Proof of Birational File Homomesy for Minuscule Posets

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This talk is being recorded.
The slides are available at the workshop website.

Birational Toggle

Let P be a finite poset and let $\widehat{P}$ be the poset obtained from P by adjoining an extra maximal element $\widehat{1}$ and an extra minimal element $\widehat{0}$:

$$\widehat{P} = P \sqcup \{\widehat{1}, \widehat{0}\}.$$

Fix positive real numbers A and B, and extend a map $F : P \to \mathbb{R}_{>0}$ to a map $\widehat{F} : \widehat{P} \to \mathbb{R}_{>0}$ by

$$\widehat{F}(\widehat{1}) = A, \quad \widehat{F}(\widehat{0}) = B.$$

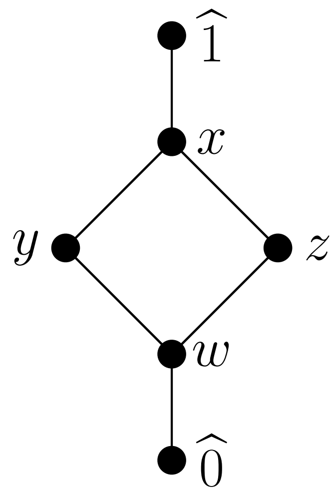
Then, for each $v \in P$, we define the birational toggle $\tau_v = \tau_v^{A,B} : (\mathbb{R}_{>0})^P \to (\mathbb{R}_{>0})^P$ at v by

$$\left(\tau_v^{A,B} F\right)(x) = \begin{cases} \dfrac{\sum_{w \lessdot v} \widehat{F}(w)}{F(v) \sum_{z \gtrdot v} 1/\widehat{F}(z)} & \text{if } x = v, \\ F(x) & \text{if } x \neq v. \end{cases}$$

Birational Toggle

$$\left(\tau_v^{A,B} F\right)(x) = \begin{cases} \dfrac{\sum_{w \lessdot v} \widehat{F}(w)}{F(v) \sum_{z \gtrdot v} 1/\widehat{F}(z)} & \text{if } x = v, \\ F(x) & \text{if } x \neq v. \end{cases}$$

Example If $P = [2] \times [2]$, then we have

$\widehat{P} =$ 

	F	$\tau_x F$	$\tau_y \tau_x F$	$\tau_z \tau_y \tau_x F$	$\tau_w \tau_z \tau_y \tau_x F$
x	X	$\frac{A(Y+Z)}{X}$	$\frac{A(Y+Z)}{X}$	$\frac{A(Y+Z)}{X}$	$\frac{A(Y+Z)}{X}$
y	Y	Y	$\frac{A(Y+Z)W}{XY}$	$\frac{A(Y+Z)W}{XY}$	$\frac{A(Y+Z)W}{XY}$
z	Z	Z	Z	$\frac{A(Y+Z)W}{XZ}$	$\frac{A(Y+Z)W}{XZ}$
w	W	W	W	W	$\frac{AB}{X}$

$$(\tau_x F)(x) = \frac{Y+Z}{X \cdot 1/A} = \frac{A(Y+Z)}{X}, \quad (\tau_y \tau_x F)(y) = \frac{W}{Y \cdot X/A(Y+Z)} = \frac{A(Y+Z)W}{XY}.$$

Birational Rowmotion

Let P be a finite poset and fix positive real numbers A and B. We define the birational rowmotion $\rho = \rho^{A,B} : (\mathbb{R}_{>0})^P \to (\mathbb{R}_{>0})^P$ by

$$\rho^{A,B} = \tau_{v_1}^{A,B} \tau_{v_2}^{A,B} \cdots \tau_{v_n}^{A,B},$$

where $(v_1, v_2, \dots, v_n)$ is any linear extension of P.

Remark The birational toggle and rowmotion are a birational lift of the combinatorial toggle t_v and rowmotion R on the poset $J(P)$ of order ideals of P given by

$$t_v(I) = \begin{cases} I \cup \{v\} & \text{if } v \notin I \text{ and } I \cup \{v\} \in J(P), \\ I \setminus \{v\} & \text{if } v \in I \text{ and } I \setminus \{v\} \in J(P), \\ I & \text{otherwise,} \end{cases}$$

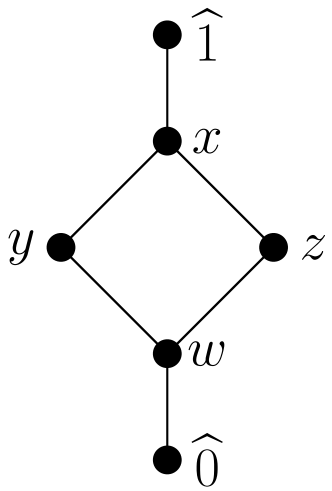
and

$$R(I) = \text{the order ideal generated by the minimal elements of } P \setminus I.$$

