Exceptional collections on moduli spaces of stable rational curves

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joint work with Jenia Tevelev

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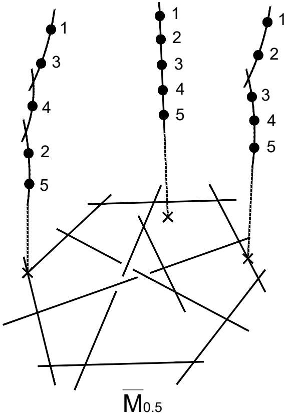
Moduli spaces of stable rational curves



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$\mathcal{M}_{0,n} \subseteq \overline{\mathcal{M}}_{0,n} =$ functorial compactification

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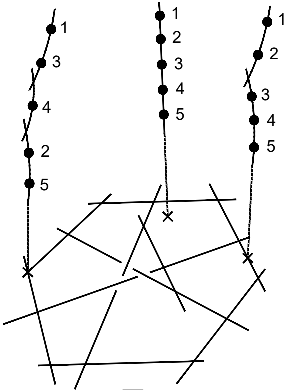

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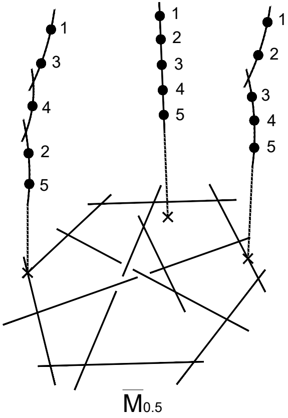

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$$\phi : \overline{\mathcal{M}}_{0,6} \to (\mathbb{P}^1)^6 /\!\!/_{\mathcal{O}(a,\dots,a)} \mathrm{PGL}_2 = X \subseteq \mathbb{P}^4 \quad \text{Segre cubic}$$

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It suffices to find full, invariant, exceptional collections on:

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For $p = 2r + 1 \geq 1$, $q \geq 0$, $\overline{M}_{p,q}$ has an $S_p \times S_q$-invariant, full exceptional collection given by the vector bundles $F_{l,E}$ ($l + e$ even) subject to

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- $l = 0$, $e_p = 0$: one line bundle $\rightsquigarrow$ $\mathcal{O}$
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$$e_p = r, \quad l + \min(e_q, q - e_q) \leq s - 1 \qquad \text{(group } 2\text{)}$$

There are 2 full $S_p \times S_q$-invariant exceptional collections on $\overline{M}_{p,q}$ obtained by combining group $1A$ (resp., $1B$) with group 2.

Not an exceptional collection if replacing $\mathcal{T}_{l,E}$ with $F_{l,E}$

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$$\mathcal{O}_{\mathbb{P}^{r+s-1} \times \mathbb{P}^{r+s-1}}(-a, -b)$$

where

- either $1 \leq a, b \leq r + s - 1$ or
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Theorem C

Let $p = 2r \geq 4$, $q + 1 = 2s + 2 \geq 0$. Consider the following objects:

- *The vector bundles $F_{l,E}$ on $\overline{M}_{p,q+1}$ for*

$$l + \min(e_p, p + 1 - e_p) \leq r - 1 \qquad \text{(group } 1A\text{)},$$

$$l + \min(e_p + 1, p - e_p) \leq r - 1 \qquad \text{(group } 1B\text{)},$$

- *The complexes $\tilde{\mathcal{T}}_{l,E}$ on $\overline{M}_{p,q+1}$ for*

$$e_p = r, \quad l + \min(e_q, q + 2 - e_q) \leq s \qquad \text{(group } 2B\text{)}.$$

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- *The complexes* $\tilde{\mathcal{T}}_{l,E}$ *on* $\overline{M}_{p,q+1}$ *for*

$$e_p = r, \quad l + \min(e_q, q + 2 - e_q) \leq s \qquad \text{(group } 2B\text{)}.$$

Then $\overline{M}_{p,q+1}$ *has two* $S_p \times S_{q+1}$*-invariant full exceptional collections of*

- *The torsion sheaves* $\mathcal{O}(-a, -b)$ *in subcategory* $\mathcal{A}$*;*
- *The bundles* $F_{l,E}$ *for pairs* (l, E) *in group* $1A$ *(alternatively* $1B$*),*
- *The complexes* $\tilde{\mathcal{T}}_{l,E}$ *for pairs* (l, E) *in group* $2B$*.*

