University of Alberta

Department of Mathematical and Statistical Sciences

Quillen metrics on modular curves

Mathieu Dutour

Saturday, November 6th

Contents

Determinant line bundle

Let X be a compact Riemann surface, and E be a holomorphic vector bundle over X, both endowed with smooth metrics.

Determinant line bundle

Let X be a compact Riemann surface, and E be a holomorphic vector bundle over X, both endowed with smooth metrics.

Definition

The determinant line bundle $\lambda(E)$ is defined as

$$\lambda(E) \quad = \quad \det H^0(X, E) \otimes \det H^1(X, E)^\vee \quad .$$

Determinant line bundle

Let X be a compact Riemann surface, and E be a holomorphic vector bundle over X, both endowed with smooth metrics.

Definition

The determinant line bundle $\lambda(E)$ is defined as

$$\lambda(E) \quad = \quad \det H^0(X, E) \otimes \det H^1(X, E)^\vee \quad .$$

Using Hodge theory, we can put the L^2-metric on $\lambda(E)$.

Quillen metric

The Quillen metric on $\lambda(E)$ is a correction of the L^2-metric to account for all the eigenvalues of the Dolbeault Laplacian $\Delta_{\overline{\partial}_E}$.

Definition

The Quillen metric $\|\cdot\|_Q$ on $\lambda(E)$ is defined as

$$\|\cdot\|_Q = \left(\det{}' \Delta_{\overline{\partial}_E}\right)^{-1/2} \|\cdot\|_{L^2} .$$

Properties

The Quillen metric, contrary to its L^2-counterpart, satisfies three properties :

Properties

The Quillen metric, contrary to its L^2-counterpart, satisfies three properties :

1. Smoothness in family

Properties

The Quillen metric, contrary to its L^2-counterpart, satisfies three properties :

1. Smoothness in family
2. Spectral interpretation

Properties

The Quillen metric, contrary to its L^2-counterpart, satisfies three properties :

1. Smoothness in family
2. Spectral interpretation
3. Riemann-Roch type theorem

The problem with modular curves

Let $X = \overline{\Gamma\backslash\mathbb{H}}$ be a compactified modular curve, where Γ is a Fuchsian group of the first kind, without torsion.

The problem with modular curves

Let $X = \overline{\Gamma \backslash \mathbb{H}}$ be a compactified modular curve, where Γ is a Fuchsian group of the first kind, without torsion.

Let E be a flat, unitary, holomorphic vector bundle of rank r over X, coming from a representation

$$\rho \quad : \quad \Gamma \quad \longrightarrow \quad U_r(\mathbb{C}) \quad .$$

The problem with modular curves

Let $X = \overline{\Gamma \backslash \mathbb{H}}$ be a compactified modular curve, where Γ is a Fuchsian group of the first kind, without torsion.

Let E be a flat, unitary, holomorphic vector bundle of rank r over X, coming from a representation

$$\rho \quad : \quad \Gamma \quad \longrightarrow \quad U_r(\mathbb{C}) \quad .$$

The canonical Hermitian metric on $\mathbb{C}^r$ induces a metric on E. Both it and the Poincaré metric on X are singular at the cusps .

Contents

The Selberg zeta function

Let $X = \overline{\Gamma \backslash \mathbb{H}}$ be a compactified modular curve without elliptic points, and E be a flat, unitary vector bundle over X.

The Selberg zeta function

Let $X = \overline{\Gamma\backslash\mathbb{H}}$ be a compactified modular curve without elliptic points, and E be a flat, unitary vector bundle over X.

Definition

The Selberg zeta function associated to X and E is defined by

$$Z(s,\Gamma,\rho) \quad = \quad \prod_{\{\gamma\}_{\mathrm{hyp}}} \prod_{k=0}^{+\infty} \det\left(I - \rho(\gamma) N(\gamma)^{-s-k}\right) \quad .$$

The Selberg zeta function

Let $X = \overline{\Gamma\backslash\mathbb{H}}$ be a compactified modular curve without elliptic points, and E be a flat, unitary vector bundle over X.

