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The Brauer group of bielliptic surfaces

Derived, Birational, and Categorical Algebraic Geometry

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I am hiring a postdoc starting July 1st 2021!!!

The Question

Let X a smooth projective variety over a field k (which will be $\mathbb{C}$ for what it concerns us today). The cohomological Brauer group of X is

$$\mathrm{Br'}(X) := \mathrm{H}^2_{\text{ét}}(X, \mathcal{O}_X^*)_{\mathrm{tor}}.$$

Given a morphism $f : X \to Y$ of smooth projective varieties, then, by pulling back classes, we get a group homomorphism

$$f_{\mathrm{Br}} : \mathrm{Br'}(Y) \longrightarrow \mathrm{Br'}(X)$$

which we call the Brauer map associated to f.

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Question

What can we say about f_{Br}?

Beauville and Enriques surfaces

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Let S be a complex Enriques surface and denote by $\hat{S}$ its universal cover. Then $\hat{S}$ is a K3 surface and there is an étale 2-1 morphism

$$\pi : \hat{S} \longrightarrow S.$$

Denote by $\sigma : \hat{S} \to \hat{S}$ the Enriques involution.
The Brauer group of an Enriques surfaces is isomomorphic $\mathbb{Z}/2\mathbb{Z}$ so there are only two possible behaviors for the Brauer map associated to π:

Theorem (Beauville 2009)

The Brauer map π_{Br} is trivial if, and only if, there is a line bundle L on $\hat{S}$ such that $\sigma^* L = L^{-1}$ and $c_1(L)^2 \equiv 2 (\mod 4)$.

Today

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We are going to study the problem for complex bielliptic surfaces. Given a complex bielliptic surface S, then there is always a cyclic cover

$$\pi : \hat{S} \to S$$

where $\hat{S}$ is an abelian surface. Sometimes (more details are coming) there is also a cyclic cover

$$\tilde{\pi} : \tilde{S} \to S$$

where $\tilde{S}$ is another bielliptic surface.

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We completely characterize the behavior of the Brauer map associated to these morphisms.

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Goal

We completely characterize the behavior of the Brauer map associated to these morphisms.

We= this is a joint work with E. Ferrari, M. Vodrup (with an appendix by me and J. Bergström).

Derived Categories?

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- Beauville work was used by Addington and Wray to study (the non existence of) twisted Fourier–Mukai partners of Enriques surfaces.
- Vodrup is using this work to do a similar investigation for bielliptic surfaces.

Plan

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1. Bielliptic Surfaces
2. The results
3. How did we do it?

Section 1
Bielliptic Surfaces

The Definition

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Definition

A biellipitic surface is a surface S with irregularity

$$q(S) := h^1(X, \mathcal{O}_S) = 1$$

and numerically trivial canonical divisor class.

- The canonical bundle is torsion, but not trivial (we are working on the complex numbers!)
- They are always constructed as quotients of a product of two elliptic curves by a finite group action.

Example

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Let A and B two elliptic curves and let $G := \mathbb{Z}/2\mathbb{Z}$.
Choose τ a point of order two in A and consider the involution

$$\sigma : A \times B \longrightarrow A \times B$$

defined by $(a, b) \mapsto (a + \tau, -b)$.
Then the surface $S := A \times B / < \sigma >$ is bielliptic.

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Then the surface $S := A \times B / < \sigma >$ is bielliptic.
The surface S admits two elliptic fibrations

$$\begin{array}{ccc} & S & \\ {}^{\mathrm{Alb}_S}\swarrow & & \searrow{}^{g} \\ A/ < \tau > & & \mathbb{P}^1 \end{array}$$

with general fibers isomorphic to A and B respectively.

Bagnera- de Franchis

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Type	G	Order of ω_S in $\mathrm{Pic}(S)$	$H^2(S, \mathbb{Z})_{\text{tor}}$
1	$\mathbb{Z}/2\mathbb{Z}$	2	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
2	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	2	$\mathbb{Z}/2\mathbb{Z}$
3	$\mathbb{Z}/4\mathbb{Z}$	4	$\mathbb{Z}/2\mathbb{Z}$
4	$\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	4	0
5	$\mathbb{Z}/3\mathbb{Z}$	3	$\mathbb{Z}/3\mathbb{Z}$
6	$\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$	3	0
7	$\mathbb{Z}/6\mathbb{Z}$	6	0

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3	$\mathbb{Z}/4\mathbb{Z}$	4	$\mathbb{Z}/2\mathbb{Z}$
4	$\mathbb{Z}/4\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$	4	0
5	$\mathbb{Z}/3\mathbb{Z}$	3	$\mathbb{Z}/3\mathbb{Z}$
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Fun Facts

- The Brauer group of a bielliptic surface is non canonically isomorphic to $H^2(S,\mathbb{Z})_{\mathrm{tor}}$ so we will disregard types 4, 6 and 7.

