H[u](\alpha) = \frac{1}{\pi}\,\mathrm{PV}\int_{-\infty}^{\infty} \frac{u(\xi)}{\alpha - \xi}\,d\xi = \sum_{\boldsymbol{j}\in\mathbb{Z}^d} (-i)\,\mathrm{sgn}(\langle \boldsymbol{j},\,\boldsymbol{k}\rangle)\hat{u}_{\boldsymbol{j}} e^{i\langle \boldsymbol{j},\,\boldsymbol{k}\rangle\alpha}$$

Torus version: want $H[u](\alpha) = H[\tilde{u}](\boldsymbol{k}\alpha)$

$$H[\tilde{u}](\boldsymbol{\alpha}) = \sum_{\boldsymbol{j}\in\mathbb{Z}^d} (-i)\,\mathrm{sgn}(\langle \boldsymbol{j},\,\boldsymbol{k}\rangle)\hat{u}_{\boldsymbol{j}} e^{i\langle \boldsymbol{j},\,\boldsymbol{\alpha}\rangle} \qquad (\text{or } H_k[\tilde{u}])$$

Projections

$$P = \mathrm{id} - P_0, \qquad P_0[u] = P_0[\tilde{u}] = \hat{u}_{\boldsymbol{0}} = \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} \tilde{u}(\boldsymbol{\alpha})\,d\alpha_1 \dots d\alpha_d$$

The analytic extension is quasi-periodic on slices of constant depth

$$f(w) = \tilde{f}(\boldsymbol{k}\alpha, \beta), \qquad (w = \alpha + i\beta\,,\ \beta \le 0)$$

$$\tilde{f}(\boldsymbol{\alpha}, \beta) = \hat{u}_{\boldsymbol{0}} + i\hat{v}_{\boldsymbol{0}} + \sum_{\langle \boldsymbol{j},\boldsymbol{k}\rangle < 0} 2[\hat{u}_{\boldsymbol{j}} e^{-\langle \boldsymbol{j},\boldsymbol{k}\rangle\beta}] e^{i\langle \boldsymbol{j},\boldsymbol{\alpha}\rangle}$$

Pseudo-spectral method in space (on torus)

Let f denote η, φ, χ etc. and let $\tilde{f}$ denote

$$f(\alpha) = \tilde{f}(\alpha, k\alpha), \qquad \tilde{f}(\alpha_1, \alpha_2) = \sum_{j_1, j_2 \in \mathbb{Z}} \hat{f}_{j_1, j_2} e^{i(j_1\alpha_1 + j_2 k \alpha_2)}, \qquad (\alpha_1, \alpha_2) \in \mathbb{T}^2$$

$$\widetilde{f_\alpha}(\alpha_1, \alpha_2) = \sum_{j_1, j_2 \in \mathbb{Z}} i(j_1 + j_2 k) \hat{f}_{j_1, j_2} e^{i(j_1\alpha_1 + j_2\alpha_2)},$$

$$\widetilde{H[f]}(\alpha_1, \alpha_2) = \sum_{j_1, j_2 \in \mathbb{Z}} (-i)\mathrm{sgn}(j_1 + j_2 k) \hat{f}_{j_1, j_2} e^{i(j_1\alpha_1 + j_2\alpha_2)}.$$

$$\tilde{f}_{m_1, m_2} = \tilde{f}(2\pi m_1/M_1\,,\, 2\pi m_2/M_2), \qquad (0 \le m_1 < M_1\,,\, 0 \le m_2 < M_2)$$

$$\hat{f}_{j_1, j_2} = \frac{1}{M_2} \sum_{m_2=0}^{M_2-1} \left(\frac{1}{M_1} \sum_{m_1=0}^{M_1-1} \tilde{f}_{m_1, m_2} e^{-2\pi i j_1 m_1/M_1} \right) e^{-2\pi i j_2 m_2/M_2}, \qquad \begin{pmatrix} 0 \le j_1 \le M_1/2 \\ -M_2/2 < j_2 \le M_2/2 \end{pmatrix}$$

High-order Runge-Kutta or ETD in time

$$\xi_\alpha = 1 + H[\eta_\alpha], \qquad \psi = -H[\varphi], \qquad J = \xi_\alpha^2 + \eta_\alpha^2, \qquad \chi = \frac{\psi_\alpha}{J},$$

choose C_1, e.g. as in (1), compute $\dfrac{dx_0}{dt}$ in (2) if necessary,

$$\eta_t = -\eta_\alpha H[\chi] - \xi_\alpha \chi + C_1 \eta_\alpha, \qquad \kappa = \frac{\xi_\alpha \eta_{\alpha\alpha} - \eta_\alpha \xi_{\alpha\alpha}}{J^{3/2}},$$

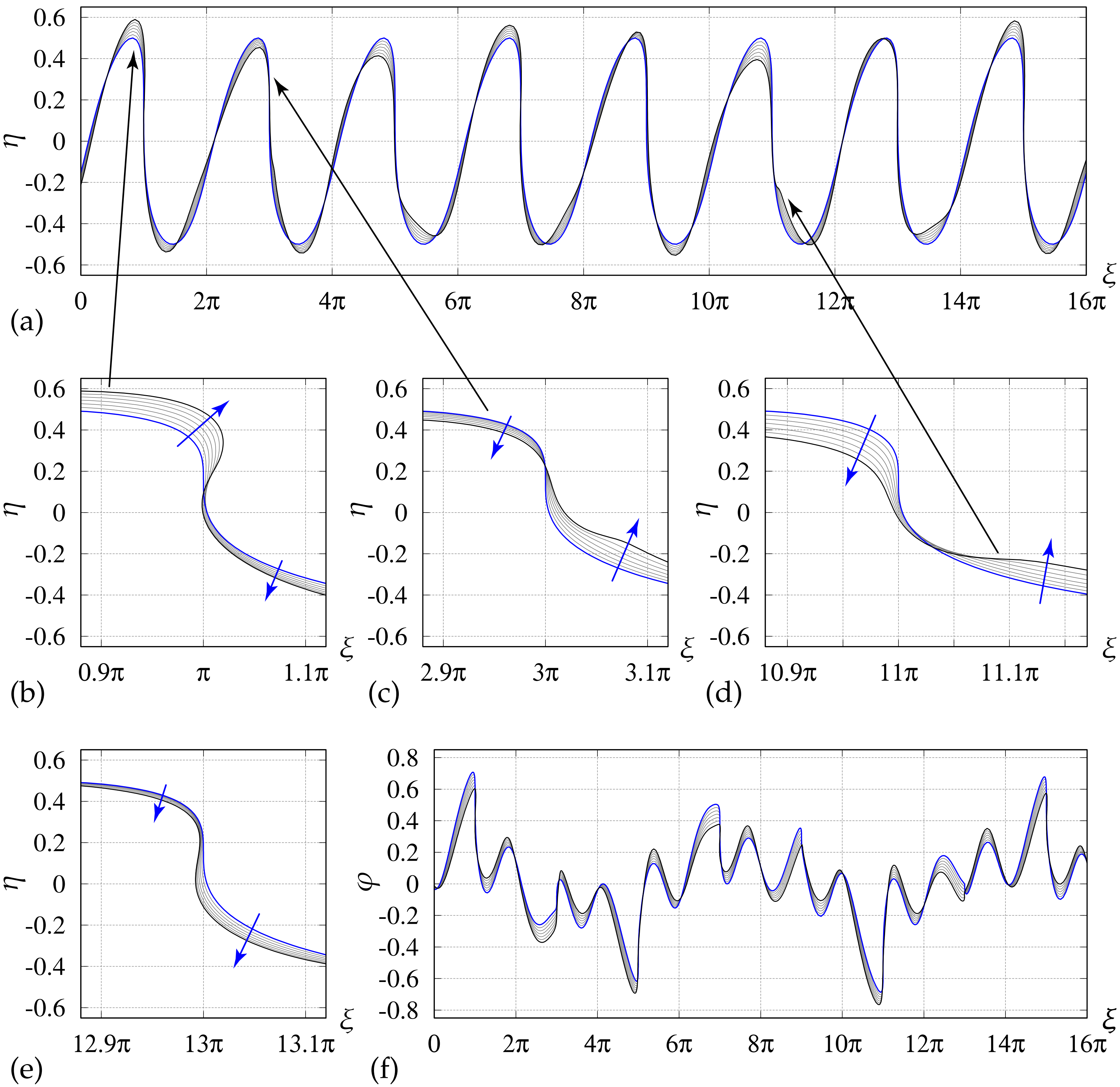
$$\varphi_t = P\left[\frac{\psi_\alpha^2 - \varphi_\alpha^2}{2J} - \varphi_\alpha H[\chi] + C_1 \varphi_\alpha - g\eta + \tau\kappa\right].$$

