Khovanov Homology Detects Split Links

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Outline

- History, statement of main theorem
- Overview of proof.
- Background on A_∞-modules.
- The basepoint action on Khovanov homology.
- Twisted coefficients and action of the exterior algebra on Heegaard Floer homology

Khovanov homology detects…

Khovanov homology detects…

- [Grigsby-Wehrli, '08] $\dim Kh(n-cable\ of\ K)$ detects the unknot $\forall\, n > 1$.
- [Hedden, '08] $\dim Kh(2-cable\ of\ K)$ detects the unknot.
- [Kronheimer-Mrowka, '10] $\dim Kh(K)$ detects the unknot.
- [Hedden-Ni, '10] $\mathrm{g}\dim Kh(L)$ detects the 2-component unlink.
- [Hedden-Ni, '12] $Kh(L)$ as a module over $\mathbb{F}_2[X_1, \ldots, X_\ell]/(X_1^2, \ldots, X_\ell^2)$ detects the unlink.
- [Batson-Seed, '13] $\mathrm{g}\dim Kh(L)$ detects the unlink.
- [Baldwin-Sivek, '18] $Kh(L; \mathbb{Z})$ detects the trefoils.
- [Baldwin-Sivek-Xie, '18] $\mathrm{g}\dim Kh(L; \mathbb{F}_2)$ detects the Hopf link.
- [Xie-B. Zhang, '19] $\dim Kh(L; \mathbb{F}_2)$ detects forests of Hopf links and unknots.
- [J. Wang, '20] $\dim Kh(L; \mathbb{F}_2)$ detects trivial band sums $K_1 \coprod K_2$ of split links, among all band sums of $K_1 \coprod K_2$.

Near-universal strategy: exploit a spectral sequence to a more geometric invariant (Heegaard / monopole Floer homology or instanton Floer homology)

Khovanov homology [also] detects…split links.

- Given a link $L, p, q \in L$, can
  - form reduced Khovanov homology $\widetilde{Kh}(L)$ using the basepoint p
  - which is a module over $\mathrm{R} = \mathbb{F}_2[X]/(X^2)$ using the basepoint q.
- **Theorem.** [L-Sarkar, '19] Given a link L and points $p, q \in L$, there is a 2-sphere in $S^3 \setminus L$ separating p and q if and only if $\widetilde{Kh}(L)$ is a free $\mathrm{R} = \mathbb{F}_2[X]/(X^2)$-module.
- (For the rest of the talk, everything is with $\mathbb{F}_2$-coefficients.)

Equivalent conditions to the main theorem

- Call a chain complex C_* over R *quasi-free* if C_* is quasi-isomorphic to a bounded chain complex of free R-modules.
- An obstruction to quasi-freeness over $\mathrm{R} = \mathbb{F}_2[X]/(X^2)$: is total complex of

$$\cdots \xleftarrow{X} C_* \xleftarrow{X} C_* \xleftarrow{X} C_* \xleftarrow{X} \cdots$$

acyclic? Call the homology of this the *unrolled homology.*
- **Theorem.** [L-Sarkar] For $p, q \in L$, the following are equivalent:
    1. There is a 2-sphere separating p and q.
    2. $\widetilde{Kh}(L)$ is a free R-module.
    3. $\tilde{C}^{Kh}(L)$ is quasi-free.
    4. The unrolled homology of $\tilde{C}^{Kh}(L)$ is trivial.

Overview of the proof

TFAE:

1. There is a 2-sphere separating p and q.
2. $\widetilde{Kh}(L)$ is a free R-module.
3. $\tilde{C}^{Kh}(L)$ is quasi-free.
4. The unrolled homology of $\tilde{C}^{Kh}(L)$ is trivial.

