Rationality and derived categories of some Fano threefolds over non-closed fields

Alexander Kuznetsov

BIRS November 02 2020

Predicting rationality criteria

Derived categories (semiorthogonal decompositions) are useful for predicting (not proving yet) rationality conditions/criteria for algebraic varieties over algebraically closed fields.

Example

If $X \subset \mathbb{P}^5$ is a smooth cubic fourfold, one has

$$\mathbf{D}(X) = \langle \mathcal{A}_X, \mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2) \rangle,$$

where $\mathcal{A}_X$ is a K3-category.

Conjecture

Cubic fourfold X is rational if and only if $\mathcal{A}_X \cong \mathbf{D}(S)$, where S is a K3-surface.

Griffiths components

In general, assume $\dim(X) = n$ and

$$\mathbf{D}(X) = \langle \mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_m \rangle$$

is a semiorthogonal decomposition with indecomposable $\mathcal{A}_i$.

Conjecture

X is rational if and only if $\mathcal{A}_i$ is a semiorthogonal component of $\mathbf{D}(Y_i)$, where Y_i is smooth projective and $\dim(Y_i) \le n - 2$.

Definition

Components $\mathcal{A}_i$ of $\mathbf{D}(X)$ for which there is no embedding $\mathcal{A}_i \hookrightarrow \mathbf{D}(Y_i)$ with $\dim(Y_i) \le n - 2$ are called Griffiths components of $\mathbf{D}(X)$.

- Expected: Griffiths componets of $\mathbf{D}(X)$ is a birational invariant of X.
- Problem: failure of Jordan–Hölder property for s.o.d.

Galois setup

What can one say when the base field k is not algebraically closed?

Galois setup: X is a Fano threefold over k, $\mathrm{char}(\mathsf{k}) = 0$, such that

1. $X_{\bar{\mathsf{k}}}$ is rational;
2. $\rho(X) := \mathrm{rk}(\mathrm{Pic}(X)) = 1$.

Remark

- $\rho(X_{\bar{\mathsf{k}}}) := \mathrm{rk}(\mathrm{Pic}(X_{\bar{\mathsf{k}}}))$ may be higher than 1, if $\mathrm{Pic}(X_{\bar{\mathsf{k}}})^{\mathrm{Gal}(\bar{\mathsf{k}}/\mathsf{k})} \subset \mathbb{Q}K_X$.
- The case $\rho(X) > 1$ reduces to the case $\rho(X) = 1$ or to lower dimensions by MMP.

Question

Find a relation between rationality of X and Griffiths components of $\mathbf{D}(X)$.

Equivariant setup

Definition

- X is a G-Fano variety if X is a Fano variety and G is a finite group acting on X *faithfully*.
- X is G-rational if there is a G-equivariant birational isomorphism $X \sim \mathbb{P}^n$ for *some* faithful action $G \curvearrowright \mathbb{P}^n$.

Equivariant setup: X is a G-Fano threefold over $\mathsf{k} = \bar{\mathsf{k}}$, $\mathrm{char}(\mathsf{k}) = 0$, s.t.

1. X is rational;
2. $\rho_G(X) := \mathrm{rk}(\mathrm{Pic}(X)^G) = 1$.

Question

Find a relation between G-rationality of X and Griffiths components of the equivariant derived category $\mathbf{D}_G(X) = \mathbf{D}([X/G])$.

General setup

General setup: $X \to S$ is a smooth family of Fano varieties such that S is connected and

1. for any geometric point s of the base S the corresponding geometric fiber X_s is rational;
2. $\rho(X/S) := \mathrm{rk}(\mathrm{Pic}(X/S)) = 1$.

Question

Find a relation between rationality of X over S and relative Griffiths components of S-linear semiorthogonal decompositions of $\mathbf{D}(X)$.

- The case $S = \mathrm{Spec}(\mathsf{k})$ is equivalent to the Galois setting.
- The case $S = [\mathrm{pt}/G]$ is equivalent to the equivariant setting.