ETD / SSD:

$$\begin{pmatrix} \eta_t \\ \varphi_t \end{pmatrix} = L \begin{pmatrix} \eta \\ \varphi \end{pmatrix} + \mathcal{N}, \qquad L = \begin{pmatrix} 0 & H\partial_\alpha \\ -(gP - \tau\partial_{\alpha\alpha}) & 0 \end{pmatrix}$$

$$\mathcal{N} = \begin{pmatrix} -\eta_\alpha H[\chi] - \left(\xi_\alpha \chi - \psi_\alpha\right) + C_1 \eta_\alpha \\ P\left[\frac{\psi_\alpha^2 - \varphi_\alpha^2}{2J} - \varphi_\alpha H[\chi] + C_1 \varphi_\alpha + \tau(\kappa - \eta_{\alpha\alpha})\right] \end{pmatrix}$$

Overturning wave example (k=1/$\sqrt{2}$)



initial parametrization
(not conformal yet)

$$\xi_1(\sigma) = \sigma + \frac{3}{5}\sin\sigma - \frac{1}{5}\sin 2\sigma,$$
$$\eta_1(\sigma) = -(1/2)\cos(\sigma + \pi/2.5),$$
$$\varphi_1(\sigma) = -(1/2)\cos(\sigma + \pi/4).$$

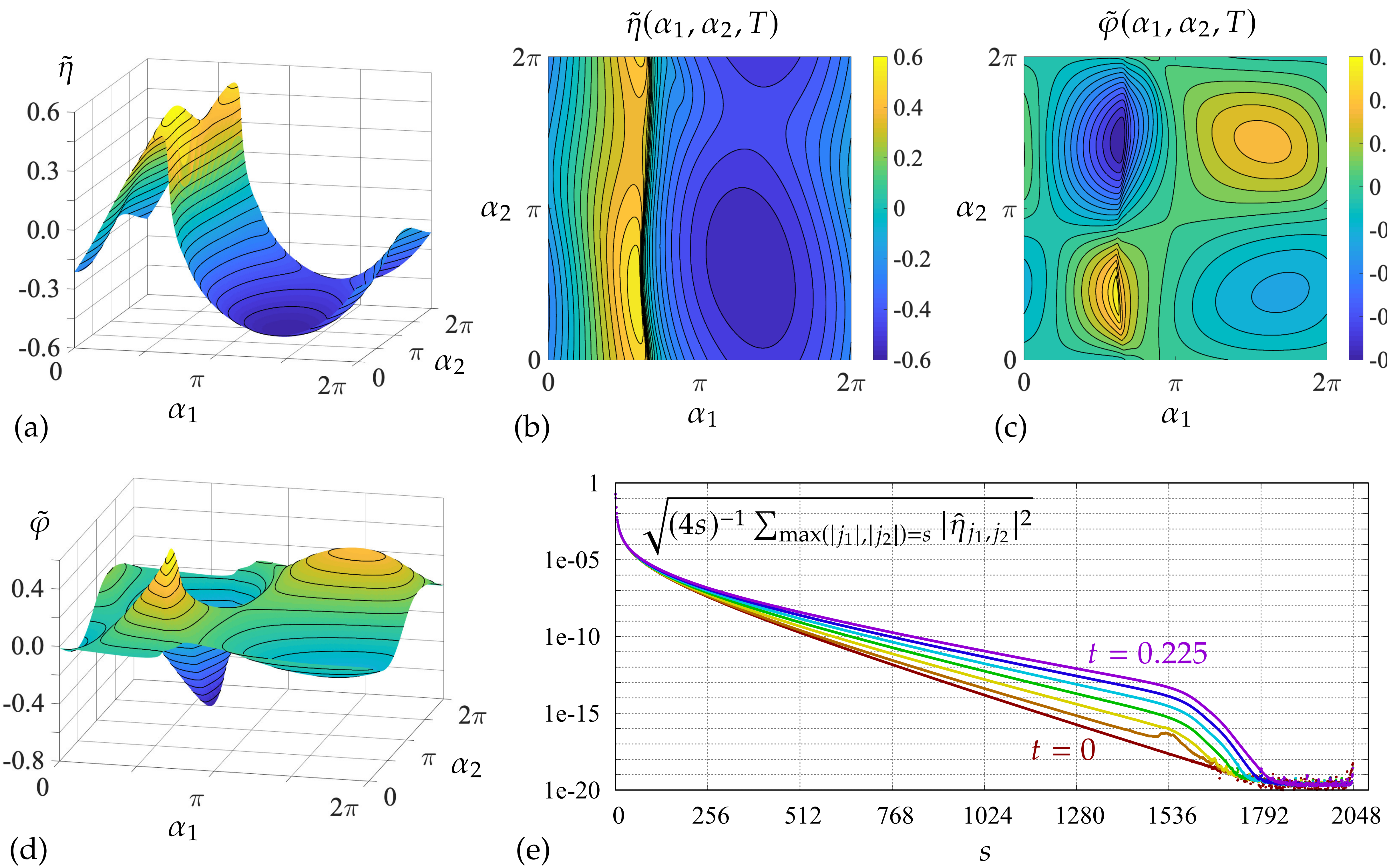
search for $\eta_3(\alpha)$, $B_3(\alpha)$, x_3 such that

$$\alpha + x_3 + H[\eta_3](\alpha) = \xi_1(\alpha + B_3(\alpha)),$$
$$\eta_3(\alpha) = \eta_1(\alpha + B_3(\alpha)),$$
$$B_3(0) = 0$$
$$\varphi_3(\alpha) = \varphi_1(\alpha + B_3(\alpha))$$

initial conditions on torus:

$$\tilde{\eta}_0(\alpha_1, \alpha_2) = \eta_3(\alpha_1),$$
$$\tilde{\varphi}_0(\alpha_1, \alpha_2) = \varphi_3(\alpha_1)\cos(\alpha_2 - q)$$
$$q = 0.6k\pi$$