Ingredients

- **Freeness.** [Shumakovitch '03] $Kh(K)$ (unreduced) is a free $R = \mathbb{F}_2[X]/(X^2)$-module.
  - Also follows from an argument in odd Khovanov homology (Ozsváth–Rasmussen–Szabó).
- **Basepoint independence.** Up to quasi-isomorphism over R, $\tilde{C}^{Kh}(L)$ depends only on L and which components contain p, q. Sarkar, Hedden-Ni $+\epsilon$
- **Spectral sequence.** The Ozsváth-Szabó spectral sequence $\widetilde{Kh}(L) \Rightarrow \widehat{HF}(\Sigma(L))$ respects the (A_∞) R-module structure. Ozsváth-Szabó, Hedden-Ni $+\epsilon$
- **Unrolled $\widehat{HF}$.** The unrolled homology of $\widehat{HF}(Y)$ is isomorphic to the Novikov twisted Heegaard Floer homology $\widehat{HF}(Y; \Lambda_\omega)$.
- **Sphere detection.** $\widehat{HF}(Y; \Lambda_\omega)$ detects homologically essential S^2s. Specifically, $\widehat{HF}(Y; \Lambda_\omega) = 0$ if and only if there is an S^2 with $\langle \omega, S^2 \rangle \neq 0$. Ni, Hedden-Ni, Alishahi-L

TFAE:

1. There is a 2-sphere separating p and q.
2. $\widetilde{Kh}(L)$ is a free R-module.
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4. The unrolled homology of $\tilde{C}^{Kh}(L)$ is trivial.

Steps

- $1 \Rightarrow 2$, $1 \Rightarrow 3$: For a split diagram, immediate from **freeness** of $Kh(K)$ and Künneth theorem. For a general diagram, follows from **basepoint independence**.
- $2 \Rightarrow 4$, $3 \Rightarrow 4$: Algebra:
  - Can compute unrolled homology from A_∞-module structure on $\widetilde{Kh}(L)$.
  - Consider the filtration on $\cdots \overset{X}{\leftarrow} C_* \overset{X}{\leftarrow} C_* \overset{X}{\leftarrow} C_* \overset{X}{\leftarrow} \cdots$ from grading on C_*.
- $4 \Rightarrow 1$:
  - **Spectral sequence**: Unrolled homology of $\tilde{C}^{Kh}(L)$ trivial implies unrolled homology of $\widehat{HF}\big(\Sigma(L)\big)$ trivial.
  - **Unrolled homology**: this is equivalent to $\widehat{HF}(\Sigma(L);\Lambda_\omega) = 0$.
  - **Sphere detection**: this is equivalent to existence of a splitting sphere.

A_∞ background

A_∞-module basics

- An A_∞-*module* over R is a (graded) vector space M and maps $m_{1+n}: M \otimes R^{\otimes n} \to M$ satisfying some compatibility conditions.
- For $\mathrm{R} = \mathbb{F}_2[X]/(X^2)$ (and M *strictly unital*) these are maps
$$m_{1+n}(\cdot, X, \cdots, X): M \to M$$
$$\sum_{i+j=n} m_{1+i} \circ m_{1+j} = 0$$
- The operation m_1 is a differential on M.
- A chain complex of R-modules gives an A_∞-module with $m_{1+n} = 0$ for $n > 1$.

