The rectangular peg problem

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Motivation.

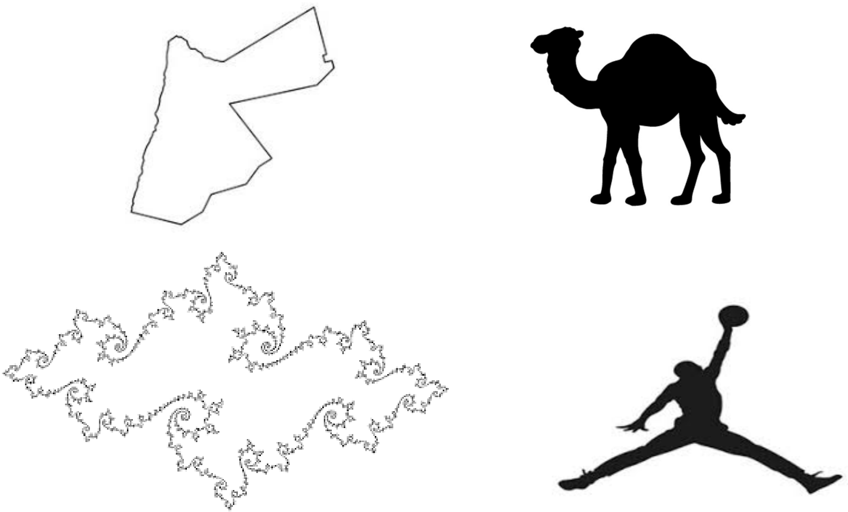
In 1911, Otto Toeplitz posed the following question:

Problem 1 (The Square Peg Problem)

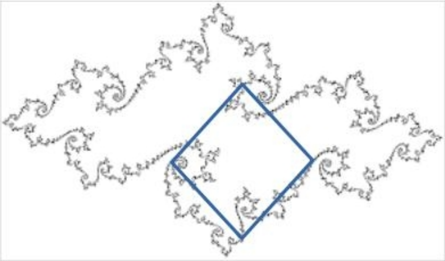
Does every continuous Jordan curve in the Euclidean plane contain four points at the vertices of a square?

It posits a striking connection between the topology and the geometry of the Euclidean plane. It remains open to this day.

Jordan curves.



Inscribed squares.



Why squares / quadrilaterals?

- Three points are ubiquitous: $\forall$ triangle T and $\forall$ Jordan curve γ, γ inscribes a triangle similar to T. (Exercise.)
- Five points are not: dissimilar ellipses inscribe dissimilar pentagons. (Distinct ellipses meet in at most four points.)
- Four is where things get interesting: a recurring theme in low-dimensional topology / geometry.

Early progress.

- Emch (1913) solved the problem for smooth convex curves. (Ideas involving configuration spaces, homology)
- Schnirelman (1929) solved it for smooth Jordan curves. In fact, a generic smooth Jordan curve contains an odd number of “inscribed” squares. (Bordism argument)

Tempting approach to original problem: a limiting argument.

Any continuous Jordan curve is a limit of smooth ones, so take a limiting sequence of squares.

Problem: the squares may shrink to points.

Variations.

- Varying regularity condition on curve (e.g. recent work of Feller-Golla, Schwartz, Tao).
- Higher dimensional analogues (e.g. inscribed octahedra in $S^2 \hookrightarrow \mathbb{R}^3$.)
- Fenn's table theorem.
- Kronheimer and son (Peter) on the tripos problem.
- Other inscribed features in Jordan curves.

See, e.g. Matschke, *Notices of the AMS*, 2014.

Step 1. Vaughan.

Theorem 1 (Vaughan 1977)

Every continuous Jordan curve contains four vertices of a rectangle.

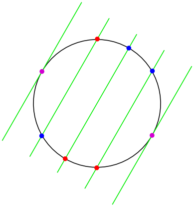
(Reference: Meyerson, *Balancing Acts*, 1981.)

Proof:

$\mathrm{Sym}^2(\gamma) = \{\{z, w\} : z, w \in \gamma\}$: unordered pairs of points on γ

It is a Möbius band:

- send $\{z, w\} \in \mathrm{Sym}^2(S^1)$ to the parallelism class of (tangent) line $\overleftrightarrow{zw}$

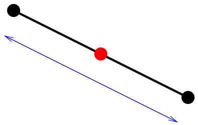


- obtain $\mathrm{Sym}^2(S^1) \to \mathbb{R}P^1$ as an I-bundle over $\mathbb{R}P^1$
- connected boundary $\partial = \{\{z, z\} : z \in \gamma\}$

Define a continuous map $v : \mathrm{Sym}^2(\gamma) \to \mathbb{R}^2 \times \mathbb{R}_{\geq 0}$:

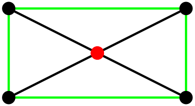
$$v(\{z,w\}) = \left(\frac{z+w}{2}, |z-w| \right).$$

The "midpoint, distance" map.