Overturning wave example



M=4096 (over 16 million degrees of freedom)
(evolved over 5400 timesteps)

Spatially quasi-periodic traveling waves

$$(1) \quad \psi_\alpha = c\eta_\alpha, \qquad \varphi_\alpha = H[\psi_\alpha] = cH[\eta_\alpha] = c(\xi_\alpha - 1)$$

$$\xi_\alpha = 1 + H[\eta_\alpha], \qquad J = \xi_\alpha^2 + \eta_\alpha^2,$$

$$\kappa = \frac{\xi_\alpha \eta_{\alpha\alpha} - \eta_\alpha \xi_{\alpha\alpha}}{J^{3/2}}, \qquad P\left[\frac{c^2}{2J} + g\eta - \tau\kappa\right] = 0. \qquad (2)$$

$$(1) \quad -\psi_\alpha/\sqrt{J} = \zeta_t \cdot \hat{\mathbf{n}} = (c,0)\cdot\hat{\mathbf{n}} = -c\eta_\alpha/\sqrt{J} \qquad \text{(kinematic condition)}$$

$$(2) \quad \breve{z} = z - ct, \quad \breve{\Phi}^{\text{phys}}(\breve{z}) = \Phi^{\text{phys}}(\breve{z} + ct, t) - c\breve{z} \quad \text{stationary (time independent)}$$

$$\breve{\Phi}(w) = \breve{\Phi}^{\text{phys}}(\breve{z}(w)) = -cw, \qquad \breve{z}(w) = z(w,0)$$

$$|\nabla\breve{\Phi}^{\text{phys}}|^2 = |\breve{\Phi}'(w)/\breve{z}'(w)|^2 = c^2/J \qquad \text{(2) then follows from steady Bernoulli}$$

see also Bridges/Dias, 1996

Linearization about zero

$$-c^2 H\eta_\alpha + g\eta - \tau\eta_{\alpha\alpha} = 0$$

dispersion relation: $$c^2 = gk^{-1} + \tau k$$

denote zeros by k_1, k_2. Non-dimensionalize so $k_1 = 1,\ k_2 = k_1 k$

$$c^2 - g - \tau = 0 \qquad\qquad c^2 k - g - \tau k^2 = 0$$

$$\left(c = \sqrt{g+\tau}, \quad k = g/\tau\right)$$

If k is rational, this leads to the classical Wilton ripple problem

If k is irrational, good place to search for quasi-periodic traveling waves

Nonlinear least squares problem (overdetermined)

Residual

$$\mathcal{R}[\tau, b, \hat{\eta}] := P\left[\frac{b}{2\tilde{J}} + g\tilde{\eta} - \tau\tilde{\kappa}\right] \qquad (b = c^2)$$

Objective function

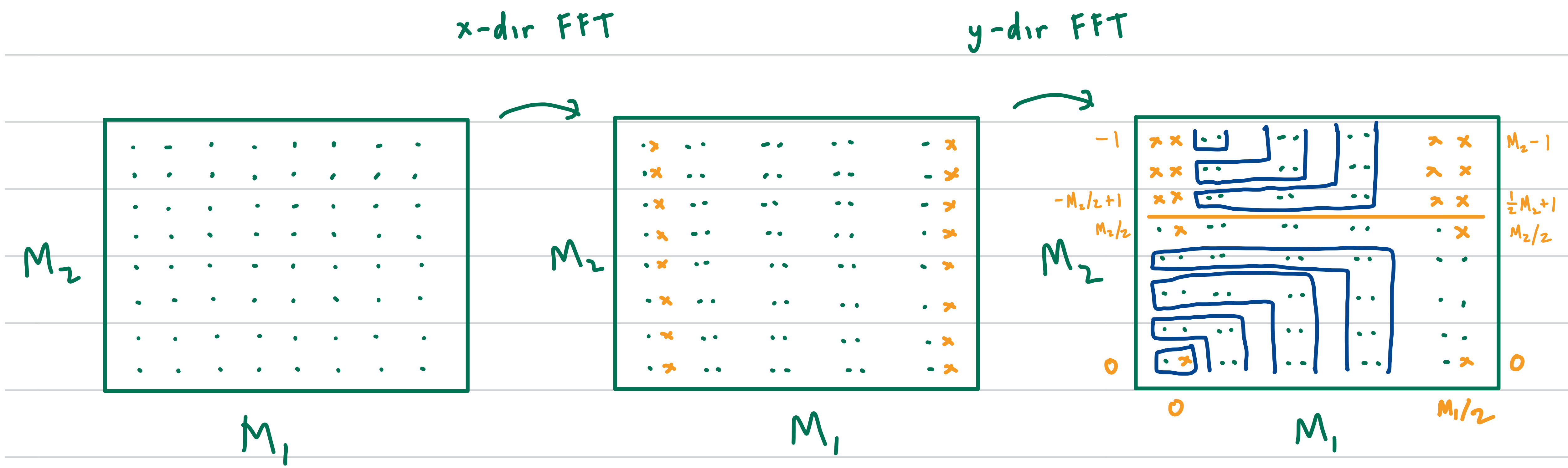
$$\mathcal{F}[\tau, b, \hat{\eta}] := \frac{1}{8\pi^2}\int_{\mathbb{T}^2} \mathcal{R}^2[\tau, b, \hat{\eta}]\, d\alpha_1\, d\alpha_2$$

Numerical version

$$r_m(p) = \mathcal{R}\left[\tau, b, \eta\right](\alpha_{m_1}, \alpha_{m_2})/M, \qquad \begin{pmatrix} m = 1 + m_1 + M m_2 \\ \alpha_{m_i} = 2\pi m_i/M \end{pmatrix}, \qquad 0 \le m_i < M.$$

$$f(p) = \frac{1}{2} r(p)^T r(p) \approx \mathcal{F}\left[\tau, b, \hat{\eta}\right]$$

Nonlinear least squares problem (overdetermined)



Numerical version

$$r_m(p) = \mathcal{R}\left[\tau, b, \eta\right](\alpha_{m_1}, \alpha_{m_2})/M, \qquad \begin{pmatrix} m = 1 + m_1 + M m_2 \\ \alpha_{m_i} = 2\pi m_i/M \end{pmatrix}, \qquad 0 \le m_i < M.$$