Geometrically rational Fano threefolds with $\rho(X_{\bar{k}}) = 1$

Rationality criteria (Galois setup) for Fano threefolds with $\rho(X_{\bar{k}}) = 1$ were established in https://arxiv.org/abs/1911.08949.

There are 8 types of Fano threefolds $X/\bar{k}$ with $\rho(X_{\bar{k}}) = 1$:

- $\mathbb{P}^3$;
- $Q^3 = (\mathbb{P}^4, \mathcal{O}(2))$;
- $V_4 = (\mathbb{P}^5, \mathcal{O}(2) \oplus \mathcal{O}(2))$;
- $V_5 = (\mathrm{Gr}(2,5), \mathcal{O}(1) \oplus \mathcal{O}(1) \oplus \mathcal{O}(1))$;
- $X_{12} = (\mathrm{OGr}_+(5,10), \mathcal{O}(1)^{\oplus 7})$;
- $X_{16} = (\mathrm{LGr}(3,6), \mathcal{O}(1)^{\oplus 3})$;
- $X_{18} = (\mathrm{G}_2\mathrm{Gr}(2,7), \mathcal{O}(1)^{\oplus 2})$;
- $X_{22} = (\mathrm{Gr}(3,7), (\wedge^2 \mathcal{U}^\vee)^{\oplus 3})$.

If X is a Fano threefold such that $X_{\bar{k}}$ is rational and $\rho(X_{\bar{k}}) = 1$ then X is a k-form of one of the Fano threefolds from this list.

Rationality results, I

Theorem (K, Prokhorov, 2019)

1. *If X is a k-form of V_5 then X is always rational.*
2. *If X is a k-form of $\mathbb{P}^3$, Q^3, X_{12}, X_{22} then X is rational if and only if*
$$X(\mathsf{k}) \neq \varnothing.$$
3. *If X is a k-form of V_4, X_{18}, X_{16} then X is rational if and only if*
$$X(\mathsf{k}) \neq \varnothing \quad \textit{and} \quad \begin{cases} \mathrm{F}_1(X)(\mathsf{k}) \neq \varnothing, & \textit{when } X \textit{ is a } \mathsf{k}\textit{-form of } V_4 \\ \mathrm{F}_2(X)(\mathsf{k}) \neq \varnothing, & \textit{when } X \textit{ is a } \mathsf{k}\textit{-form of } X_{18} \\ \mathrm{F}_3(X)(\mathsf{k}) \neq \varnothing, & \textit{when } X \textit{ is a } \mathsf{k}\textit{-form of } X_{16} \end{cases}$$

In all these cases $X(\mathsf{k}) \neq \varnothing$ implies that X is unirational.

Here $\mathrm{F}_d(X)$ is the Hilber scheme of rational curves of degree d on X.

Geometrically rational Fano threefolds with $\rho(X_{\bar{k}}) > 1$

There are 6 types of Fano threefolds with $\rho(X) = 1$ and $\rho(X_{\bar{k}}) > 1$:

- $X_{\bar{k}} \cong \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$;
- $X_{\bar{k}} \cong (\mathbb{P}^2 \times \mathbb{P}^2, \mathcal{O}(1,1))$;
- $X_{\bar{k}} \cong (\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1,1,1,1))$;
- $X_{\bar{k}} \cong (\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2, \mathcal{O}(1,1,0) \oplus \mathcal{O}(1,0,1) \oplus \mathcal{O}(0,1,1))$;
- $X_{\bar{k}} \cong (\mathbb{P}^3 \times \mathbb{P}^3, \mathcal{O}(1,1)^{\oplus 3})$;
- $X_{\bar{k}} \cong \mathrm{Bl}_{C_1}(Q_1) \cong \mathrm{Bl}_{C_2}(Q_2) \subset Q_1 \times Q_2 \subset \mathbb{P}^4 \times \mathbb{P}^4$, where
  - Q_1 and Q_2 are smooth 3-dimensional quadrics,
  - C_1 and C_2 are rational twisted quartic curves,
  - the isomorphism of the blowups is given by a Cremona transformation.