- $\mathrm{im}(v)$ hits $\mathbb{R}^2 \times \{0\}$ in $v(\partial) = \gamma \times \{0\}$

$$v(\{z,w\}) = v(\{x,y\}) \iff$$



$$\iff \{z,w\} \text{ and } \{x,y\} \text{ span diagonals of a rectangle}$$

Principle:

$$\boxed{\{\text{inscribed rectangles in } \gamma\} \leftrightarrow \{\text{self-intersections of } v\}}$$

reflect $\mathrm{im}(v)$ across $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$:

get continuous map $v \cup \overline{v}$ of the Klein bottle to $\mathbb{R}^3$, 1-to-1 at $\gamma \times \{0\}$.

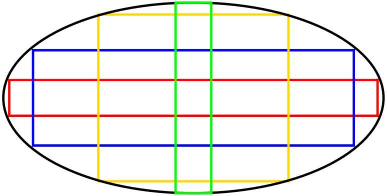
v contains a point of self-intersection $\implies$ γ inscribes a rectangle. $\square$

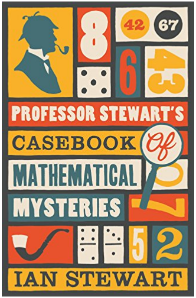
Any map of the Klein bottle to $\mathbb{R}^3$ must contain "a lot" of self-intersection, so there should exist many inscribed rectangles in γ.

How to quantify?

Problem 2 (The rectangular peg problem)

For every (smooth) Jordan curve and every rectangle in the Euclidean plane, do there exist four points on the curve at the vertices of a rectangle similar to the one given?





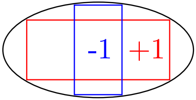
Step −1.

Published “solution” in 1991.

Idea: intersection theory / bordism argument.

Each inscribed rectangle in γ gets a sign; signed count of inscribed rectangles in γ similar to a given one is 2; hence there exist at least two.

In 2008, Matschke found a mistake: the signed count is 0.



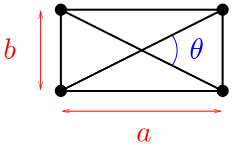
It suggests a limit to the efficacy of intersection theory / bordism arguments.

Step 2. Hugelmeyer.

In 2018, Cole Hugelmeyer recovered some new cases of the rectangular peg problem:

Theorem 2 (Hugelmeyer 2018)

Every smooth Jordan curve contains four points at the vertices of a rectangle with aspect angle equal to an integer multiple of π/n*, for all* $n \geq 3$*.*
In particular, every smooth Jordan curve inscribes a rectangle of aspect ratio $\sqrt{3}$*.*



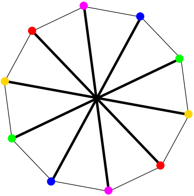
aspect ratio: a/b

aspect angle: θ

Resolve v into a 4D version:

$$h_n : \mathrm{Sym}^2(\gamma) \to \mathbb{C} \times \mathbb{C},$$

$$h_n(\{z, w\}) = \left(\frac{z+w}{2}, (z-w)^{2n} \right)$$



$$\left\{ \begin{array}{c} \text{inscribed rectangles in } \gamma \\ \text{with aspect angle } k\pi/n, k \in \mathbb{Z} \end{array} \right\} \leftrightarrow \{\text{self-intersections of } h_n\}$$

Blow up: $\tilde{h}_n : \mathrm{Sym}^2(\gamma) \to X = \mathbb{C} \times \mathbb{R}_{\geq 0} \times S^1$,

$$\tilde{h}_n(z,w) = \left(\frac{z+w}{2}, |z-w|^{2n}, \frac{(z-w)^{2n}}{|z-w|^{2n}} \right), \quad z \neq w$$

$$\tilde{h}_n(z,z) = \left(z, 0, u(z)^{2n}\right), u(z) \text{ unit tangent to } \gamma \text{ at } z.$$

$M = \mathrm{im}(\tilde{h}_n)$ hits $\partial X = \mathbb{C} \times \{0\} \times S^1$ in a $(1, 2n)$-curve.

insert X into $S^3 \times \mathbb{R}_{\geq 0}$, matching ∂X with an open solid torus in $S^3 \times \{0\}$ by an axial twist.

∂M maps onto the torus knot $T(2n, 2n-1)$.