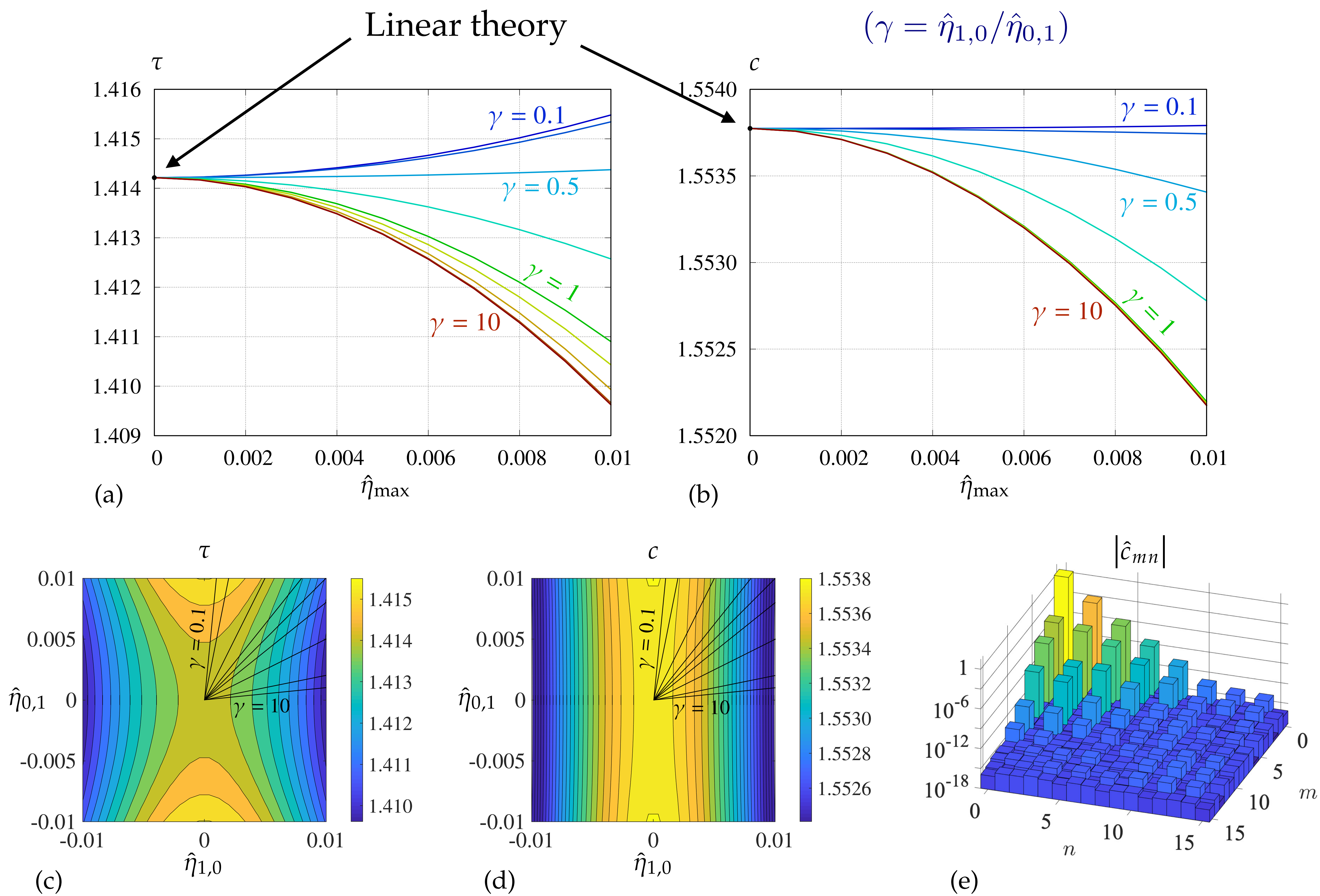
$$f(p) = \frac{1}{2} r(p)^T r(p) \approx \mathcal{F}\left[\tau, b, \hat{\eta}\right]$$

(also set $\hat{\eta}_{00} = 0$)

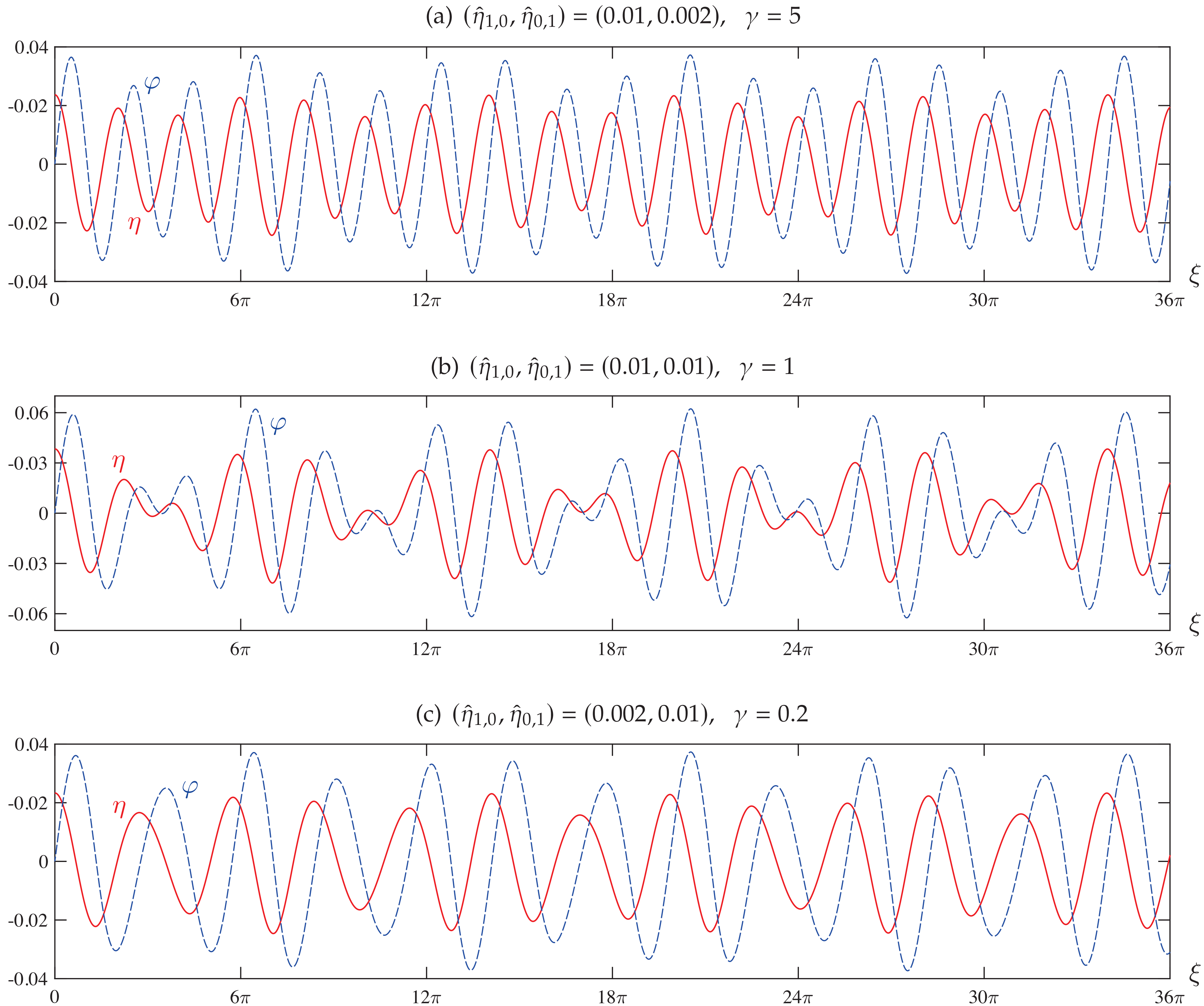
unknowns: fix $\hat{\eta}_{1,0}$ and $\hat{\eta}_{0,1}$ (replace them with τ and $b = c^2$ in the list)

$$p_1 = \tau, \quad p_2 = \hat{\eta}_{1,1}, \quad p_3 = b, \quad p_4 = \hat{\eta}_{1,-1}, \quad p_5 = \hat{\eta}_{0,2}, \quad \ldots, \quad p_{N(N/2+1)} = \hat{\eta}_{1,-N/2}$$