A_∞-module basics

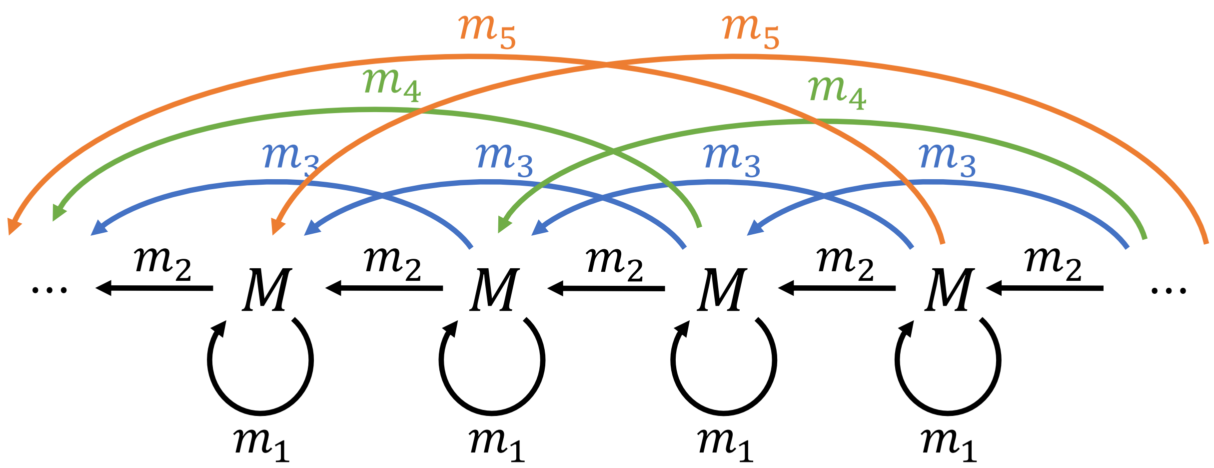
- **Homological Perturbation Lemma**. Given an A_∞-module M, a chain complex N, and a chain homotopy equivalence $M \simeq N$ over $\mathbb{F}_2$ there is an induced A_∞-module structure on N so that M and N are homotopy equivalent A_∞-modules.
- In particular, homology of any chain complex of R-modules is an A_∞-module.
- **Derived category is the A_∞ homotopy category**. Given chain complexes of R-modules M and N, M and N are quasi-isomorphic chain complexes of R-modules if and only if they are homotopy equivalent as A_∞-modules.

1. There is a 2-sphere separating p and q.
2. $\widetilde{Kh}(L)$ is a free R-module.
3. $\tilde{C}^{Kh}(L)$ is quasi-free.
4. The unrolled homology of $\tilde{C}^{Kh}(L)$ is trivial.

Unrolled complex of an A_∞-module

- The *unrolled complex* of M is $M \otimes_{\mathbb{F}_2} \mathbb{F}_2[Y^{-1}, Y]]$ with differential

$$\partial\left(x \otimes Y^i\right) = \sum_n m_{1+n}(x, X, \dots, X) Y^{i+n} \ .$$

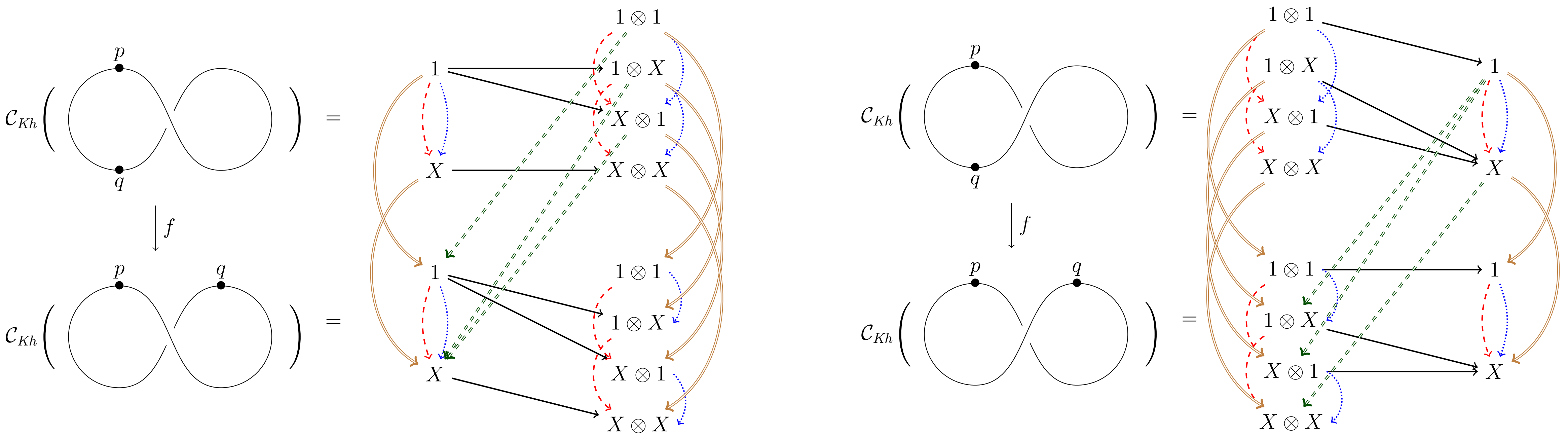


- Clearly invariant under A_∞ homotopy equivalence. So, by homological perturbation lemma, unrolled complex of $\widetilde{Kh}(L)$ and $\tilde{C}^{Kh}(L)$ agree.
- $2 \Rightarrow 4$ obvious. $3 \Rightarrow 4$ follows by filtering by grading on M.