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Fun Facts

- The Brauer group of a bielliptic surface is non canonically isomorphic to $H^2(S,\mathbb{Z})_{\text{tor}}$ so we will disregard types 4, 6 and 7.
- To construct types 3 and 5 we cannot choose freely the elliptic curve B: for type 3 we have $j(B) = 1728$, and for type 5 we have $j(B) = 0$.

Canonical Covers

Let S be a bielliptic surface and denote by n the order of its canonical bundle. Then ω_S induces an étale cyclic cover $\pi_S \colon \hat{S} \to S$, called the *canonical cover of S*.
If we let $\lambda_S := |G|/|(\omega_S)|$, we have that $G \simeq \mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/\lambda_S\mathbb{Z}$, and X is the abelian surface sitting as an intermediate quotient

$$\begin{array}{ccc} A \times B & \longrightarrow & S \simeq A \times B/G \\ & \searrow \qquad \nearrow_{\pi} & \\ & X \simeq A \times B/H & \end{array}$$

where $H \simeq \mathbb{Z}/\lambda_S\mathbb{Z}$.

Bielliptic covers (after Nuer)

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When G is not cyclic or when G is cyclic, of non prime order number, then the bielliptic surface S admits a cyclic cover $\tilde{\pi} : \tilde{S} \to S$, where $\tilde{S}$ is another bielliptic surface.

Example

1. If S is a bielliptic surface of type 3, then the canonical bundle has order 4. In addition the canonical cover $\hat{S}$ of S is a product of elliptic curves, that is $X \simeq A \times B$. By taking the cover associated with $\omega_S^{\otimes 2}$ we get $\tilde{S}$ which is a bielliptic surface of type 1.
2. Suppose that S is a bielliptic surface of type 2, so the group G is isomorphic to the product $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Then we obtain $\tilde{S}$ from $A \times B$ by taking the quotient with respect to $(x, y) \mapsto (x + \tau, -y)$. Thus $\tilde{S}$ is a again bielliptic surface of type 1.

Section 2
The results

Those easily stated...

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Theorem (Ferrari, :-), Vodrup)

Let S be a bielliptic surface which admits a bielliptic cover $\tilde{\pi} : \tilde{S} \to S$.

1. If S is of type 2, then $\tilde{\pi}_{\mathrm{Br}} : \mathrm{Br}(S) \to \mathrm{Br}(\tilde{S})$ is trivial.
2. If S is of type 3, then $\tilde{\pi}_{\mathrm{Br}} : \mathrm{Br}(S) \to \mathrm{Br}(\tilde{S})$ is injective.

Theorem (Ferrari, :-), Vodrup)

Let $S := A \times B/G$ be a bielliptic surface and denote by $\pi : \hat{S} \to S$ its canonical cover. If the elliptic curves A and B are not isogenous, then the Brauer map π_{Br} is trivial.

1. If S is of type 2, then $\tilde{\pi}_{\mathrm{Br}} : \mathrm{Br}(S) \to \mathrm{Br}(\tilde{S})$ is trivial.
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The description of the behavior of the Brauer map associated to the canonical cover in the isogeny case is far from being neat.

Type 1 bielliptic surfaces:

There are two main subcases:

1. When B (and so A) does not have complex multiplication. Here we will see has the map can be non injective, but it is never trivial.
2. When B (and so A) has complex multiplication. Here the map can be trivial.

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Notation

Recall that in this case G is cyclic of order 2 acting by translation by a point τ on A and by $-\operatorname{id}_B$ on B. We identify the dual of A with A using the isomorphism associated to $\mathcal{O}_A(0_A)$ and we denote by P_τ the topologically trivial line bundle on A associated to τ.

A whiteboard

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Non CM case

If A and B are isogenous and do not have complex multiplication, then $\mathrm{Hom}(B, A)$ is a free $\mathbb{Z}$-module of rank 1.

Theorem

The Brauer map associated to the canonical cover is not injective if, and only if, for a (and hence all) choice of a generator $\phi : B \to A$ we have that $\phi^* P_\tau$ is trivial. In this case the kernel of the Brauer map will be isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

CM case

If A and B are isogenous and do not have complex multiplication, then $\mathrm{Hom}(B, A)$ is a free $\mathbb{Z}$-module of rank 2.

Theorem

The Brauer map associated to the canonical cover is not injective if, and only if, for a (and hence all) choice of a generators $\phi_1,\ \phi_2 : B \to A$ we have that one of the following line bundles is trivial

$$\phi_1^* P_\tau, \quad \phi_2^* P_\tau, \quad (\phi_1 + \phi_2)^* P_\tau \tag{1}$$

In addition the Brauer map is trivial if, and only if, two (and hence all) line bundles in are trivial. (1)

Examples

- Suppose that $A \simeq B$. If A does not have complex multiplication, then we can take $\psi = \pm 1_A$. In particular we have that $\psi^* P_\tau$ is never trivial and the Brauer map is injective.