Definition

The Selberg zeta function associated to X and E is defined by

$$Z(s, \Gamma, \rho) = \prod_{\{\gamma\}_{\mathrm{hyp}}} \prod_{k=0}^{+\infty} \det\left(I - \rho(\gamma) N(\gamma)^{-s-k}\right) .$$

This function exists on the half-plane $\mathrm{Re}\, s > 1$, and can be meromorphically continued to the complex plane. The location and multiplicity of the zeros and poles is given by the Selberg trace formula.

First attempt at a Quillen metric

Assuming E is stable, Takhtajan and Zograf defined in 2007 a Quillen metric on the determinant line bundle $\lambda\left(\mathrm{End}\left(E\right)\right)$ of the endomorphism bundle of E.

First attempt at a Quillen metric

Assuming E is stable, Takhtajan and Zograf defined in 2007 a Quillen metric on the determinant line bundle $\lambda\left(\mathrm{End}\left(E\right)\right)$ of the endomorphism bundle of E.

Definition (Takhtajan-Zograf, 2007)

The regularized determinant is defined as

$$\det \Delta = \frac{\partial}{\partial s}_{|s=1} Z\left(s, \Gamma, Ad\,\rho\right)$$

where $Ad\,\rho$ is the adjoint representation, and the Quillen metric by

$$\left\|\cdot\right\|_{Q} = \left(\det \Delta\right)^{-1/2} \left\|\cdot\right\|_{L^2} \; .$$

First attempt at a Quillen metric

Assuming E is stable, Takhtajan and Zograf defined in 2007 a Quillen metric on the determinant line bundle $\lambda\left(\mathrm{End}\left(E\right)\right)$ of the endomorphism bundle of E.

Definition (Takhtajan-Zograf, 2007)

The regularized determinant is defined as

$$\det\Delta \quad = \quad \frac{\partial}{\partial s}_{|s=1} Z\left(s, \Gamma, Ad\,\rho\right)$$

where $Ad\,\rho$ is the adjoint representation, and the Quillen metric by

$$\left\|\cdot\right\|_Q \quad = \quad \left(\det\Delta\right)^{-1/2} \left\|\cdot\right\|_{L^2} \quad .$$

Their aim was to get a curvature formula.

Properties of the Quillen metric

As inspired by the compact case, this Quillen metric should satisfy :

1. Smoothness in family
2. Spectral interpretation
3. Riemann-Roch type theorem

Properties of the Quillen metric

As inspired by the compact case, this Quillen metric should satisfy :

1. Smoothness in family
2. Spectral interpretation
3. Riemann-Roch type theorem

Properties of the Quillen metric

As inspired by the compact case, this Quillen metric should satisfy :

1. Smoothness in family
2. Spectral interpretation
3. Riemann-Roch type theorem

Properties of the Quillen metric

As inspired by the compact case, this Quillen metric should satisfy :

1. Smoothness in family
2. Spectral interpretation
3. Riemann-Roch type theorem

Properties of the Quillen metric

As inspired by the compact case, this Quillen metric should satisfy :

1. Smoothness in family
2. Spectral interpretation
3. Riemann-Roch type theorem

We will work to get a functorial Riemann-Roch theorem on modular curves, similar to the one proved by Deligne in 1987.

Contents

Functorial isomorphism

Let $f : X \longrightarrow S$ be a family of compact Riemann surfaces of genus g, and E be a holomorphic vector bundle over X of rank r.

Functorial isomorphism

Let $f : X \longrightarrow S$ be a family of compact Riemann surfaces of genus g, and E be a holomorphic vector bundle over X of rank r.

Theorem (Deligne, 1987)

We have an isomorphism of line bundles over S

$$\lambda(E)^{12}_{X/S} \quad \simeq \quad \langle \omega_{X/S}, \omega_{X/S} \rangle^{r} \langle \det E, \det E \otimes \omega^{-1}_{X/S} \rangle^{6} IC2_{X/S}(E)^{-12}$$

which is compatible with base change.