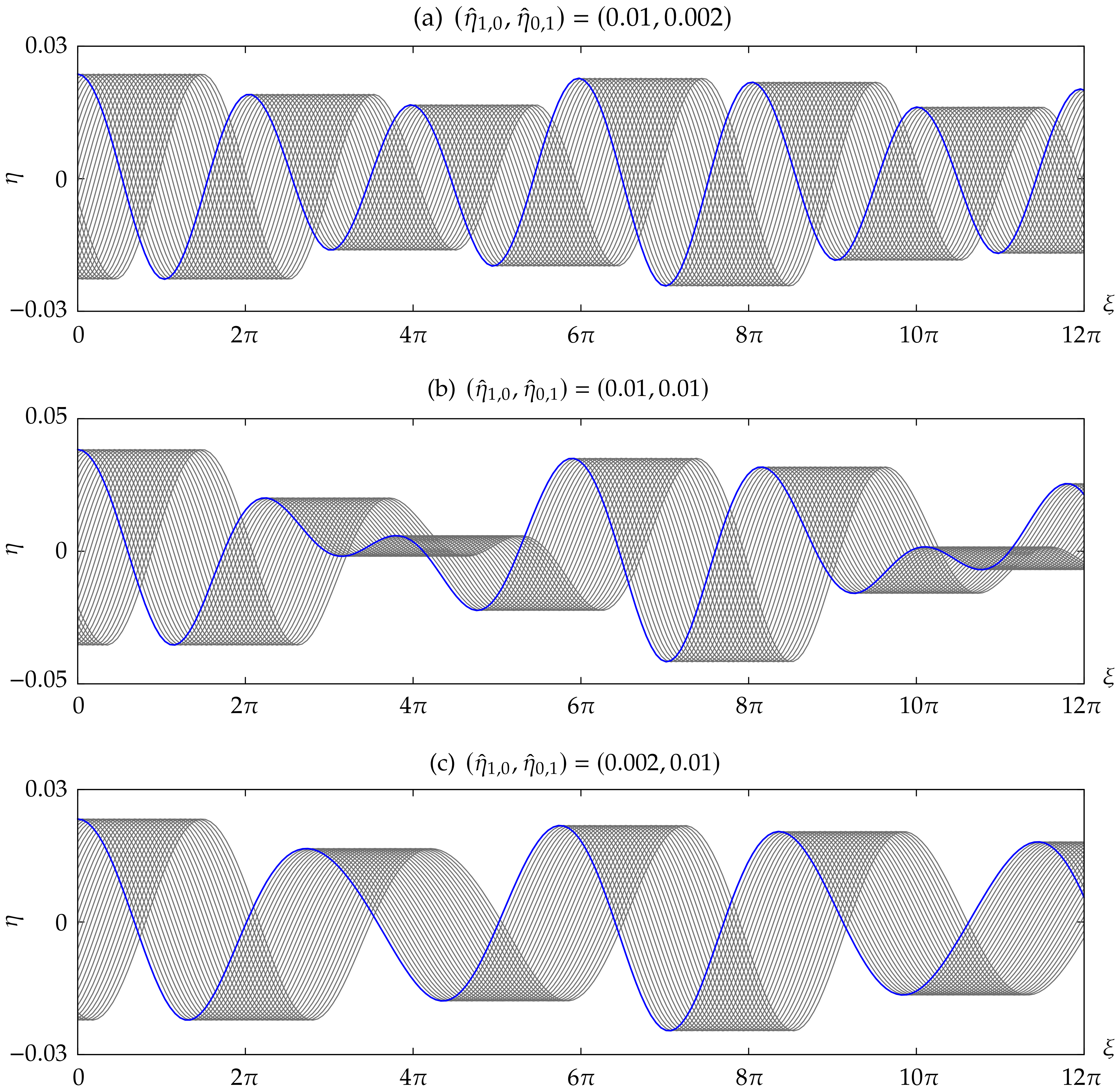
Two-parameter family of spatially quasi-periodic traveling waves



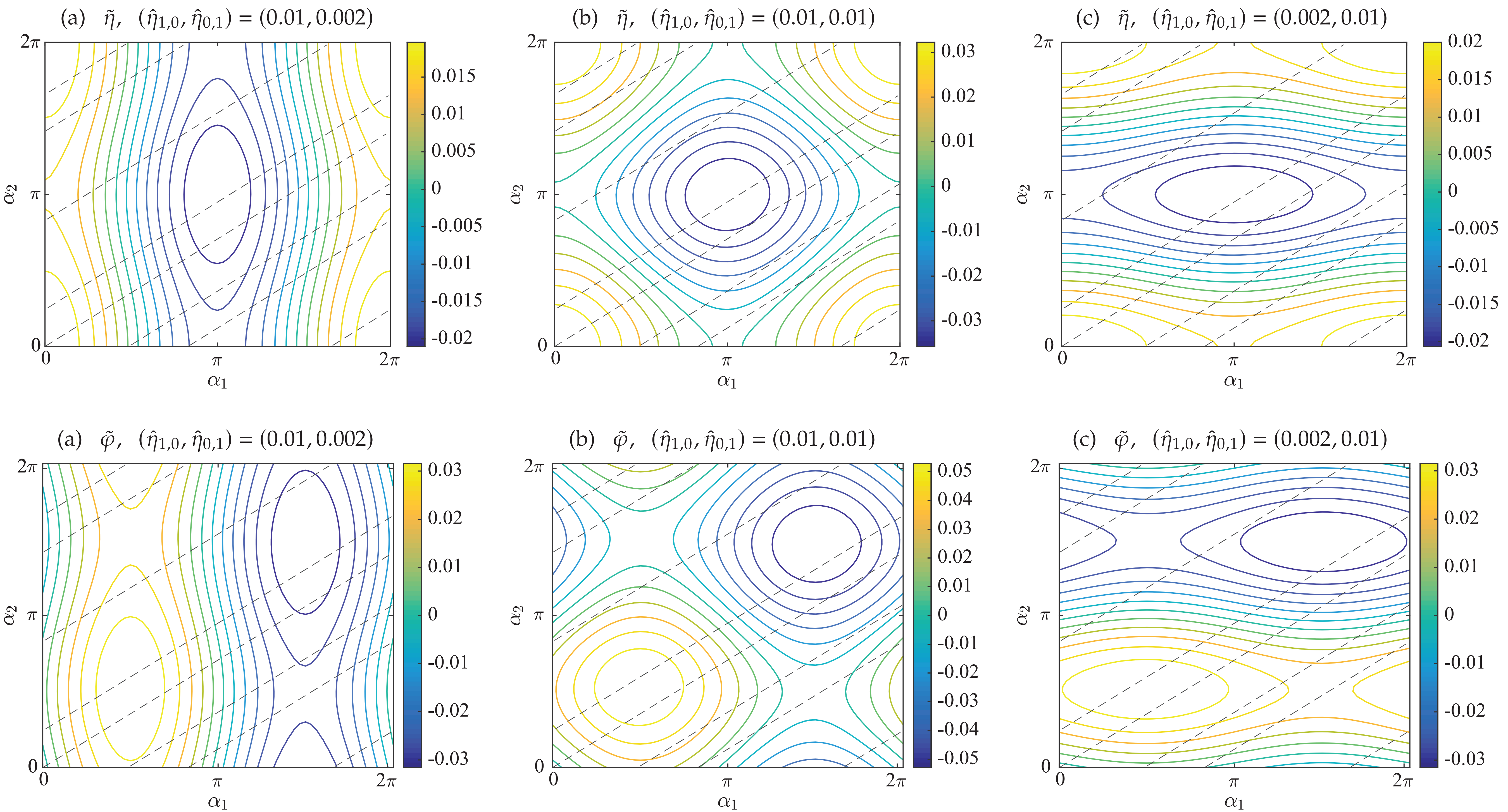
Spatially quasi-periodic traveling waves ($t = 0$)