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- Suppose again that $A \simeq B$ and that the j-invariant of A is 1728. Then $\mathrm{End}(A) \simeq \mathbb{Z}[i]$ and the multiplication by i induces an automorphism ω of A of order 4, and we can take 1_A and ω as generators of $\mathrm{End}(A)$. Suppose that P_τ is a fixed point of the dual automorphism ω^* (For example we can identify A with its dual and ω^* with ω and take $\tau = (\frac{1}{2}, \frac{1}{2}) + \Lambda$, where $\Lambda =< 1, i >$ $A \simeq \mathbb{C}/\Lambda$). Then $(1_A + \omega)^* P_\tau$ is zero and the Brauer map is not injective (and is neither trivial!!!)

Examples II

- We can also use a similar argument to construct uncountably many Type 1 byelliptic surfaces with non injective Brauer map. Let *B* any elliptic curve without complex multiplication and chose θ a point of over 2 on *B*. Let $A := B/\theta$ and $\psi : B \to A$ the quotient map. This is a degree 2 isogeny, so it is primitive and hence generating. If τ is the only point of order 2 in $\mathrm{Ker}\,\psi^*$, then we have that the data *A*, τ, *B* uniquely identify a Type 1 bielliptic surface which has a non injective Brauer map.

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- Let now $A \simeq \mathbb{C}/\mathbb{Z}[2i]$ and let τ the point $(0, i) + \mathbb{Z}[2i]$. The elliptic curve $B := A/ < \tau >$ has j-invariant 1728 and $\mathrm{Hom}(B, A)$ is generated by the isogenies $\psi_1 := \varphi_2$ and $\psi_2 := \varphi_2 \circ \lambda_B$, where $\varphi_2 : B \to A$ denotes the isogeny induced by multiplication by 2. Observe that

$$\varphi_2^*(P_\tau) \simeq \varphi_2^*(\mathcal{O}_A(\tau - p_0) \simeq \mathcal{O}_A(\varphi_2(\tau) - \varphi(p_0)) \simeq \mathcal{O}_B$$

  Thus we have that $\psi_1^* P_\tau \simeq \psi_2^* P_\tau \simeq \mathcal{O}_B$ and the Brauer map is trivial.

The "Moduli" Picture

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Type 1 bielliptic surfaces are constructed by choosing two elliptic curves *A* and *B* and a 2-torsion point on *A*. Thus the moduli space has dimension 2. In order to have a non injective Brauer map one can choose freely the elliptic curve *B*, but has only finitely many possibilities for *A* and the 2-torsion point. Thus we obtain a 1-dimensional family. On the other hand only countably many type 1 bielliptic surfaces can have a trivial Brauer map to their canonical cover. In fact, to obtain a trivial Brauer map one has to choose the ellipic curve *B* among those having complex multiplication.

The other types

Type 2: These surfaces are constructed by choosing two elliptic curves A and B and two 2-torsion points, one on A and one on B. Hences the moduli space has dimension 1. Similarly to what happens in the previous case, in order to have a trivial Brauer map only the choice of the curve B can be made freely, while A must be taken among finitely many possibilities.

Type 3: These surfaces are constructed by choosing one elliptic curve A and a 4-torsion point on it. Therefore the moduli space has dimension 1. In order to have a non injective (and hence trivial) Brauer map, A must be isogenous to the curve with j-invariant 1728. Thus there are only finitely many such surfaces.

Type 5: These surfaces are constructed by choosing one elliptic curve A and a 3-torsion point on it. We deduce that the moduli space has dimension 1. In order to have a non injective (and hence trivial) Brauer map, A must be isogenous to the curve with j-invariant 0. Thus, as in the previous case, there are only finitely many such surfaces.

Section 3
How did we do it?

Beauville's Strategy

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Let $\pi : X \to Y$ be a finite locally free morphism of projective varieties of degree n. To it we can associate a group homomorphism

$$\mathrm{Nm}_\pi : \mathrm{Pic}(X) \to \mathrm{Pic}(Y)$$

called the norm homomorphism associated to π

Proposition (Beauville 2009)

Let $\pi : X \to S$ be an étale cyclic covering of smooth projective varieties. Let σ be a generator of the Galois group of π, $\mathrm{Nm}_\pi : \mathrm{Pic}(X) \to \mathrm{Pic}(S)$ be the norm map and $\pi_{\mathrm{Br}} : \mathrm{Br}(S) \to \mathrm{Br}(X)$ be the pullback. Then we have a canonical isomorphism

$$\mathrm{Ker}(\pi_{\mathrm{Br}}) \simeq \mathrm{Ker}\,\mathrm{Nm}_\pi/(1-\sigma^*)\,\mathrm{Pic}(X).$$

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$$\mathsf{Ker}(\pi_{\mathsf{Br}}) \simeq \mathsf{Ker}\,\mathrm{Nm}_\pi/(1-\sigma^*)\,\mathsf{Pic}(X).$$

So we studied the quotient in the RHS in the various cases.