Batson (2014): $T(2n, 2n-1)$ does not bound a smoothly embedded Möbius band in $S^3 \times \mathbb{R}_{\geq 0}$ for any $n \geq 3$.

Hence M self-intersects $\implies \exists$ asserted inscribed rectangle. □

(The case of a square does not follow: e.g. $T(4,3)$ bounds a Möbius band in B^4.)

Feller and Golla (2020): recovered Hugelmeyer's result, and the case of a square, for curves obeying a weaker regularity condition than smoothness.
Proof based on branched covering / intersection form arguments (free of gauge theory / symplectic geometry).

Step 3. Hugelmeyer v2.0.

In 2019, Hugelmeyer recovered 1/3 of the rectangular peg problem:

Theorem 3 (Hugelmeyer 2019)

For every smooth Jordan curve γ, the set of angles $\phi \in (0, \pi/2]$ such that γ contains an inscribed rectangle of aspect angle ϕ has Lebesgue measure $\geq (1/3)(\pi/2)$.

Proof:

Reconsider $h = h_2 : \mathrm{Sym}^2(\gamma) \to \mathbb{C} \times \mathbb{C}$,

$$h(\{z, w\}) = \left(\frac{z+w}{2}, (z-w)^2 \right)$$

It is a smooth embedding. Write $M = \mathrm{im}(h)$.
For $\phi \in \mathbb{R}$, let $R_\phi : \mathbb{C} \times \mathbb{C} \to \mathbb{C} \times \mathbb{C}$ denote rotation by ϕ in the second coordinate:

$$R_\phi(z, w) = (z, e^{i\phi} \cdot w).$$

$$\boxed{\left\{ \begin{matrix} \text{inscribed rectangles in } \gamma \\ \text{with aspect angle } \phi \end{matrix} \right\} \leftrightarrow \mathring{M} \cap R_{2\phi}(\mathring{M})}$$

Goal: show non-empty for $\geq 1/3$ of angles $\phi \in (0, \pi/2]$.

Blow up as before (γ is smooth).
M_1, M_2 - rotations of M with disjoint interiors.
Define a comparison $M_1 \prec M_2$ based on linking number.
Fact 1. $\prec$ is antisymmetric.
(Linking number argument.)
M_1, M_2, M_3 - rotations of M with pairwise disjoint interiors.
Fact 2. $\prec$ is transitive on M_1, M_2, M_3.
(Milnor triple linking number.)
$\prec$ + additive combinatorics (Kemperman / Cauchy-Davenport) delivers the result. $\square$

In fact $\exists M$ (not derived from any γ) s.t. $\mathring{M} \cap R_\phi(\mathring{M}) \neq \emptyset$ for 1/3 of angles ϕ.
How to ensure that $\mathring{M} \cap R_\phi(\mathring{M}) \neq \emptyset$ for all ϕ, $M = \text{im}(h)$?

Step 4. Shift in perspective: symplectic geometry.

Idea: place a symplectic form on $\mathbb{C} \times \mathbb{C}$ so that M is Lagrangian and R_ϕ form a family of Hamiltonian symplectomorphisms.
"Optimistic" Arnold-Givental:

$$|\mathring{M} \cap R_\phi(\mathring{M})| \geq \dim H_*(M; \mathbb{Z}/2\mathbb{Z}) = 2.$$

Technicality: M is nonorientable and has boundary.
Shortcut: nonembeddability of the Klein bottle.

The rectangular peg problem.

Theorem 4 (G-Lobb 2020)

For every smooth Jordan curve and rectangle in the Euclidean plane, there exist four points on the curve that form the vertices of a rectangle similar to the one given.

Proof, minus details:

Define $f : \mathrm{Sym}^2(\gamma) \to \mathbb{C} \times \mathbb{C}$,

$$f(\{z, w\}) = \left(\frac{z+w}{2}, \frac{(z-w)^2}{2\sqrt{2}|z-w|} \right) \quad (z \neq w)$$

Möbius band $M = \mathrm{im}(f)$.

M hits $\mathbb{C} \times \{0\}$ in $\partial M = \gamma \times \{0\}$.

Away from ∂, M is smooth and Lagrangian w.r.t. symplectic form $\omega_{std} = \frac{i}{2}(dz \wedge d\overline{z} + dw \wedge d\overline{w})$ on $\mathbb{C}^2$.

Let $\phi \in (0, \pi/2]$.

$$\boxed{\left\{ \begin{array}{c} \text{inscribed rectangles in } \gamma \\ \text{with aspect angle } \phi \end{array} \right\} \leftrightarrow \mathring{M} \cap R_{2\phi}(\mathring{M})}$$

R_ϕ is a symplectomorphism.
It fixes ∂M.
Hence M and $R_{2\phi}(M)$ are Möbius bands, smooth and Lagrangian away from their common boundary $\gamma \times \{0\}$, where they meet in a controlled way.