Birational Rowmotion

Example If $P = [2] \times [2]$, then we have

$\widehat{P} =$



	F	ρF	$\rho^2 F$	$\rho^3 F$	$\rho^4 F$
x	X	$\frac{A(Y+Z)}{X}$	$\frac{A(Y+Z)W}{YZ}$	$\frac{AB}{W}$	X
y	Y	$\frac{A(Y+Z)W}{XY}$	$\frac{AB}{Z}$	$\frac{BXZ}{(Y+Z)W}$	Y
z	Z	$\frac{A(Y+Z)W}{XZ}$	$\frac{AB}{Y}$	$\frac{BXY}{(Y+Z)W}$	Z
w	W	$\frac{AB}{X}$	$\frac{BX}{Y+Z}$	$\frac{BYZ}{(Y+Z)W}$	W

Note that F has a finite order 4 and

$$\prod_{k=0}^{3} \left(\rho^k(F)\right)(x) \left(\rho^k(F)\right)(w) = A^4 B^4,$$

$$\prod_{k=0}^{3} \left(\rho^k(F)\right)(y) = \prod_{k=0}^{3} \left(\rho^k(F)\right)(z) = A^2 B^2,$$

$$F(x) \cdot (\rho F)(w) = F(y) \cdot \left(\rho^2 F\right)(y) = F(z) \cdot \left(\rho^2 F\right)(z) = F(w) \cdot \left(\rho^3 F\right)(x) = AB.$$

Minuscule Posets

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra (over $\mathbb{C}$). For a dominant integral weight λ, we put

$$L_\lambda = \text{the set of all weights in } V_\lambda,$$

where V_λ is the irreducible $\mathfrak{g}$-module with highest weight λ. A dominant integral weight λ is called minuscule if

$$L_\lambda = W\lambda, \quad W = \text{the Weyl group of } W.$$

We define the minuscule poset associated to a minuscule weight λ by

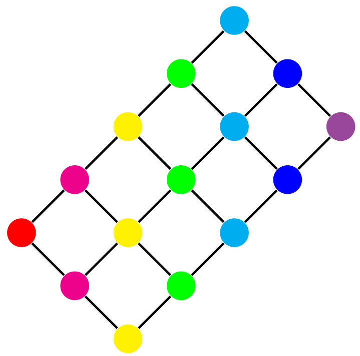
$$P_\lambda = \{\beta^\vee \in \Phi_+^\vee : \langle \beta^\vee, \lambda \rangle = 1\},$$

where $\Phi_+^\vee$ is the positive coroot system of $\mathfrak{g}$. Then there exist a unique map (coloring) $c : P_\lambda \to \Pi$, where Π is the set of simple roots, such that that map

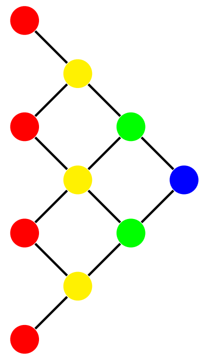
$$J(P_\lambda) \ni I \mapsto \lambda - \sum_{v \in I} c(v) \in L_\lambda$$

give an isomorphism of posets.

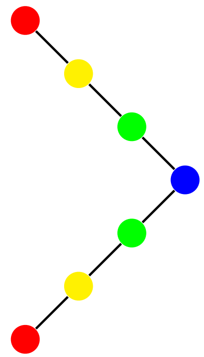
Classification of Minuscule Posets with Colorings (1/2)



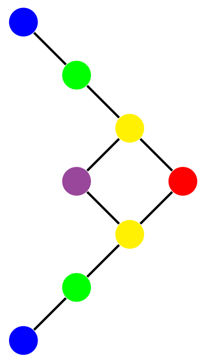
(A_n, ϖ_r)
$(1 \leq r \leq n)$



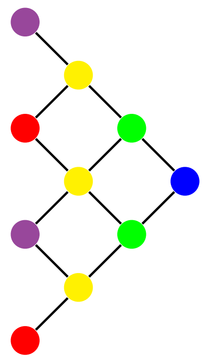
(B_n, ϖ_n)



(C_n, ϖ_1)