The basepoint action on Khovanov homology

Invariance of the module structure.

- Fix points $p, q \in L$. Endows $C^{Kh}(L)$ with structure of a $(\mathbb{F}_2[W]/(W^2), \mathbb{F}_2[X]/(X^2))$-bimodule (or $\mathbb{F}_2[W,X]/(W^2,X^2)$-module).
- **Theorem**. [Hedden-Ni; LS] Up to quasi-isomorphism, the differential bimodule $C^{Kh}(L)$ depends only on the components containing p, q.
- **Corollary.** Up to quasi-isomorphism, the module structure on $\tilde{C}^{Kh}(L)$ and A_∞-module structure on $Kh(L)$ and $\widetilde{Kh}(L)$ depend only on the components containing p, q.
- To prove the theorem, it suffices to construct an A_∞ homotopy equivalence (or quasi-isomorphism) associated to moving a basepoint through a crossing.



Solid: differential. Dashed: $m_2(W,\cdot)$. Dotted: $m_2(\cdot,X)$. Double: $f_{0,1,0}$. Double-dashed: $f_{0,1,1}(\cdot,X)$.

Module structures on Heegaard Floer homology

A tale of two twistings

- Fix Y^3, homomorphism $\omega: H_2(Y) \to \mathbb{Z}$.
- Ozsváth-Szabó construct:
  - An action of $\Lambda^*\mathbb{F}_2 = \mathbb{F}_2[X]/(X^2)$ on (untwisted) $\widehat{HF}(Y)$. (More generally, a $\Lambda^*(H_1(Y)/tors)$-action.)
  - Twisted $\underline{\widehat{HF}}(Y;\Lambda_\omega)$, a module over $\mathbb{F}_2[t^{-1},t]$ or $\mathbb{F}_2[t^{-1},t]]$
- Ni, Hedden-Ni, Alishahi-L: $\underline{\widehat{HF}}(Y;\Lambda_\omega)$ vanishes if and only if Y has an S^2 with $\omega([S^2]) \neq 0$.
- Hedden-Ni, LS: The spectral sequence $\widehat{Kh}(L) \Rightarrow \widehat{HF}\big(\Sigma(L)\big)$ respects the $\mathbb{F}_2[X]/(X^2)$-action.
- Goal: relate $\widehat{HF}(Y)_{\mathbb{F}_2[X]/(X^2)}$ and $\underline{\widehat{HF}}(Y;\Lambda_\omega)_{\mathbb{F}_2[t^{-1},t]]}$.
- (cf. earlier work of Sarkar, work of Zemke.)

The $H_1/torsion$-action

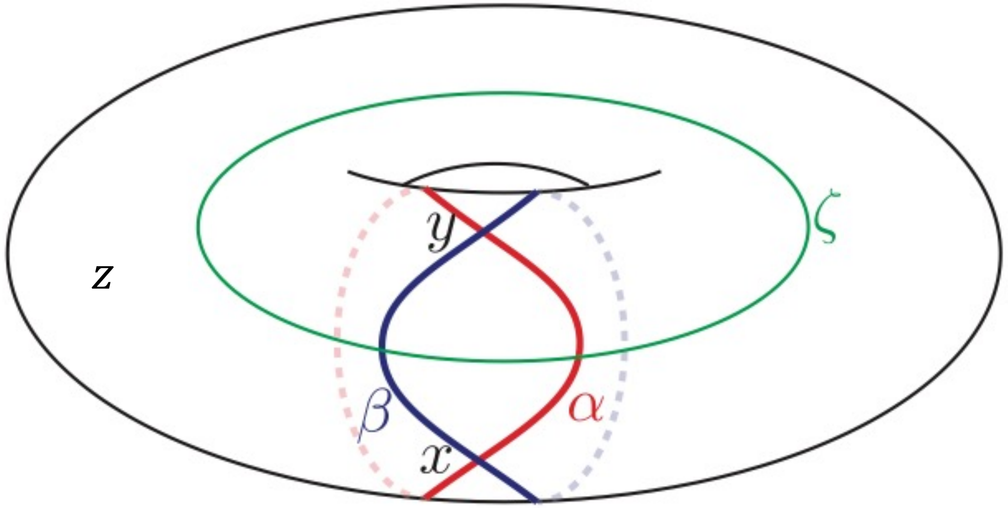
- Differential on $\widehat{HF}(\Sigma, \alpha, \beta, z)$:

