Generalized braid ~~group~~ category actions

(joint work with Timothy Logvinenko)

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Weak braid group actions

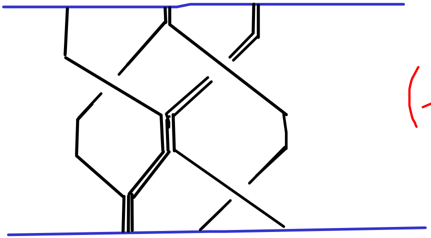
A weak action of a group G on a triangulated category T is an assignment of an exact functor $\Phi(g) : T \to T$ to every $g \in G$ so that $\Phi(g_1)\Phi(g_2) \simeq g_1 g_2$.

Examples of braid group actions on $D^b(Coh\ X)$:

- $X = \mathbb{C}^2$ with the action of S_3, and therefore Br_3. The action on $D^b(Coh\ X)$ is induced by the automorphisms of X.
- X is the minimal resolution of a Kleinian singularity (of type A_n for Br_n). The braid group action is generated by spherical twists in the structure sheaves of the exceptional curves. (Seidel-Thomas'00)
- $X = T^*FL_n$. The braid group action is induced by spherical twists in certain functors from $D^b(T^*P_{k,n})$, where $P_{k,n}$ is the variety of partial flags missing the space of dimension k. (Khovanov-Thomas'06)

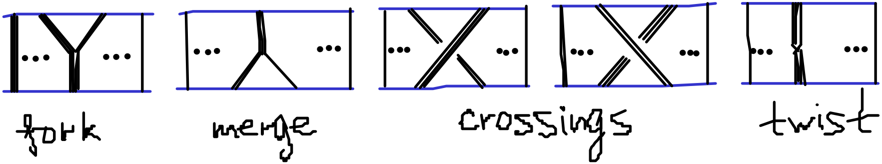
Generalized braid category $FGBr_n$

Objects: $(k_1, \dots, k_m)$ with $k_1 + \dots + k_m = n$.
Morphisms:



(framed)

Generators:

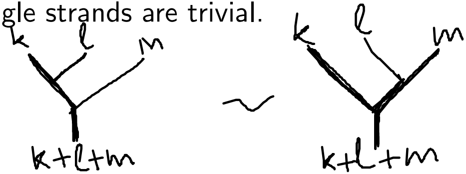


Skein triangulated representations of $FGBr_n$

We would like to consider weak triangulated representations of GBr_n s.t.:

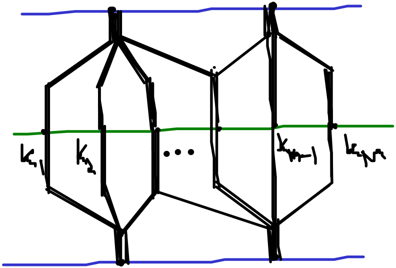
- Merge and fork generators are adjoint;
- With a set of adjoint merge and fork generators satisfying the multifork relation we can construct complexes $L^{r,s}_{p,q}$ where $p + q = r + s$ generalizing Rickard complexes. We need $L^{r,s}_{p,q}$ to be equivalences when $p - s \equiv 0 \mod (p + q)$ and acyclic otherwise.
- Moreover, the convolution of $L^{q,p}_{p,q}$ must be isomorphic to the (p, q) crossing, and the convolution of $L^{m,0}_{m,0}$, to the twisting of the multiplicity m strand.
- The twists of single strands are trivial.

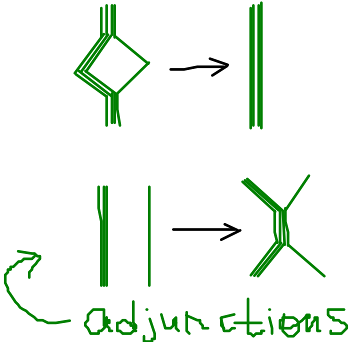
multifork:



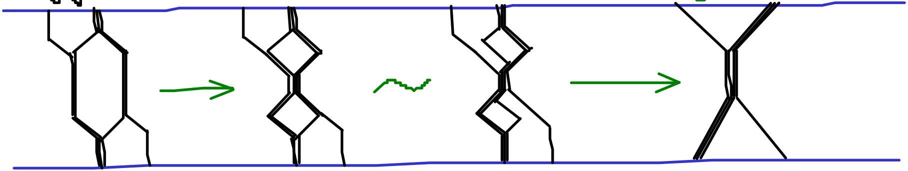
The complexes

The terms of the complex $L^{r,s}_{p,q}$ are enumerated by all $k_1 + \ldots + k_m = n$.