We call these threefolds $X_{1,1,1}$, $X_{2,2}$, $X_{1,1,1,1}$, $X_{2,2,2}$, $X_{3,3}$, and $X_{4,4}$, respectively.

Rationality results, II

Theorem (K, Prokhorov)

1. *$X_{1,1,1}$, $X_{2,2}$, $X_{2,2,2}$, and $X_{4,4}$ are rational if and only if $X(\mathsf{k}) \neq \varnothing$.*
2. *$X_{3,3}$ is never rational.*

Conjecture

$X_{1,1,1,1}$ is never rational.

- $G_X := \mathrm{Im}\left(\mathrm{Gal}(\bar{\mathsf{k}}/\mathsf{k}) \to \mathfrak{S}_4 \subset \mathrm{Aut}(\mathrm{Pic}(X_{\bar{\mathsf{k}}}))\right)$;
- G_X is a transitive subgroup of $\mathfrak{S}_4$, i.e., $G_X \in \{\mathfrak{S}_4, \mathfrak{A}_4, \mathrm{D}_4, \mathrm{V}_4, \mathrm{C}_4\}$.

Theorem (K, Prokhorov)

1. *For $G_X \in \{\mathfrak{S}_4, \mathfrak{A}_4, \mathrm{D}_4, \mathrm{V}_4\}$:*
   a very general $X = X_{1,1,1,1}$ with $X(\mathsf{k}) \neq \varnothing$ is not stably rational.
2. *For $G \in \{\mathfrak{S}_4, \mathfrak{A}_4\}$ any G-Fano $X = X_{1,1,1,1}$ is not G-rational.*

Griffiths components for threefolds over a base

Recall that an S-linear admissible subcategory $\mathcal{A} \subset \mathbf{D}(X)$ is not a Griffiths component for a family X/S of smooth threefolds if $\mathcal{A}$ is admissible and S-linear in $\mathbf{D}(Y)$, where $\dim(Y/S) \leq 1$.

Assume Y is connected, flat over S, and

- $\dim(Y/S) = 0$, or
- $\dim(Y/S) = 1$ and $\mathrm{g}(Y/S) > 0$,

then $\mathbf{D}(Y)$ has no S-linear decompositions.

- If $\dim(Y/S) = 1$ with the Stein factorization $Y \xrightarrow{\mathbb{P}^1} T \xrightarrow{\text{finite}} S$, where $Y \to T$ is a $\mathbb{P}^1$-bundle then $\mathbf{D}(Y) = \langle \mathbf{D}(T), \mathbf{D}(T, \beta) \rangle$, where $\beta \in \mathrm{Br}(T)_2$ is the corresponding **2-torsion** Brauer class.

Thus, non-Griffiths components are:

1. $\mathbf{D}(T)$, where T/S is finite;
2. $\mathbf{D}(T, \beta)$, where T/S is finite and $\beta \in \mathrm{Br}(T)_2$;
3. $\mathbf{D}(\Gamma)$, where Γ/S is a family of smooth curves of genus $\mathrm{g} > 0$.

Derived categories, I

Theorem

Let X/S be a family of smooth Fano threefolds such that

- *geometric fibers X_s are rational, and*
- $\rho(X_s) = 1$.