$$\partial x = \sum_{y} \sum_{\substack{\phi \in \pi_2(x,y) \\ n_z(\phi)=0 \\ \mu(\phi)=1}} \left(\#\mathcal{M}(\phi)\right) y$$

- Action of $\zeta \in H_1(Y)$:

$$x \cdot \zeta = \sum_{y} \sum_{\substack{\phi \in \pi_2(x,y) \\ n_z(\phi)=0 \\ \mu(\phi)=1}} \left(\#\mathcal{M}(\phi)\right)(\zeta \cdot \partial_\alpha \phi) y$$

$$\partial(x) = 2y = 0$$
$$\partial(y) = 0$$



$$x \cdot \zeta = y$$
$$y \cdot \zeta = 0$$

The A_∞ $H_1/torsion$-action

$$\partial(x) = 2y = 0$$
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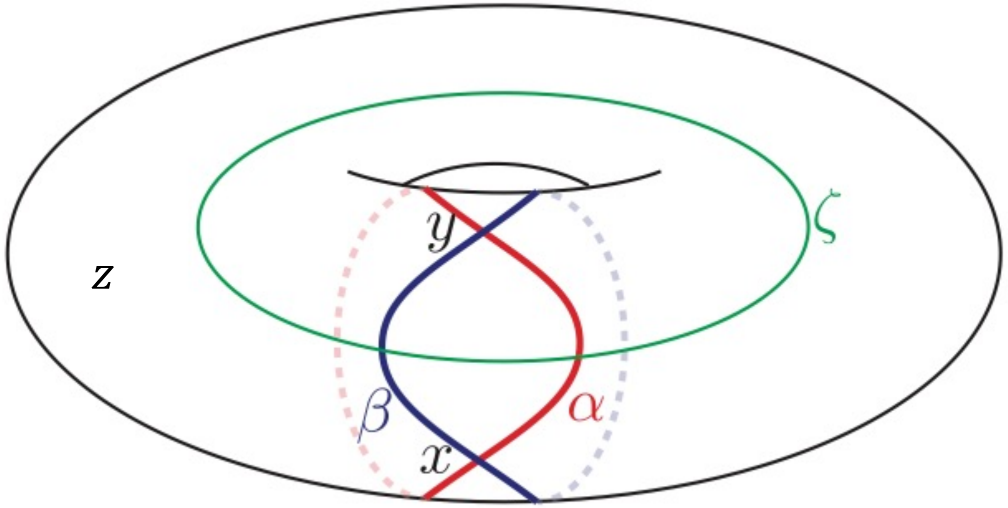
- Action of $\zeta \in H_1(Y)$:

$$x \cdot \zeta = \sum_{y} \sum_{\substack{\phi \in \pi_2(x,y) \\ n_z(\phi)=0 \\ \mu(\phi)=1}} \big(\#\mathcal{M}(\phi)\big)(\zeta \cdot \partial_\alpha \phi) y$$



- Equivalently, $x \cdot \zeta$ counts disks
  $$\text{u: } [0,1] \times \mathbb{R} \to Sym^g(\Sigma)$$
  with $u(1,0) \in \zeta \times Sym^{g-1}(\Sigma)$.
- At the level of homology, $(x \cdot \zeta) \cdot \zeta = 0$ by considering 1D moduli space of u with $u(1,0) \in \zeta$, $u(1,t) \in \zeta'$ for some $t > 0$ (ζ' a pushoff of ζ).