Example: $p = 6$, $q = 0$ in Theorem C

S_6-invariant full, exceptional collection on $\overline{\mathrm{M}}_6 \cong \overline{\mathcal{M}}_{0,6}$:

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S_6-invariant full, exceptional collection on $\overline{\mathrm{M}}_6 \cong \overline{\mathcal{M}}_{0,6}$:

- The torsion sheaves $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-1, -1)$
- The vector bundles $F_{l,E}$ with $l + \min(e_p + 1, 6 - e_p) \leq 2$

Example: $p = 6,\ q = 0$ in Theorem C

S_6-invariant full, exceptional collection on $\overline{\mathrm{M}}_6 \cong \overline{\mathcal{M}}_{0,6}$:

- The torsion sheaves $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,-1)$
- The vector bundles $F_{l,E}$ with $l + \min(e_p + 1, 6 - e_p) \leq 2$
  - $l = 0,\ e_p = 0$: one line bundle $\rightsquigarrow$ $\mathcal{O}$
  - $l = 0,\ e_p = 4$: $\binom{6}{2}$ line bundles $\rightsquigarrow$ $\{\pi_{ij}^*\mathcal{O}(1)\}_{i,j}$

    $$\pi_{ij} : \overline{\mathcal{M}}_{0,6} \to \overline{\mathcal{M}}_{0,4} = \mathbb{P}^1 \quad \text{forget markings} \quad i, j$$

  - $l = 1,\ e_p = 5$: 6 rank 2 vector bundles $\rightsquigarrow$ $\{\pi_i^*\Omega_{\overline{\mathcal{M}}_{0,5}}(\log)\}_i,$

    $$\pi_i : \overline{\mathcal{M}}_{0,6} \to \overline{\mathcal{M}}_{0,5} \quad \text{forget marking} \quad i$$

  - $l = 2,\ e_p = 6$: one rank 3 vector bundle $\rightsquigarrow$ $\Omega_{\overline{\mathcal{M}}_{0,6}}(\log)$

Map of Proof

Exceptionality of $F_{l,E}$'s:

- Theorem A for p odd, $q = 0$: window calculation
- Theorem A for p odd, $q > 0$: forgetful maps

$$\overline{\mathrm{M}}_{p,q} \to \overline{\mathrm{M}}_{p,q-1} \quad \text{is a} \quad \mathbb{P}^1 \quad \text{bundle}$$

- Theorem B for $\overline{\mathrm{M}}_{p,q} \Rightarrow$ Theorem C for $\overline{\mathrm{M}}_{p,q+1}$ (p even, q odd)

  Compare $R\,\mathrm{Hom}(F_{l,E}, F_{l',E'})$'s via forgetful maps

$$\overline{\mathrm{M}}_{p,q+1} \to \overline{\mathrm{M}}_{p,q}$$

- Theorem C for $\overline{\mathrm{M}}_{p,q-1} \Rightarrow$ Theorem B for $\overline{\mathrm{M}}_{p,q}$ (p even, q odd)

  There is no forgetful map $\overline{\mathrm{M}}_{p,q} \to \overline{\mathrm{M}}_{p,q-1}$. Use instead:

  universal family $\mathcal{U} \to \overline{\mathrm{M}}_{p,q-1} +$ a new reduction map $\mathcal{U} \to \overline{\mathrm{M}}_{p,q}$

Map of Proof

Exceptionality of $\mathcal{T}_{I,E}$'s with $F_{I,E}$'s, $\mathcal{T}_{I,E}$'s (p even, any q)

- window calculation on Z_R (support of torsion sheaf $\mathcal{T}_{I,E}$)

Map of Proof

Exceptionality of $\mathcal{T}_{I,E}$'s with $F_{I,E}$'s, $\mathcal{T}_{I,E}$'s (p even, any q)

- window calculation on Z_R (support of torsion sheaf $\mathcal{T}_{I,E}$)

Fullness (all p, all q)

- Prove that all $F_{I,E}$'s generate $\mathrm{D}^b(\overline{\mathrm{M}}_{p,q})$
- Generate all vector bundles $F_{I,E}$ with the given collection:
- Use forgetful maps+ universal families+ new reduction map
- Use Koszul resolutions of $\mathcal{T}_{I,E}$'s by $F_{I,E}$'s