Using geometric realizations to construct non-Fourier-Mukai functors

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joint with Theo Raedschelders and Michel Van den Bergh

A non-Fourier-Mukai functor

Theorem (R.,Neeman,Van den Bergh)

There exists a non-Fourier-Mukai functor

$$D^b(Q) \to D^b(\mathbb{P}^4)$$

here Q is a non-singular quadric in $\mathbb{P}^4$.

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General construction for non-Fourier-Mukai functors:

$$D^b(X) \xrightarrow{L} D(X_\eta)$$

working for any smooth projective scheme X of dimension ≥ 3.

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For any smooth projective scheme X with a tilting bundle, $\dim(X) \geq 3$ *construct a non-Fourier-Mukai functor* $D^b(\operatorname{coh} X) \to D^b(\operatorname{coh} Y)$*, for some scheme Y.*

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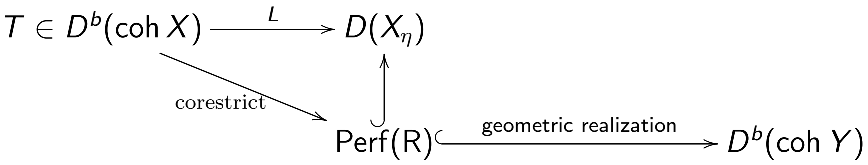

$$R = \text{End}(L(T))$$

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Note that Γ_Λ contains the following algebra:

$$\begin{pmatrix} \Lambda & J & J^2 & \ldots & J^{n-1} \\ \Lambda/J^{n-1} & \Lambda/J^{n-1} & J/J^{n-1} & \ldots & J^{n-2}/J^{n-1} \\ \Lambda/J^{n-2} & \Lambda/J^{n-2} & \Lambda/J^{n-2} & \ldots & J^{n-3}/J^{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \Lambda/J & \Lambda/J & \Lambda/J & \ldots & \Lambda/J \end{pmatrix}$$

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Then

- (Auslander) There exists a fully faithful functor $\text{Perf}(\Lambda) \hookrightarrow \text{Perf}(\Gamma_\Lambda)$ and Γ_Λ has finite global dimension
- (Orlov) $\text{Perf}(\Gamma_\Lambda)$ has a semiorthogonal decomposition

$$\text{Perf}(\Gamma_\Lambda) = \langle \text{Perf}(D_1), \ldots, \text{Perf}(D_N) \rangle$$

with D_i central simple algebras.

Auslander (A_∞)-category

Let R be a finite-dimensional A_∞-algebra equipped with a (decreasing) filtration $R \supset F^1 \supset F^2 \ldots \supset F^{n-1} \supset F^n = 0$ compatible with the A_∞ structure.

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Let Γ be the A_∞-category with objects $0, \ldots, n$ and morphisms

$$\begin{pmatrix} \mathrm{R} & F^1 & F^2 & \cdots & F^{n-1} \\ \mathrm{R}/F^{n-1} & \mathrm{R}/F^{n-1} & F^1/F^{n-1} & \cdots & F^{n-2}/F^{n-1} \\ \mathrm{R}/F^{n-2} & \mathrm{R}/F^{n-2} & \mathrm{R}/F^{n-2} & \cdots & F^{n-3}/F^{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathrm{R}/F^1 & \mathrm{R}/F^1 & \mathrm{R}/F^1 & \mathrm{R}/F^1 & \mathrm{R}/F^1 \end{pmatrix}$$

Proposition (RRVdB)

There is a fully faithful functor $\Gamma \overset{\infty}{\otimes}_{\mathrm{R}} - : \mathrm{Perf}(\mathrm{R}) \to \mathrm{Perf}(\Gamma)$ *and a semi-orthogonal decomposition*

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The SOD is obtained as $\mathrm{Perf}(\Gamma) = \langle \langle S_0 \rangle, \dots, \langle S_{n-1} \rangle \rangle$ with $S_i \in D_\infty \Gamma$ given by

$$S_i = \begin{pmatrix} F^i/F^{i+1} \\ F^{i-1}/F^i \\ \vdots \\ \mathsf{R}/F^1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Corollary (RRvDb)

Let R *be a finite dimensional* A_∞*-algebra equipped with a finite descending filtration such that* $\mathrm{R}/F^1\mathrm{R}$ *is geometric. Then there exists a fully faithful Fourier-Mukai functor* $\operatorname{Perf}\mathrm{R} \hookrightarrow D^b(\operatorname{coh}(Y))$ *where* Y *is a smooth projective* k*-scheme.*

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Proof: By Orlov one can glue geometric realizations, so we take

$$\operatorname{Perf}(\mathrm{R}) \to \operatorname{Perf}(\Gamma) = \langle \underbrace{\operatorname{Perf}(\mathrm{R}/F^1\mathrm{R}), \ldots, \operatorname{Perf}(\mathrm{R}/F^1\mathrm{R})}_{n} \rangle \to \operatorname{Perf}(Y).$$

Let X be a smooth projective scheme of dimension $m \geq 3$ and $M = \omega_X^{\otimes 2}$.

- Construct a category $\mathcal{X}$ encoding the data of an affine covering of X, so that $\text{Mod}(\mathcal{X})$ is roughly the category of presheaves associated to the covering;

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  - The objects are the same as the objects of $\mathcal{X}$
  - Morphisms are $\mathcal{X}(-,-) \oplus \Sigma^{2m-2} M(-,-)$
  - A_∞ structure on morphisms is given by composition in $\mathcal{X}$ (m_2) and by $\eta : \mathcal{X} \otimes \ldots \otimes \mathcal{X} \to M \hookrightarrow \mathcal{X}_\eta$ (m_{2m})

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The filtration by degree $F^p = \bigoplus_{i \geq p} \mathrm{R}^{-i}$ is compatible with the A_∞ structure so we can apply our geometrization result:

$$D^b(X) \xrightarrow{L} \mathrm{Perf}(\mathrm{R}) \to D^b(Y).$$