Outline:

1. We first find numerical conditions for a line bundle to be in the kernel of the norm map, and we get V a subspace of the NS upstair.

Example

- If upstair and downstair we have the same Picard rank, then $L \in \mathrm{Ker\,Nm}$ iff $L \equiv 0$.
- If the group G is cyclic and $\pi : A \times B \to S$ is the canonical cover, then $\mathrm{Nm}_\pi(L)$ is trivial iff the numerical class of L is in $\mathrm{Hom}(A, B)$.

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2. We study $V/(1-\sigma^*)V$ and get "numerical candidates".
3. Study the norm of line bundles equivalent to the numerical candidates.

The Pic^0 trick

How to construct elements in the kernel of the norm

Let $\pi : X \to Y$ be an étale morphism of degree n and suppose that there is a line bundle L on X such that $\mathrm{Nm}_\pi(L) \in \mathrm{Pic}^0(Y)$. Then there is an element $\alpha \in \mathrm{Pic}^0(X)$ such that $\mathrm{Nm}_\pi(L \otimes \alpha)$ is trivial.

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Proof:

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Proof:

1. By Norm-calculus we have that $\mathrm{Nm}_\pi(\pi^* M) = M^{\otimes n}$;'

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2. $\mathrm{Pic}^0(Y)$ is a divisible group so we can find $\beta \in \mathrm{Pic}^0(Y)$ such that $\beta^{\otimes n} \simeq \mathrm{Nm}_\pi(L)^{-1}$;

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2. $\mathrm{Pic}^0(Y)$ is a divisible group so we can find $\beta \in \mathrm{Pic}^0(Y)$ such that $\beta^{\otimes n} \simeq \mathrm{Nm}_\pi(L)^{-1}$;
3. then we have

$$\mathrm{Nm}_\pi(L \otimes \pi^* \beta) \simeq \mathrm{Nm}_\pi(L) \otimes \beta^{\otimes n} \simeq \mathcal{O}_Y.$$

Bielliptic covers

Let S be a Type 2 or 3 bielliptic surfaces, then there is a Type 1 bielliptic surface $\tilde{S}$ and an involution $\tilde{\sigma}$ such that $S \simeq \tilde{S}/\tilde{\sigma}$.

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1. The fibration $g : \tilde{S} \to \mathbb{P}^1$ has four multiple fibers all of multiplicity 2 which we will denote by $D_1, \ldots, D_4$. Let $\tau_{ij} := \mathcal{O}_{\tilde{S}}(D_i - D_j)$.

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2. We have that
$$H^2(\tilde{S}, \mathbb{Z})_{\text{tor}} = \{0, [\tau_{1j}]\}_{j \neq 1}\}$$

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2. We have that

$$H^2(\tilde{S}, \mathbb{Z})_{\text{tor}} = \{0, [\tau_{1j}]\}_{j \neq 1}\}$$

Type 2

If S is of type 2, then by seeing how the involution acts on the D_i's we get that $\text{Nm}(\tau_{13}) \in \text{Pic}^0(S)$, but $\tau_{13} \otimes \alpha \notin \text{Im}(1 - \tilde{\sigma}^*)$ for every $\alpha \in \text{Pic}^0(\tilde{S})$. We conclude by the Pic^0-trick.

Bielliptic Cover of Type 3

Why they are different

The deep reason for the different behavior of the Brauer map in the two cases is how the $\tilde{\sigma}$ acts on the D_i's, which affect the computation of the norm map.

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Lemma

Let n and m be two integers. Then the norm of the line bundle $\tau_{13}^{\otimes n} \otimes \tau_{14}^{\otimes m}$ is zero if and only if n and m have the same parity. In addition we have that $\mathrm{Nm}(\tau_{13}^{\otimes n} \otimes \tau_{14}^{\otimes m})$ is not in $\mathrm{Pic}^0(S)$ if n and m are not congruent modulo 2.

Preposition

If L is in the Kernel of the norm map, then L is in $\mathrm{Pic}^{\tau}(\tilde{S})$.

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Preposition

If L is in the Kernel of the norm map, then L is in $\mathrm{Pic}^{\tau}(\tilde{S})$.

(Depends from the fact that $\tilde{S}$ and S have the same Picard rank).

Questions?

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Thank you for your attention!