Isometry

Assuming $\omega_{X/S}$ and E are endowed with smooth metrics, every factor in Deligne's isomorphism can be metrized, and we have the following.

Isometry

Assuming $\omega_{X/S}$ and E are endowed with smooth metrics, every factor in Deligne's isomorphism can be metrized, and we have the following.

Theorem (Deligne, 1987)

We have an isometry of line bundles over S

$$\lambda(E)^{12}_{X/S} \quad \simeq \quad \langle \omega_{X/S}, \omega_{X/S} \rangle^r \langle \det E, \det E \otimes \omega_{X/S}^{-1} \rangle^6 IC2_{X/S}(E)^{-12}$$

up to a factor depending only on g, which is compatible with base change.

Isometry

Assuming $\omega_{X/S}$ and E are endowed with smooth metrics, every factor in Deligne's isomorphism can be metrized, and we have the following.

Theorem (Deligne, 1987)

We have an isometry of line bundles over S

$$\lambda(E)^{12}_{X/S} \quad \simeq \quad \langle \omega_{X/S}, \omega_{X/S} \rangle^r \langle \det E, \det E \otimes \omega^{-1}_{X/S} \rangle^6 IC2_{X/S}(E)^{-12}$$

up to a factor depending only on g, which is compatible with base change.

The isometry part of this can be checked above each point of S.

Contents

Singularity of the metrics

Let $X = \overline{\Gamma \backslash \mathbb{H}}$ be a compactified modular curve without elliptic points, and E be a flat unitary vector bundle over X of rank r.

Singularity of the metrics

Let $X = \overline{\Gamma\backslash\mathbb{H}}$ be a compactified modular curve without elliptic points, and E be a flat unitary vector bundle over X of rank r.

Deligne's result cannot be applied directly, as the metrics on X and E are singular at the cusps.

Singularity of the metrics

Let $X = \overline{\Gamma\backslash\mathbb{H}}$ be a compactified modular curve without elliptic points, and E be a flat unitary vector bundle over X of rank r.

Deligne's result cannot be applied directly, as the metrics on X and E are singular at the cusps.

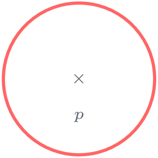
An open neighborhood of a cusp p can be seen as a punctured disk, and in the associated z coordinate, the metrics are as follows

Singularity of the metrics

Let $X = \overline{\Gamma\backslash\mathbb{H}}$ be a compactified modular curve without elliptic points, and E be a flat unitary vector bundle over X of rank r.

Deligne's result cannot be applied directly, as the metrics on X and E are singular at the cusps.

An open neighborhood of a cusp p can be seen as a punctured disk, and in the associated z coordinate, the metrics are as follows



$|z| = \varepsilon$

$$\text{Poincaré metric} \quad : \quad ds^2_{\mathrm{hyp}} = \frac{|dz|^2}{(|z|\log|z|)^2}$$

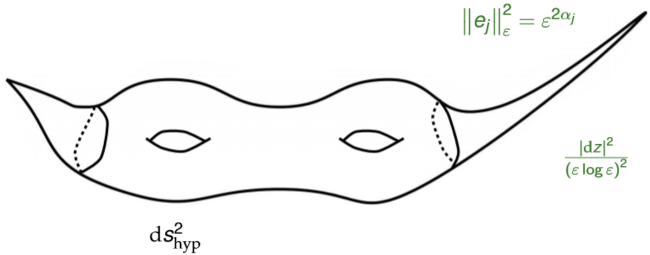
$$\text{Metric on } E \quad : \quad \|e_j\|^2_z = |z|^{2\alpha_j}$$

Truncation of the metrics

In order to get around the singularity of the metric, we truncate the metric, *i.e.* we replace it by the constant value they take on the boundary of each circle of radius ε.

Truncation of the metrics

In order to get around the singularity of the metric, we truncate the metric, *i.e.* we replace it by the constant value they take on the boundary of each circle of radius ε.