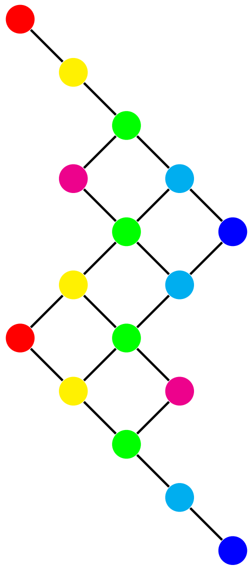


(D_n, ϖ_1)

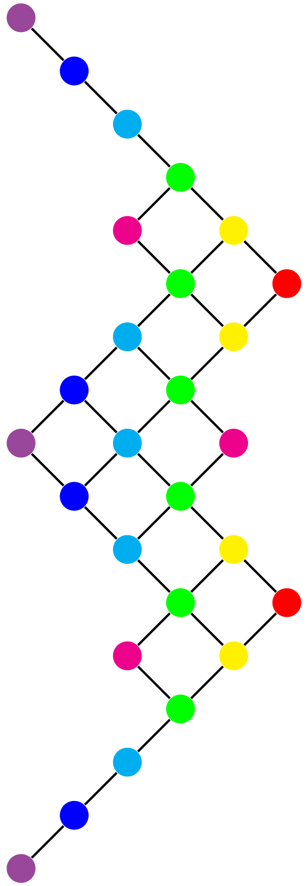


(D_n, ϖ_{n-1})
(D_n, ϖ_n)

Classification of Minuscule Posets with Colorings (2/2)



(E_6, ϖ_1)
(E_6, ϖ_6)



(E_7, ϖ_7)

Results for Birational Rowmotion

Theorem (Grinberg–Roby, Musiker–Roby, Einstein–Propp, Okada) Let

P : a minuscule poset associated with a minuscule weight λ of $\mathfrak{g}$,
$c : P \to \Pi$: coloring, $\mathrm{ht} : P \to \{1, 2, \dots\}$: height,
$\rho = \rho^{A,B}$: the birational rowmotion map on P (A, $B \in \mathbb{R}_{>0}$).

Then

(1) (periodicity) The order of ρ is equal to the Coxeter number h of $\mathfrak{g}$.

(2) (reciprocity) For $v \in P$, we have

$$\left(\rho^{\mathrm{ht}(v)} F\right)(v) = \frac{AB}{F(w_\lambda v)},$$

where w_λ is the longest element of $W_\lambda = \{w \in W : w\lambda = \lambda\}$.

(3) (file homomesy) For a simple root $\alpha \in \Pi$, we have

$$\prod_{k=0}^{h-1} \prod_{c(v)=\alpha} \left(\rho^k F\right)(v) = A^{h\langle \varpi^\vee, -w_0\lambda \rangle} B^{h\langle \varpi^\vee, \lambda \rangle},$$

where $\varpi^\vee$ is the fundamental coweight corresponding to α.

Remark

- The periodicity (1) was obtained by Grinberg–Roby except for the case (E_7, ϖ_7). We can use the Musiker–Roby's "A-variables" to settle the E_7 case with the help of computer.
- The proof of the reciprocity (2) is based on a case-by-case analysis. The type A case (rectangle posets) was proved by Grinberg–Roby and Musiker–Roby.
- The homomesy result (3) in type A was proved by Musiker–Roby and Eisenstein–Propp.
- For a simple root α, we put

$$P^\alpha = \{v \in P : c(v) = \alpha\}, \quad \Phi_\alpha(F) = \prod_{v \in P^\alpha} F(v),$$

  then the statistic Φ_α is a birational lift of the refined order ideal cardinality $\#(I \cap P^\alpha)$. For the combinatorial rowmotion, Rush–Wang gave a uniform proof to the homomesy result (3).

Proof sketch of homomesy (1/3)

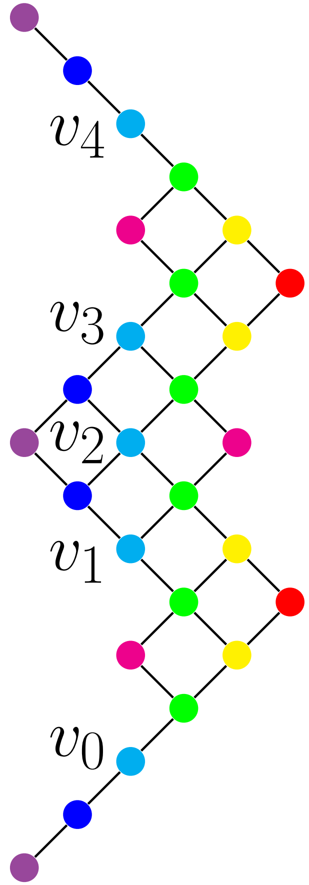
Let P be a minuscule poset with coloring $c : P \to \Pi$ and height $\mathrm{ht} : P \to \{1, 2, \dots\}$. For $\alpha \in \Pi$ and $F : P \to \mathbb{R}_{>0}$, we put $P^\alpha = \{v \in P : c(v) = \alpha\}$, and work with