Spatially quasi-periodic traveling waves



Spatially quasi-periodic traveling waves (torus view)

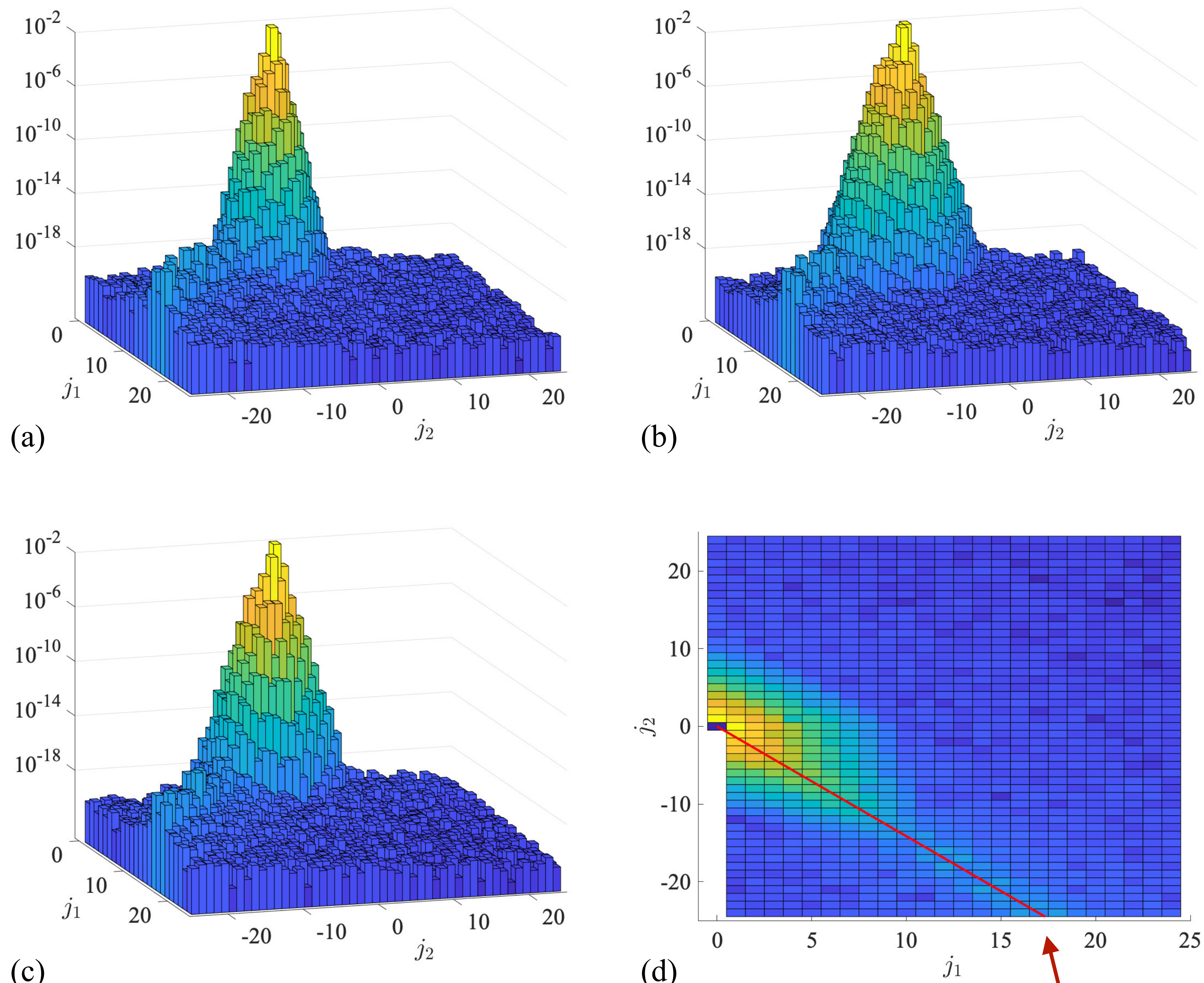


$\gamma = 5$ $\gamma = 1$ $\gamma = 0.2$

Spatially quasi-periodic traveling waves (Fourier modes)