ε-Quillen metric

Setting aside the fact that the truncated metrics are only continuous, and not smooth, we can make the following definition

ε-Quillen metric

Setting aside the fact that the truncated metrics are only continuous, and not smooth, we can make the following definition

Definition

The ε-Quillen metric on $\lambda(E)$ is defined to be

$$\|\cdot\|_{Q,\varepsilon} = \left(\det \Delta_{\overline{\partial}_E,\varepsilon}\right)^{-1/2} \|\cdot\|_{L^2} \ ,$$

where $\Delta_{\overline{\partial}_E,\varepsilon}$ is the Dolbeault Laplacian acting on functions associated to the truncated metric.

ε-isometry

This ε-Quillen metric now fits into Deligne's result, which yields

$$\lambda(E)^{12}_{Q,\varepsilon} \quad \simeq \quad \langle \omega_{X,\varepsilon}, \omega_{X,\varepsilon} \rangle^{r} \left\langle \det E_{\varepsilon}, \det E_{\varepsilon} \otimes \omega^{-1}_{X,\varepsilon} \right\rangle^{6} IC2\,(E_{\varepsilon})^{-12},$$

where every index ε means the metric has been truncated at radius ε at each cusp.

ε-isometry

This ε-Quillen metric now fits into Deligne's result, which yields

$$\lambda\left(E\right)_{Q,\varepsilon}^{12} \quad \simeq \quad \left\langle \omega_{X,\varepsilon}, \omega_{X,\varepsilon} \right\rangle^{r} \left\langle \det E_{\varepsilon}, \det E_{\varepsilon} \otimes \omega_{X,\varepsilon}^{-1} \right\rangle^{6} IC2\left(E_{\varepsilon}\right)^{-12},$$

where every index ε means the metric has been truncated at radius ε at each cusp.

The aim is now to let ε go to 0

Twisting the isometry

Since none of the factors in the ε-isometry converges as ε goes to 0, we will need to regularize them, so as to extract the divergent part.

Twisting the isometry

Since none of the factors in the ε-isometry converges as ε goes to 0, we will need to regularize them, so as to extract the divergent part.

For instance, one can regularize $\omega_{X,\varepsilon}$, by writting it as

$$\omega_{X,\varepsilon} \quad = \quad \omega_{X,\varepsilon}(D) \otimes \mathcal{O}_{X,\varepsilon}(-D) \quad ,$$

Twisting the isometry

Since none of the factors in the ε-isometry converges as ε goes to 0, we will need to regularize them, so as to extract the divergent part.

For instance, one can regularize $\omega_{X,\varepsilon}$, by writting it as

$$\omega_{X,\varepsilon} \quad = \quad \omega_{X,\varepsilon}(D) \otimes \mathcal{O}_{X,\varepsilon}(-D) \quad ,$$

where D is the divisor

$$D \quad = \quad \sum_{p \text{ cusp}} p \quad .$$

Twisting the isometry

Since none of the factors in the ε-isometry converges as ε goes to 0, we will need to regularize them, so as to extract the divergent part.

For instance, one can regularize $\omega_{X,\varepsilon}$, by writting it as

$$\omega_{X,\varepsilon} \quad = \quad \omega_{X,\varepsilon}(D) \otimes \mathcal{O}_{X,\varepsilon}(-D) \quad ,$$

where D is the divisor

$$D \quad = \quad \sum_{p \text{ cusp}} p \quad .$$

The Deligne pairing $\langle \omega_{X,\varepsilon}(D), \omega_{X,\varepsilon}(D) \rangle$ then converges as ε goes to 0.

Spectral side

The last step is to understand the determinant of the Laplacian $\Delta_{\overline{\partial}_E,\varepsilon}$ as ε goes to 0.

Spectral side

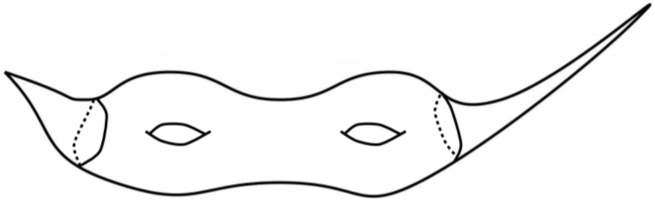
The last step is to understand the determinant of the Laplacian $\Delta_{\overline{\partial}_E, \varepsilon}$ as ε goes to 0.