We can smooth $M \cup R_{2\phi}(M)$ nearby $\gamma \times \{0\}$ to get a smoothly mapped, Lagrangian Klein bottle.

Theorem 5 (Shevchishin, Nemirovski 2007)

There does not exist a smooth, Lagrangian embedding of the Klein bottle in $(\mathbb{C}^2, \omega)$.

Hence $\mathring{M} \cap R_{2\phi}(\mathring{M}) \neq \emptyset \implies \exists$ inscribed rectangle in γ of aspect angle ϕ. $\square$

Details.

1. Why is M Lagrangian?

$\gamma \subset \mathbb{C}$ is Lagrangian
$\implies \gamma \times \gamma \subset \mathbb{C} \times \mathbb{C}$ is
$\implies \mathrm{Sym}^2(\gamma) - \Delta \subset \mathrm{Sym}^2(\mathbb{C}) - \Delta$ is.
The map f is just $\mathbb{C} \times \mathbb{C} \xrightarrow{\pi} \mathrm{Sym}^2(\mathbb{C}) \xrightarrow{\sim} \mathbb{C} \times \mathbb{C}$ written explicitly:
$f = g \circ l$, where $g, l : \mathbb{C} \times \mathbb{C} \to \mathbb{C} \times \mathbb{C}$,

$$l(z, w) = \left(\frac{z+w}{2}, \frac{z-w}{2} \right), \; g(z, r, \theta) = (z, r/\sqrt{2}, 2\theta).$$

l is a diffeomorphism and $l^*(\omega) = \omega/2$.
g is smooth and $g^*(\omega) = \omega$ away from $\mathbb{C} \times \{0\}$.
$M = f(\gamma \times \gamma)$ is Lagrangian (away from $\mathbb{C} \times \{0\}$).

Details.

2. Why is the smoothing possible?

Work with Lagrangian tori $L = l(\gamma \times \gamma)$ and $R_\phi(L)$.
They intersect cleanly at $\gamma \times \{0\} \subset \mathbb{C} \times \{0\}$.
They are invariant under R_π.
Apply equivariant Weinstein theorem à la Poźniak:
$\exists$ $\mathbb{Z}/2$-equivariant symplectomorphism of neighborhood of intersection to $S^1 \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ with

- coordinates: x_1, x_2, y_1, y_2
- symplectic form: $dx_1 \wedge dy_1 + dx_2 \wedge dy_2$
- $\mathbb{Z}/2$ action: $(x_1, x_2, y_1, y_2) \leftrightarrow (x_1, -x_2, y_1, -y_2)$
- Lagrangians: $S^1 \times \mathbb{R} \times \{0\} \times \{0\}$ and $S^1 \times \{0\} \times \{0\} \times \mathbb{R}$.

smooth the intersection $\mathbb{Z}/2$-equivariantly, then project via g

Details.

3. Nonexistence of Lagrangian Klein bottles in $\mathbb{C}^2$.

This had been a question of Givental.

Nemirovski's proof:

Given smoothly embedded Lagrangian Klein bottle $K \subset (X, \omega)$, $[K] = 0 \in H_2(X; \mathbb{Z}/2)$, do Luttinger surgery.

Get dual Klein bottle $K' \subset (X', \omega')$, $[K'] \neq 0 \in H_2(X'; \mathbb{Z}/2)$.

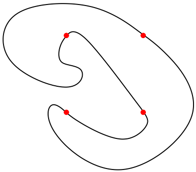
$(X - N(K), \omega) \approx (X' - N(K'), \omega')$.

Gromov: any symplectic 4-manifold asymptotic to $(\mathbb{C}^2, \omega_{std})$ at ∞ with $\pi_2 = 0$ is actually $(\mathbb{C}^2, \omega_{std})$.

So could not have been in $(\mathbb{C}^2, \omega_{std})$ in the first place (else get $\mathbb{C}^2 = X = X'$ and $[K'] \neq 0 \in H_2(\mathbb{C}^2; \mathbb{Z}/2)$ $\lightning$).

Beyond.

1. Does every smooth Jordan curve inscribe a rectangle of each aspect ratio whose vertices appear in the same cyclic order around both the curve and the rectangle? (“Yes” for the square: Schwartz.)



2. Does every smooth Jordan curve inscribe every cyclic quadrilateral?
3. Is there an “algorithm” to locate an inscribed square in a smooth Jordan curve? Compare: finding a fixed point of a continuous map from the disk to itself.