Then

1. - $\mathbf{D}(V_5/S) = \langle \mathbf{D}(S), \mathbf{D}(S), \mathbf{D}(S), \mathbf{D}(S) \rangle$
2. a. $\mathbf{D}(\mathbb{P}^3/S) = \langle \mathbf{D}(S), \underline{\mathbf{D}(S, \beta_4)}, \mathbf{D}(S, \beta_4^2), \underline{\mathbf{D}(S, \beta_4^3)} \rangle$
   b. $\mathbf{D}(Q^3/S) = \langle \mathbf{D}(S), \mathbf{D}(S), \mathbf{D}(S), \underline{\mathbf{D}(S, \beta_4)} \rangle$
   c. $\mathbf{D}(X_{12}/S) = \langle \mathbf{D}(S), \mathbf{D}(S), \mathbf{D}(\Gamma_7/S) \rangle$
   d. $\mathbf{D}(X_{22}/S) = \langle \mathbf{D}(S), \mathbf{D}(S), \mathbf{D}(S, \beta_2), \mathbf{D}(S) \rangle$
3. a. $\mathbf{D}(V_4/S) = \langle \mathbf{D}(S), \mathbf{D}(S, \beta_2), \underline{\mathbf{D}(\Gamma_2/S, \beta_{\Gamma,4})} \rangle$
   b. $\mathbf{D}(X_{18}/S) = \langle \mathbf{D}(S), \mathbf{D}(S), \underline{\mathbf{D}(\Gamma_2/S, \beta_{\Gamma,3})} \rangle$
   c. $\mathbf{D}(X_{16}/S) = \langle \mathbf{D}(S), \mathbf{D}(S), \underline{\mathbf{D}(\Gamma_3/S, \beta_{\Gamma,2})} \rangle$

Griffiths components are underlined.

Derived categories, II

Theorem

Let X/S be a family of smooth Fano threefolds such that

- *geometric fibers X_s are rational,*
- $\rho(X/S) = 1$, *and* $r := \rho(X_s) > 1$.

Monodromy $\pi_0(S) \to \mathfrak{S}_r \subset \mathrm{Aut}(\mathrm{Pic}(X_s))$ *gives étale* $S_r \xrightarrow{r:1} S$. *Then*

1.
   a. $\mathbf{D}(X_{1,1,1}/S) = \langle \mathbf{D}(S), \mathbf{D}(S_3, \beta_2), \mathbf{D}(S, \mathrm{Nm}(\beta_2)), \mathbf{D}(S_3, \beta_2 + \mathrm{Nm}(\beta_2)) \rangle$
   b. $\mathbf{D}(X_{2,2}/S) = \langle \mathbf{D}(S), \underline{\mathbf{D}(S_2, \beta_3)}, \mathbf{D}(S), \underline{\mathbf{D}(S_2, \beta_3)} \rangle$
   c. $\mathbf{D}(X_{2,2,2}/S) = \langle \mathbf{D}(S), \mathbf{D}(S, \beta_2), \mathbf{D}(S_3), \mathbf{D}(S_3) \rangle$
   d. $\mathbf{D}(X_{4,4}/S) = \langle \mathbf{D}(S), \mathbf{D}(S), \mathbf{D}(S_2), \mathbf{D}(S_2, \beta_2) \rangle$
2.
   a. $\mathbf{D}(X_{3,3}/S) = \langle \mathbf{D}(S), \underline{\mathbf{D}(S_2, \beta_4)}, \underline{\mathcal{A}_X} \rangle$,
      $\mathcal{A}_{X_s} = \langle \mathbf{D}(\Gamma_3), \mathbf{D}(\bar{\mathrm{k}}) \rangle$;
   b. $\mathbf{D}(X_{1,1,1,1}/S) = \langle \mathbf{D}(S), \mathbf{D}(S_4, \beta_2), \underline{\mathcal{A}_X} \rangle$,
      $\mathcal{A}_{X_s} = \langle \mathbf{D}(\Gamma_1), \mathbf{D}(\bar{\mathrm{k}}), \mathbf{D}(\bar{\mathrm{k}}), \mathbf{D}(\bar{\mathrm{k}}) \rangle$;

Griffiths components are underlined.

Derived categories, III

Theorem

Let X/S be a family of smooth Fano threefolds such that

- *geometric fibers X_s are rational, and*
- $\rho(X_s) = 1$.