$$\Psi_\alpha(F) = \prod_{v \in P^\alpha} \left(\rho^{(\mathrm{ht}(v) - \mathrm{ht}(v_0))/2} F\right)(v),$$

where v_0 is the minimum element of P^α, instead of $\Phi_\alpha(F) = \prod_{v \in P^\alpha} F(v)$.

Example If P is the minuscule poset of type E_7 and α is the simple root corresponding to ●, then

$$\Psi_\alpha(F) = F(v_0) \cdot \left(\rho^2 F\right)(v_1) \times \left(\rho^3 F\right)(v_2) \cdot \left(\rho^4 F\right)(v_3) \cdot \left(\rho^6 F\right)(v_4).$$



Proof sketch of homomesy (2/3)

Key Lemma We have

$$\Psi_\alpha(F) \cdot \Psi_\alpha(\rho F) = A^{\delta_{\alpha,\alpha_{\max}}} B^{\delta_{\alpha,\alpha_{\min}}} \prod_{\beta \in \Pi,\, \beta \neq \alpha} \Psi_\beta\big(\rho^{m(\alpha,\beta)} F\big)^{-\langle \beta, \alpha^\vee \rangle},$$

where $\alpha_{\max}$ (resp. $\alpha_{\min}$) is the color of the maximal (resp. minimum) element of P, and

$$m(\alpha, \beta) = \begin{cases} 1 & \text{if } \min P^\alpha < \min P^\beta, \\ 0 & \text{if } \min P^\alpha > \min P^\beta. \end{cases}$$

Proof It is enough to consider the values of $\rho^k F$ on the neighborhood $\widehat{N}^\alpha$ of P^α in the Hasse diagram of $\widehat{P}$. We can analyze the structure of $\widehat{N}^\alpha$ and prove the claim in a case-by-case manner.

Proof sketch of homomesy (3/3)

Suppose that ρ has a finite order h. We put

$$\widetilde{\Phi}_\alpha(F) = \prod_{k=0}^{h-1} \prod_{v \in P^\alpha} \big(\rho^k F\big)(v) = \prod_{k=0}^{h-1} \Psi_\alpha\big(\rho^k F\big),$$

$$\widetilde{\mu}(F) = \sum_{\alpha \in \Pi} \log \widetilde{\Phi}_\alpha(F) \cdot \alpha.$$

Key Lemma implies $\prod_{\beta \in \Pi} \widetilde{\Phi}_\beta(F)^{\langle \beta, \alpha^\vee \rangle} = A^{\delta_{\alpha,\alpha_{\max}} h} B^{\delta_{\alpha,\alpha_{\min}} h}$ and

$$s_\alpha \widetilde{\mu}(F) = \widetilde{\mu}(F) - \big(\delta_{\alpha,\alpha_{\max}} h \log A + \delta_{\alpha,\alpha_{\min}} h \log B\big) \alpha.$$

By computing the action of a Coxeter element on $\widetilde{\mu}(F)$, we can show that

$$\widetilde{\mu}(F) = h \log A \cdot \varpi_{\max} + h \log B \cdot \varpi_{\min},$$

where $\varpi_{\max}$ (resp. $\varpi_{\min}$) is the fundamental weight corresponding to $\alpha_{\max}$ (resp. $\alpha_{\min}$).

Coxeter-motion

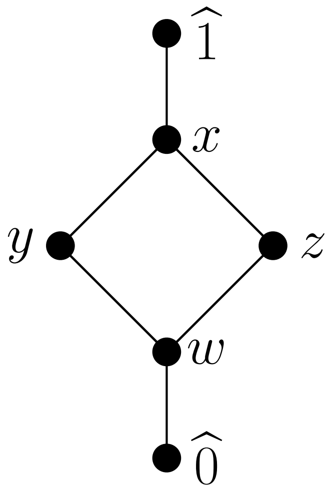
Let P be a minuscule poset with coloring $c : P \to \Pi$. We put

$$\sigma_\alpha^{A,B} = \prod_{c(x)=\alpha} \tau_x^{A,B} \quad (\alpha \in \Pi).$$