$$x \cdot \zeta = y$$
$$\text{y} \cdot \zeta = 0$$

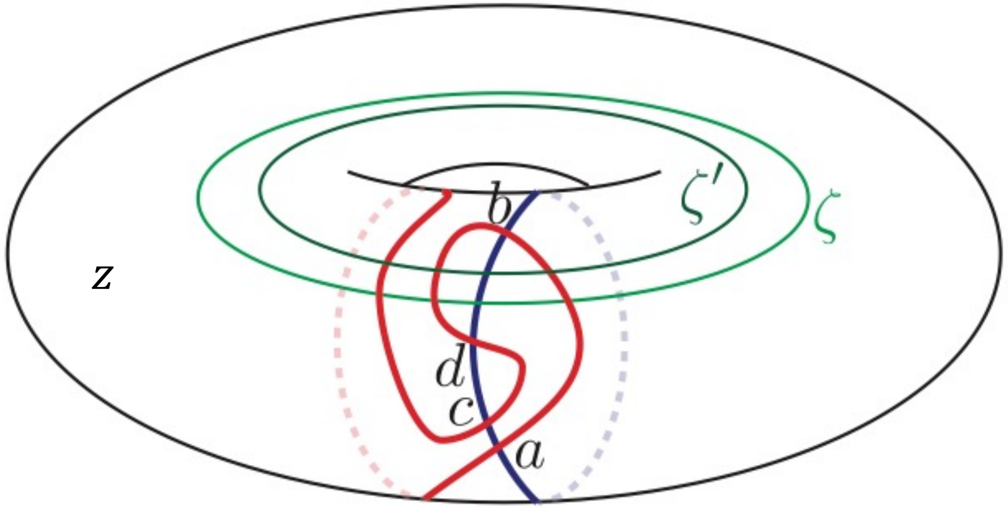
The A_∞ $H_1/torsion$-action

- Action of $\zeta \in H_1(Y)$:

$$x \cdot \zeta = \sum_{y} \sum_{\substack{\phi \in \pi_2(x,y) \\ n_z(\phi)=0 \\ \mu(\phi)=1}} \big(\#\mathcal{M}(\phi)\big)(\zeta \cdot \partial_\alpha \phi) y$$

- Equivalently, $x \cdot \zeta$ counts disks
  $$\text{u: } [0,1]\times\mathbb{R} \to Sym^g(\Sigma \setminus \{z\})$$
  with $u(1,0) \in \zeta \times Sym^{g-1}(\Sigma)$.
- At the level of homology, $(x \cdot \zeta) \cdot \zeta = 0$ by considering 1D moduli space of u with $u(1,0) \in \zeta$, $u(1,t) \in \zeta'$ for some $t > 0$ (ζ' a pushoff of ζ).
- Define $m_3(x, \zeta, \zeta)$ by counting 0D moduli space of this form.
- Define $m_n(x, \zeta, \dots, \zeta)$, $n > 3$, similarly.

$\partial(a) = 2b + 2c = 0$
$\partial(b) = d$
$\partial(c) = d$
$\partial(d) = 0$