If there is a rational section $S \dashrightarrow X$ then

2. (a) $\beta_4 \mapsto 1$, *hence* $\mathbf{D}(\mathbb{P}^3/S) = \langle \mathbf{D}(S), \mathbf{D}(S), \mathbf{D}(S), \mathbf{D}(S) \rangle$
   (b) $\beta_4 \mapsto \beta_2$, *hence* $\mathbf{D}(Q^3/S) = \langle \mathbf{D}(S), \mathbf{D}(S), \mathbf{D}(S), \mathbf{D}(S, \beta_2) \rangle$
3. (a) $\beta_2 \mapsto 1$, $\beta_{\Gamma,4} \mapsto \beta_{\Gamma,2}$, *hence* $\mathbf{D}(V_4/S) = \langle \mathbf{D}(S), \mathbf{D}(S), \underline{\mathbf{D}(\Gamma_2/S, \beta_{\Gamma,2})} \rangle$
   (b) $\mathbf{D}(X_{18}/S) = \langle \mathbf{D}(S), \mathbf{D}(S), \underline{\mathbf{D}(\Gamma_2/S, \beta_{\Gamma,3})} \rangle$
   (c) $\mathbf{D}(X_{16}/S) = \langle \mathbf{D}(S), \mathbf{D}(S), \underline{\mathbf{D}(\Gamma_3/S, \beta_{\Gamma,2})} \rangle$

If also $\mathrm{F}_1(V_4/S) \to S$, $\mathrm{F}_2(X_{18}/S) \to S$, *and* $\mathrm{F}_3(X_{16}/S) \to S$ *have rational sections, then* $\beta_{\Gamma,d} \mapsto 1$ *and all Griffiths components disappear.*

Derived categories, IV

Theorem

Let X/S be a family of smooth Fano threefolds such that

- *geometric fibers X_s are rational,*
- $\rho(X/S) = 1$, *and* $\rho(X_s) > 1$.

If there is a rational section $S \dashrightarrow X$ then

1.
   a. $\beta_2 \mapsto 1$, *hence* $\mathbf{D}(X_{1,1,1}/S) = \langle \mathbf{D}(S), \mathbf{D}(S_3), \mathbf{D}(S), \mathbf{D}(S_3) \rangle$
   b. $\beta_3 \mapsto 1$, *hence* $\mathbf{D}(X_{2,2}/S) = \langle \mathbf{D}(S), \mathbf{D}(S_2), \mathbf{D}(S), \mathbf{D}(S_2) \rangle$
2.
   a. $\beta_4 \mapsto 1$, *hence* $\mathbf{D}(X_{3,3}/S) = \langle \mathbf{D}(S), \mathbf{D}(S_2), \underline{\mathcal{A}_X} \rangle$,
   $\mathcal{A}_{X_s} = \langle \mathbf{D}(\Gamma_3), \mathbf{D}(\bar{\mathsf{k}}) \rangle$;
   b. $\beta_2 \mapsto 1$, *hence* $\mathbf{D}(X_{1,1,1,1}/S) = \langle \mathbf{D}(S), \mathbf{D}(S_4), \underline{\mathcal{A}_X} \rangle$,
   $\mathcal{A}_{X_s} = \langle \mathbf{D}(\Gamma_1), \mathbf{D}(\bar{\mathsf{k}}), \mathbf{D}(\bar{\mathsf{k}}), \mathbf{D}(\bar{\mathsf{k}}) \rangle$;

Rationality conjecture

Conjecture

Let X be a smooth projective threefold over k, $\mathsf{char}(\mathsf{k}) = 0$. Assume

- $X_{\bar{\mathsf{k}}}$ is rational,
- $X(\mathsf{k}) \neq \varnothing$.

Then X is rational if and only if it has no Griffiths components, i.e.,

$$\mathbf{D}(X) = \langle \mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_m \rangle$$

and for each i one has

$$\mathcal{A}_i \cong \mathbf{D}(\mathsf{k}') \quad \text{or} \quad \mathcal{A}_i \cong \mathbf{D}(\mathsf{k}', \beta_2) \quad \text{or} \quad \mathcal{A}_i \cong \mathbf{D}(\Gamma),$$

where k'/k is a finite field extension, $\beta_2 \in \mathrm{Br}(\mathsf{k}')_2$ is a 2-torsion Brauer class, and Γ is a smooth projective curve of positive genus.

Thanks for attention!