A birational Coxeter-motion $\gamma^{A,B}$ is a composition of all $\sigma_\alpha^{A,B}$'s:

$$\gamma^{A,B} = \sigma_{\alpha_1}^{A,B} \cdots \sigma_{\alpha_n}^{A,B}, \quad \Pi = \{\alpha_1, \dots, \alpha_n\}.$$

Example If $P = [2] \times [2]$ and $\gamma = \tau_z \cdot (\tau_x \tau_w) \cdot \tau_y$, then we have

$\widehat{P} =$ 

	F	γF	$\gamma^2 F$	$\gamma^3 F$	$\gamma^4 F$
x	X	$\frac{A(XW+YZ)}{XY}$	$\frac{AB}{W}$	$\frac{A(XW+YZ)}{XZ}$	X
y	Y	$\frac{XW}{Y}$	$\frac{ABZ}{XW}$	$\frac{AB}{Z}$	Y
z	Z	$\frac{A(XW+YZ)}{XZ}$	$\frac{AB}{Z}$	$\frac{XW}{Z}$	Z
w	W	$\frac{BXZ}{XW+YZ}$	$\frac{AB}{X}$	$\frac{BXY}{XW+YZ}$	W

Results for Coxeter-motion

Theorem Let

P : a minuscule poset associated with a minuscule weight λ of $\mathfrak{g}$,
$c : P \to \Pi$: coloring, $\mathrm{ht} : P \to \{1, 2, \dots\}$: height,
$\gamma = \gamma^{A,B}$: a birational Coxeter-motion map on P (A, $B \in \mathbb{R}_{>0}$).

Then

(1) (periodicity) The order of γ is equal to the Coxeter number h of $\mathfrak{g}$.
(2) (file homomesy) For a simple root $\alpha \in \Pi$, we have

$$\prod_{k=0}^{h-1} \prod_{c(v)=\alpha} (\gamma^k F)(v) = A^{h\langle \varpi^\vee, -w_0\lambda \rangle} B^{h\langle \varpi^\vee, \lambda \rangle},$$

where $\varpi^\vee$ is the fundamental coweight corresponding to α.

Remark Einstein–Propp proved the homomesy result (2) for the birational promotion in type A.

Proof sketch of homomesy

We put put

$$\Phi_\alpha(F) = \prod_{c(v)=\alpha} F(v), \quad \mu(F) = \sum_{\alpha\in\Pi} \log \Phi_\alpha(F) \cdot \alpha.$$

Then we have

$$\Phi_\alpha(F) \cdot \Phi_\alpha(\sigma_\alpha F) = A^{\delta_{\alpha,\alpha_{\max}}} B^{\delta_{\alpha,\alpha_{\min}}} \prod_{\beta\in\Pi,\, \beta\neq\alpha} \Phi_\beta(F)^{-\langle \beta, \alpha^\vee \rangle}.$$

Hence we have

$$\mu(\sigma_\alpha F) = s_\alpha \mu(F) + \big(\delta_{\alpha,\alpha_{\max}} \log A + \delta_{\alpha,\alpha_{\min}} \log B\big)\alpha$$

If c is the corresponding Coxeter element, then we have

$$\mu(\gamma F) = c\mu(F) + \log A \cdot (\varpi_{\max} - c\varpi_{\max}) + \log B \cdot (\varpi_{\min} - c\varpi_{\min}),$$

which implies

$$\sum_{k=0}^{h-1} \mu(\gamma^k F) = h \log A \cdot \varpi_{\max} + h \log B \cdot \varpi_{\min}.$$

Conjecture on Reciprocity (Half-periodicity)

Let P be a minuscule poset. Let $\Pi = \Pi_1 \sqcup \Pi_2$ be a decomposition of the corresponding simple system into a disjoint union of orthogonal subsets (i.e., each Π_i consists of pairwise orthogonal simple roots). Then we put

$$\gamma_1^{A,B} = \prod_{\alpha \in \Pi_1} \sigma_\alpha^{A,B}, \quad \gamma_2^{A,B} = \prod_{\beta \in \Pi_2} \sigma_\beta^{A,B},$$

and define

$$\delta^{A,B} = \gamma_1^{A,B} \gamma_2^{A,B} \gamma_1^{A,B} \gamma_2^{A,B} \gamma_1^{A,B} \dots \quad (h \text{ factors}),$$

where h is the Coxeter number of the corresponding Lie algebra.

Conjecture For any $F \in (\mathbb{R}_{>0})^P$, we have

$$\left(\delta^{A,B}(F) \right)(v) = \frac{1}{F(\iota v)} \quad (v \in P),$$

where ι is the order-reversing involution of P induced from the longest element of W_λ.