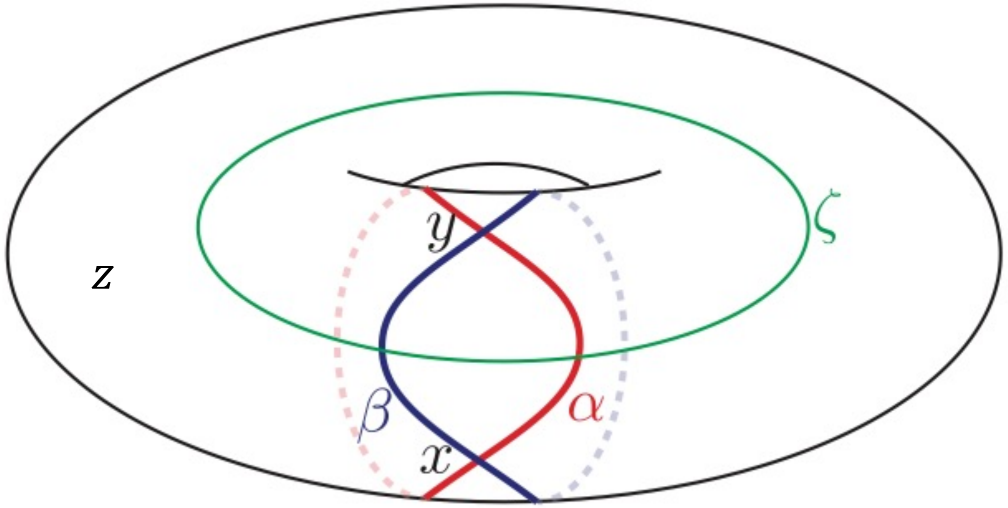


$a \cdot \zeta = b + 2b + c + 2c = b + c$
$\text{b} \cdot \zeta = d$
$\text{c} \cdot \zeta = 0$
$\text{d} \cdot \zeta = 0$
$m_3(a, \zeta, \zeta) = b + 2b + 2c = b$
$m_3(\partial(a), \zeta, \zeta) + \partial\big(m_3(a, \zeta, \zeta)\big) = 0 + d$

Twisted coefficient $\widehat{HF}$

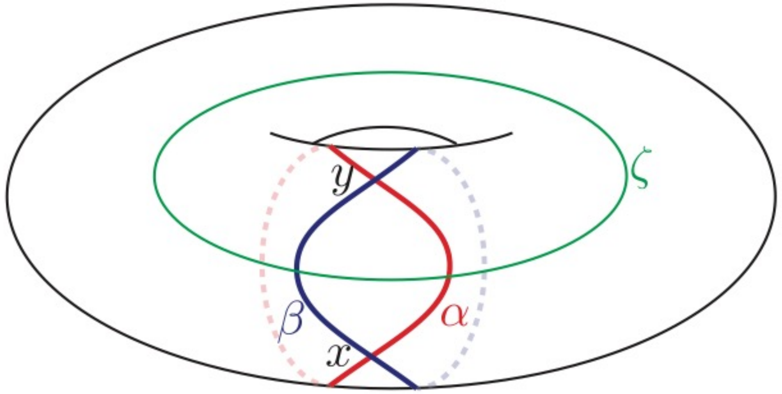
- Fix Y^3, homomorphism $\omega: H_2(Y) \to \mathbb{Z}$.
- Choose $\zeta \in H_1(\Sigma)$ with $\omega(B) = \zeta \cdot \partial_\alpha(B)$.
- Define
$$\underline{\partial} x = \sum_y \sum_{\substack{\phi \in \pi_2(x,y) \\ n_z(\phi)=0 \\ \mu(\phi)=1}} \left(\#\mathcal{M}(\phi)\right) t^{(\zeta \cdot \partial_\alpha \phi)} y .$$
- (This is a module over $\mathbb{F}_2[t^{-1}, t]$.)
- Notice that
$$\partial = \underline{\partial}\Big|_{t=1} .$$