$\gamma = 5$

$\gamma = 1$

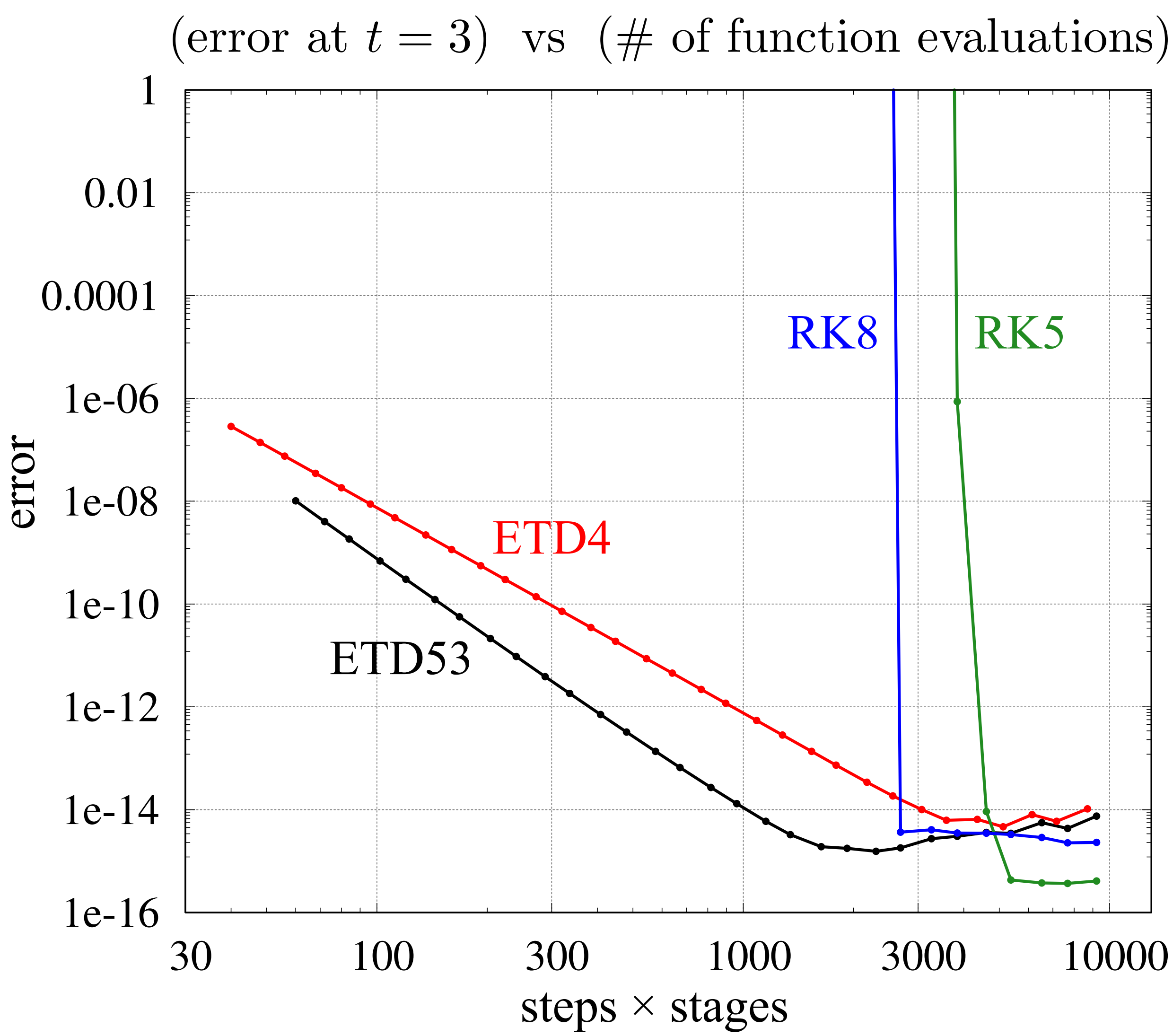


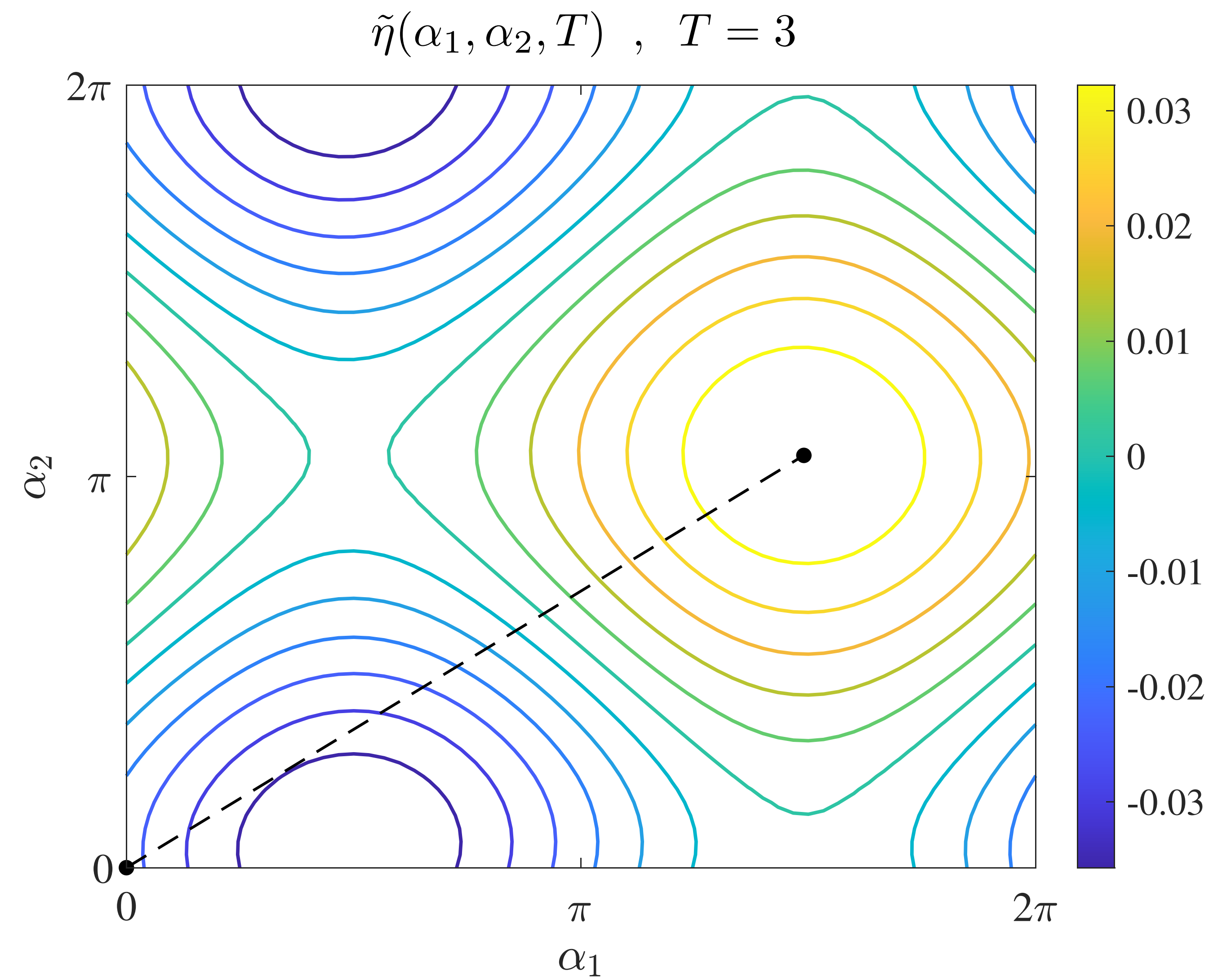
$\gamma = 0.2$

$\gamma = 1$

$j_1 + j_2 k = 0$

Error in evolving the solution from $t = 0$ to $t = 3$





$$\mathrm{err} = \sqrt{\|\tilde{\eta} - \tilde{\eta}_{\mathrm{exact}}\|^2 + \|\tilde{\varphi} - \tilde{\varphi}_{\mathrm{exact}}\|^2}, \qquad \|\tilde{\eta}\|^2 = \frac{1}{M_1 M_2} \sum_{m_1, m_2} \tilde{\eta}\left(\frac{2\pi m_1}{M_1}, \frac{2\pi m_2}{M_2}, T\right)^2$$

$$\tilde{\eta}_{\mathrm{exact}}(\alpha_1, \alpha_2, t) = \tilde{\eta}_0\big(\alpha_1 - ct + A(t), \alpha_2 - kct + kA(t)\big) + A(t),$$
$$\tilde{\varphi}_{\mathrm{exact}}(\alpha_1, \alpha_2, t) = \tilde{\varphi}_0\big(\alpha_1 - ct + A(t), \alpha_2 - kct + kA(t)\big)$$

$$A(t) = \mathcal{A}(-ct, -kct)$$

Families of quasi-periodic solutions and a return to physical space

Theorem B.1. *The solution pair* $(\tilde{\zeta}, \tilde{\varphi})$ *on the torus represents an infinite family of quasi-periodic solutions on* $\mathbb{R}$ *given by*

$$\text{(B.1)} \qquad \begin{aligned} \zeta(\alpha, t\,;\, \theta_1, \theta_2, \delta) &= \alpha + \delta + \tilde{\zeta}(\theta_1 + \alpha, \theta_2 + k\alpha, t), \\ \varphi(\alpha, t\,;\, \theta_1, \theta_2) &= \tilde{\varphi}(\theta_1 + \alpha, \theta_2 + k\alpha, t), \end{aligned} \qquad \begin{pmatrix} \alpha \in \mathbb{R},\ t \geqslant 0 \\ \theta_1, \theta_2, \delta \in \mathbb{R} \end{pmatrix}.$$