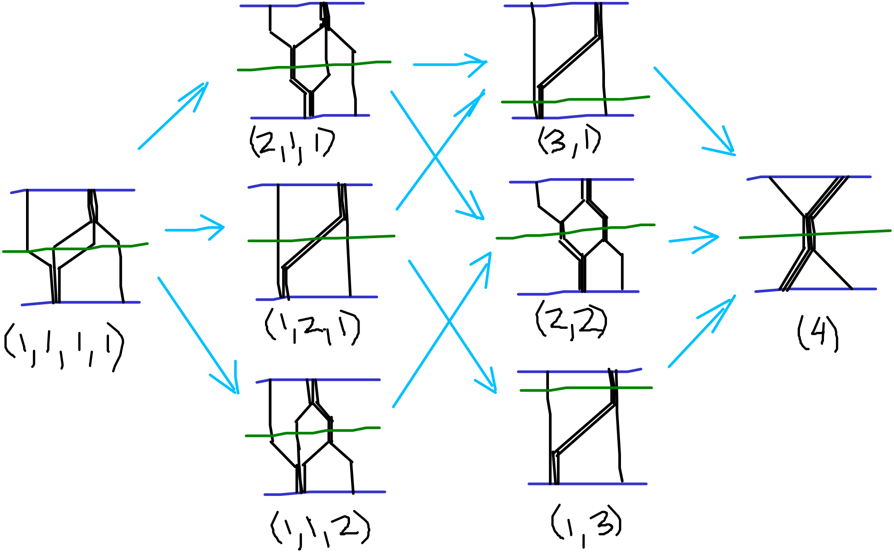




differential:



Example: $L^{1,3}_{3,1}$



DG enhancements

A DG category $\mathcal{A}$ is an enhancement of the triangulated category T if $H^0(\mathcal{A}) \simeq T$ and a standard (resp. large) Morita enhancement if $H^0(\mathcal{P}^{Perf}(\mathcal{A})) \simeq T$, (resp. $H^0(\mathcal{P}(\mathcal{A})) \simeq T$) where $\mathcal{P}^{Perf}(\mathcal{A})$ (resp. $\mathcal{P}(\mathcal{A})$) is the category of h-projective perfect (resp. h-projective) $\mathcal{A}$-modules.

For a quasi-compact, quasi-separated scheme X, the category $D_{qc}(X)$ has a large Morita enhancement $\mathcal{A} = \mathrm{End}(I)$, where I is the h-injective resolution of the compact generator. Any continuous functor $D_{qc}(X) \to D_{qc}(Y)$ can be represented by a bimodule in $\mathcal{P}(\mathcal{A}\text{-}\mathcal{B})$.

Similarly, for a separated scheme of finite type $D^b(Coh\ X)$ has a Morita enhancement and every continuous functor can be represented by a $\mathcal{B}$-perfect bimodule in $\mathcal{P}(\mathcal{A}\text{-}\mathcal{B})$.

Nil Hecke algebras

Let $W = S_n$ (full generality for this slide: W is a Coxeter group).
The nil Hecke algebra $\mathcal{H}(W)$ is the complex algebra with generators $h_1, \ldots, h_{n-1}$ and relations

$$h_i h_j = h_j h_i \qquad \text{for } |i-j| > 1;$$
$$h_i h_j h_i = h_j h_i h_j \qquad \text{for } |i-j| = 1;$$
$$h_i^2 = 0.$$

Alternative description: let $s_i \in W$ permute i and $i+1$. Denote by $l(w)$ the length of each $w \in W$: the length of the minimal presentation $w = s_{i_1} \ldots s_{i_N}$. Then $l(w_1 w_2) \leq l(w_1) + l(w_2)$.