$$\underline{\partial}(x) = t^1 y + t^0 y$$
$$\underline{\partial}\,(y) = 0$$



$$\partial(x) = 2y = 0$$
$$\partial(y) = 0$$

First relation: Hasse derivatives



- $\underline{\partial}x = \sum_y \sum_{\substack{\phi\in\pi_2(x,y)\\ n_z(\phi)=0\\ \mu(\phi)=1}} \left(\#\mathcal{M}(\phi)\right)t^{(\zeta\cdot\partial_\alpha\phi)}y$.
- $x\cdot\zeta = \sum_y \sum_{\substack{\phi\in\pi_2(x,y)\\ n_z(\phi)=0\\ \mu(\phi)=1}} \left(\#\mathcal{M}(\phi)\right)(\zeta\cdot\partial_\alpha\phi)y = \frac{d}{dt}\big|_{t=1}\left(\underline{\partial}x\right)$
- $\frac{d^2}{dt^2} = 0$ over $\mathbb{F}_2$. But there's an analogue D^2 of $\frac{1}{2}\left(\frac{d^2}{dt^2}\right)$, called the *Hasse derivative.* (More generally, D^n is an analogue of $\frac{1}{\mathrm{n!}}\left(\frac{d^n}{dt^n}\right)$.)
- **Proposition.** [LS] $m_{1+n}(x,\zeta,\cdots,\zeta) = D^n|_{t=1}\left(\underline{\partial}x\right)$.
- **Corollary.** [LS] The unrolled homology of $\widehat{CF}(Y)$ (or $\widehat{HF}(Y)$) is isomorphic to $\underline{\widehat{HF}}(Y;\Lambda_\omega)_{\mathbb{F}_2[t^{-1},t]]}$ (so detects homologically essential S^2s).

Second relation: Koszul duality

- Can view $\mathbb{F}_2$ as an $\mathbb{F}_2[t^{-1}, t]$-module where t acts as 1.
- There is an isomorphism of algebras (or A_∞-algebras)

$$\mathbb{F}_2\,[X]\,/\,(X^2) \cong Ext_{\mathbb{F}_2[t^{-1},t]}(\mathbb{F}_2, \mathbb{F}_2)$$

- **Proposition.** [LS] There is an isomorphism of A_∞-modules over $\mathbb{F}_2\,[X]\,/\,(X^2)$

$$\widehat{HF}(Y) \cong Tor_{\mathbb{F}_2[t^{-1},t]}\left(\underline{\widehat{CF}}(Y), \mathbb{F}_2\right)$$

- **Proof.** See that tensoring with a free resolution of $\mathbb{F}_2$ leads to the formula with Hasse derivatives from the previous slide.

The Ozsváth-Szabó spectral sequence respects the A_∞-module structure

- **Proposition.** [LS] There is a filtered A_∞-module C so that:
  1. As an unfiltered A_∞-module, C is quasi-isomorphic to $\widehat{CF}\big(\Sigma(L)\big)$.
  2. The differential strictly increases the filtration.
  3. There is an isomorphism of modules $C \cong \tilde{C}_{Kh}(L)$ taking filtration to homological grading.
  4. To first order, the differential on C agrees with the Khovanov differential.
  5. To zeroth order, m_2 agrees with the action of X on $\tilde{C}_{Kh}(L)$.
- **Corollary.** [LS] If the unrolled homology of the Khovanov complex is trivial then the unrolled homology of the $\widehat{CF}\big(\Sigma(L)\big)$ is trivial.
- In that case, $\Sigma(L)$ has a homologically essential S^2 so L is split.

TFAE:

1. There is a 2-sphere separating p and q.
2. $\widetilde{Kh}(L)$ is a free R-module.
3. $\tilde{C}^{Kh}(L)$ is quasi-free.
4. The unrolled homology of $\tilde{C}^{Kh}(L)$ is trivial.

Review of the proof

- $1 \Rightarrow 2$, $1 \Rightarrow 3$: For a split diagram, immediate from **FREENESS** of $Kh(K)$ and Künneth theorem. For a general diagram, follows from **BASEPOINT INDEPENDENCE**.
- $2 \Rightarrow 4$, $3 \Rightarrow 4$: Algebra:
  - Can compute unrolled homology from A_∞-module structure on $\widetilde{Kh}(L)$.
  - Consider the filtration on $\cdots \xleftarrow{X} C_* \xleftarrow{X} C_* \xleftarrow{X} C_* \xleftarrow{X} \cdots$ from grading on C_*.
- $4 \Rightarrow 1$:
  - **SPECTRAL SEQUENCE**: Unrolled homology of $\tilde{C}^{Kh}(L)$ trivial implies unrolled homology of $\widehat{HF}\big(\Sigma(L)\big)$ trivial.
  - **UNROLLED HOMOLOGY**: this is equivalent to $\widehat{HF}(\Sigma(L); \Lambda_\omega) = 0$.
  - **SPHERE DETECTION**: this is equivalent to existence of a splitting sphere.

That’s all. Thanks for listening!