Theorem B.2. *Fix* $t \geqslant 0$ *and suppose* $\xi_\alpha(\alpha, t\,;\, 0, \theta, 0) > 0$ *for* $\alpha \in [0, 2\pi)$ *and* $\theta \in [0, 2\pi)$. *Then there is a periodic, real analytic function* $\mathcal{A}(x_1, x_2, t)$ *defined on* $\mathbb{T}^2$ *satisfying*

$$\text{(B.6)} \qquad \mathcal{A}(x_1, x_2, t) + \tilde{\xi}\big(x_1 + \mathcal{A}(x_1, x_2, t)\,,\, x_2 + k\mathcal{A}(x_1, x_2, t)\,,\, t\big) = 0, \qquad (x_1, x_2) \in \mathbb{T}^2.$$

Given $\theta \in [0, 2\pi)$, *the change of variables* $\alpha = x + \mathcal{A}(x, \theta + kx, t)$ *satisfies*

$$\text{(B.7)} \qquad \xi(\alpha, t\,;\, 0, \theta, 0) = \alpha + \tilde{\xi}(\alpha, \theta + k\alpha, t) = x, \qquad (x \in \mathbb{R}).$$

This allows us to express solutions in the family (B.4) as functions of x *and* t,

$$\text{(B.8)} \qquad \begin{aligned} \eta^{phys}(x, t\,;\, 0, \theta, 0) &= \eta(\alpha, t\,;\, 0, \theta, 0), \\ \varphi^{phys}(x, t\,;\, 0, \theta, 0) &= \varphi(\alpha, t\,;\, 0, \theta), \end{aligned} \qquad \big(\alpha = x + \mathcal{A}(x, \theta + kx, t)\big).$$

These functions are real analytic, quasi-periodic functions of x *in the sense that*

$$\text{(B.9)} \qquad \begin{aligned} \eta^{phys}(x, t; 0, \theta, 0) &= \tilde{\eta}^{phys}(x, \theta + kx, t) \\ \varphi^{phys}(x, t; 0, \theta, 0) &= \tilde{\varphi}^{phys}(x, \theta + kx, t) \end{aligned}$$

with

$$\text{(B.10)} \qquad \begin{aligned} \tilde{\eta}^{phys}(x_1, x_2, t) &= \tilde{\eta}(x_1 + \mathcal{A}(x_1, x_2, t),\ x_2 + k\mathcal{A}(x_1, x_2, t),\ t), \\ \tilde{\varphi}^{phys}(x_1, x_2, t) &= \tilde{\varphi}(x_1 + \mathcal{A}(x_1, x_2, t),\ x_2 + k\mathcal{A}(x_1, x_2, t),\ t). \end{aligned}$$

Conformal mapping theorems

Theorem A.1. *Suppose $\varepsilon > 0$ and $z(w)$ is analytic on the half-plane $\mathbb{C}_\varepsilon^- = \{w : \operatorname{Im} w < \varepsilon\}$. Suppose there is a constant $M > 0$ such that $|z(w) - w| \leqslant M$ for $w \in \mathbb{C}_\varepsilon^-$, and that the restriction $\zeta = z|_{\mathbb{R}}$ is injective. Then the curve $\zeta(\alpha)$ separates the complex plane into two regions, and $z(w)$ is an analytic isomorphism of the lower half-plane onto the region below the curve $\zeta(\alpha)$.*

Corollary A.3. *Suppose $k > 0$ is irrational, $\tilde{\eta}(\alpha_1, \alpha_2) = \sum_{(j_1,j_2)\in\mathbb{Z}^2} \hat{\eta}_{j_1,j_2} e^{i(j_1\alpha_1 + j_2\alpha_2)}$, and there exist constants C and $\varepsilon > 0$ such that*

$$\text{(A.7)} \qquad \hat{\eta}_{-j_1,-j_2} = \overline{\hat{\eta}_{j_1,j_2}}, \qquad \left|\hat{\eta}_{j_1,j_2}\right| \leqslant Ce^{-3\varepsilon K \max(|j_1|,|j_2|)}, \qquad (j_1, j_2) \in \mathbb{Z}^2,$$

where $K = \max(k, 1)$. Let x_0 be real and define $\tilde{\xi} = x_0 + H[\tilde{\eta}]$, $\tilde{\zeta} = \tilde{\xi} + i\tilde{\eta}$ and

$$\text{(A.8)} \qquad \tilde{z}(\alpha_1, \alpha_2, \beta) = x_0 + i\hat{\eta}_{0,0} + \sum_{j_1+j_2k<0} 2i\hat{\eta}_{j_1,j_2} e^{-(j_1+j_2k)\beta} e^{i(j_1\alpha_1+j_2\alpha_2)}, \qquad (\beta < \varepsilon),$$

where the sum is over all integer pairs (j_1, j_2) satisfying the inequality. Suppose also that for each fixed $\theta \in [0, 2\pi)$, the function $\alpha \mapsto \zeta(\alpha; \theta) = \alpha + \tilde{\zeta}(\alpha, \theta + k\alpha)$ is injective from $\mathbb{R}$ to $\mathbb{C}$ and $\zeta_\alpha(\alpha; \theta) \neq 0$ for $\alpha \in \mathbb{R}$. Then for each $\theta \in \mathbb{R}$, the curve $\zeta(\alpha; \theta)$ separates the complex plane into two regions and

$$\text{(A.9)} \qquad z(\alpha + i\beta; \theta) = (\alpha + i\beta) + \tilde{z}(\alpha, \theta + k\alpha, \beta), \qquad (\beta < \varepsilon)$$

is an analytic isomorphism of the lower half-plane onto the region below $\zeta(\alpha; \theta)$. Moreover, there is a constant $\delta > 0$ such that $|z_w(w; \theta)| \geqslant \delta$ for $\operatorname{Im} w \leqslant 0$ and $\theta \in \mathbb{R}$.

Part 0: Introduction

Part 1: spatially quasi-periodic water waves

- Equations of motion
- Initial value problem
- Traveling waves (two quasi-periods)

Part 2: temporally Quasi-periodic solutions (ran out of time)

- two quasi-periods (methods and examples)
- three quasi-periods (method and example)