The monomial basis

$\mathcal{H}(W)$ has a basis h_w, $w \in W$, with $h_{w_1} h_{w_2} = h_{w_1 w_2}$ if $l(w_1 w_2) = l(w_1) + l(w_2)$ and $h_{w_1} h_{w_2} = 0$ otherwise. For $w = s_{i_1} \ldots s_{i_N}$,

$$h_w = h_{i_1} \ldots h_{i_N}.$$

Nil Hecke bimodules

Now let $\mathcal{A}$ be a small DG category. Suppose there are DG $\mathcal{A}$-$\mathcal{A}$-bimodules $H_1, \ldots, H_{n-1}$ such that

$$H_i \otimes_{\mathcal{A}} H_j \sim H_j \otimes_{\mathcal{A}} H_i \qquad \text{for } |i-j| > 1;$$

$$H_i \otimes_{\mathcal{A}} H_j \otimes_{\mathcal{A}} H_i \sim H_j \otimes_{\mathcal{A}} H_i \otimes_{\mathcal{A}} H_j \qquad \text{for } |i-j| = 1.$$

To each basic monomial $h_\sigma = h_{i_1} \ldots h_{i_m}$ in $\mathcal{H}(W)$ we assign a twisted complex $\mathcal{H}_\sigma$ of $\mathcal{A}$-$\mathcal{A}$-bimodules homotopy equivalent to $H_{i_1} \otimes_{\mathcal{A}} \ldots \otimes_{\mathcal{A}} H_{i_m}$. We define a structure of an algebra in the category of $\mathcal{A}$-$\mathcal{A}$-bimodules on $\mathcal{H} = \bigoplus \mathcal{H}_\sigma$ by sending $\mathcal{H}_{\sigma_1} \otimes_{\mathcal{A}} \mathcal{H}_{\sigma_2}$ to zero or to a subcomplex of $\mathcal{H}_{\sigma_1 \sigma_2}$.

Example: for $n = 3$, the bimodule $\mathcal{H}$ will be

$$\mathcal{A} \;\oplus\; H_1 \;\oplus\; H_2 \;\oplus\; H_1 \otimes H_2 \;\oplus\; H_2 \otimes H_1$$

$$\oplus \{H_1 \otimes H_2 \otimes H_1 \xrightarrow{\mathrm{Id} \oplus \sim} H_1 \otimes H_2 \otimes H_1 \;\oplus\; H_2 \otimes H_1 \otimes H_2\}.$$

The block subalgebras

For each subset $I \subset \{1, \ldots, n-1\}$ of generators (or even for any subgroup of W) let $\mathcal{H}_I$ be the direct sum of the "monomials" in $\mathcal{H}$ generated by H_i, $i \in I$. This is an algebra object in $\mathcal{A}$-Mod-$\mathcal{A}$ as well.

$\mathcal{H}$, or each of its "block subalgebras" $\mathcal{H}_I$ can be viewed as a DG category with $\mathrm{Ob}\ \mathcal{H} = \mathrm{Ob}\ \mathcal{A}$. There are natural functors $\mathcal{H}_I \to \mathcal{H}_J$ when $I \subset J$. Each $\mathcal{H}_I$ can be viewed as an $\mathcal{H}_J$-$\mathcal{H}_K$-bimodule if $J, K \subseteq I$.

Restriction and induction

The functors $\mathcal{H}_J \to \mathcal{H}_I$ for $J \subset I$ induce adjoint pairs of functors $((-) \otimes_{\mathcal{H}_J} \mathcal{H}_I,\ (-) \otimes_{\mathcal{H}_I} \mathcal{H}_I)$ between the categories Mod-$\mathcal{H}_J$ and Mod-$\mathcal{H}_I$, where $\mathcal{H}_I$ is viewed as a $\mathcal{H}_J$-$\mathcal{H}_I$ and $\mathcal{H}_I$-$\mathcal{H}_J$-bimodule respectively. In particular, the monad algebra of this adjunction is $\mathcal{H}_I$ viewed as an algebra over $\mathcal{H}_J$.

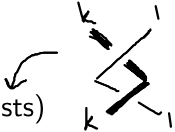
Expanding the braid group action

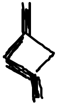
Let $(-) \otimes H_i$ generate a weak braid group action on $H^0(\mathrm{Mod}\text{-}\mathcal{A})$.

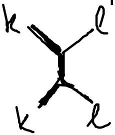
Hypothesis (A.-Logvinenko)

The categories $H^0(\mathrm{Mod}\text{-}\mathcal{H}_I)$ form a skein triangulated representation of $FGBr_n$. The forks correspond to induction functors, the merges to restriction functors, and the crossings are induced by convolutions of the complexes $L^{q,p}_{p,q}$ and their right adjoints.

- Theorem for $n = 3$ and partially for general n
- Twists are <u>not</u> shifts
- $(1, k)$ forks are <u>not</u> $\mathbb{P}^n$ functors (there still are twists)
- fork-merge "bubbles" don't split, but merge-fork compositions do







Thank you!