For that, we need analytic surgery.

Spectral side

The last step is to understand the determinant of the Laplacian $\Delta_{\overline{\partial}_E,\varepsilon}$ as ε goes to 0.

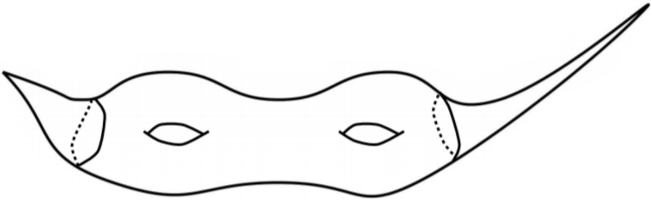
For that, we need analytic surgery. More precisely, we use Mayer-Vietoris formulas to relate the determinant on the whole modular curve to determinants on parts of it.



Spectral side

The last step is to understand the determinant of the Laplacian $\Delta_{\overline{\partial}_E,\varepsilon}$ as ε goes to 0.

For that, we need analytic surgery. More precisely, we use Mayer-Vietoris formulas to relate the determinant on the whole modular curve to determinants on parts of it.

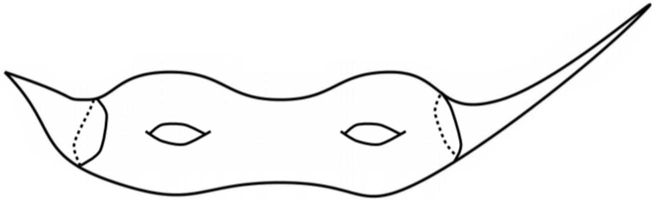


We need to chose boundary conditions on the smaller parts of the modular curve. The only ones that work are the Alvarez–Wentworth boundary conditions.

Spectral side

The last step is to understand the determinant of the Laplacian $\Delta_{\overline{\partial}_E,\varepsilon}$ as ε goes to 0.

For that, we need analytic surgery. More precisely, we use Mayer-Vietoris formulas to relate the determinant on the whole modular curve to determinants on parts of it.



We need to chose boundary conditions on the smaller parts of the modular curve. The only ones that work are the Alvarez–Wentworth boundary conditions. To complete the study, one needs the Selberg trace formula , and local computations around cusps.

Quillen metrics on modular curves

Taking the dominant terms as ε goes to 0 on both sides of the regularized ε-isometry, we can make the following definition.

Quillen metrics on modular curves

Taking the dominant terms as ε goes to 0 on both sides of the regularized ε-isometry, we can make the following definition.

Definition

The Quillen metric on the determinant line bundle $\lambda(E)$ is defined as

$$\|\cdot\|_Q = \left(C(\Gamma,\rho)\, Z^{(d)}(1,\Gamma,\rho)\right)^{-1/2} \|\cdot\|_{L^2} \quad ,$$

where d is the dimension of the kernel of the Laplacian $\Delta_{\overline{\partial}_E}$. The constant $C(\Gamma,\rho)$ is explicit in terms of the group and the representation.

Quillen metrics on modular curves

Taking the dominant terms as ε goes to 0 on both sides of the regularized ε-isometry, we can make the following definition.

Definition

The Quillen metric on the determinant line bundle $\lambda(E)$ is defined as

$$\|\cdot\|_Q \quad = \quad \left(C(\Gamma,\rho)\, Z^{(d)}(1,\Gamma,\rho)\right)^{-1/2} \|\cdot\|_{L^2} \quad ,$$

where d is the dimension of the kernel of the Laplacian $\Delta_{\overline{\partial}_E}$. The constant $C(\Gamma,\rho)$ is explicit in terms of the group and the representation.

This Quillen metric satisfies all three properties from the compact case :

1. Smoothness in family
2. Spectral interpretation
3. Riemann-Roch type theorem