Monopoles and difference modules

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Introduction

It is interesting to obtain a natural correspondence between objects in differential geometry and objects in algebraic geometry.

Theorem (rough statement)

Differential Geometry		*Algebraic Geometry*
Periodic monopoles	$\longleftrightarrow$	**Additive difference modules** (**Difference modules on** $\mathbb{C}$)
Doubly periodic monopoles	$\longleftrightarrow$	**Multiplicative difference modules** (q-**Difference modules, Difference modules on** $\mathbb{C}^*$)
Triply periodic monopoles	$\longleftrightarrow$	**Elliptic difference modules** (**Difference modules on elliptic curves**)

Monopoles

M : an oriented 3-dimensional Riemannian manifold

(E,h) : a vector bundle with a Hermitian metric on M

∇ : a unitary connection of (E,h)

ϕ : an anti-Hermitian endomorphism of E (called Higgs field)

Definition (E,h,∇,ϕ) is called *monopole* on M if

$$F(\nabla) = *\nabla\phi \qquad \textit{(Bogomolny equation)}.$$

Here, $*$ denote the Hodge star operator.

Let Γ be a discrete subgroup of $\mathbb{R}^3$. Set $\mathscr{M}_\Gamma := \mathbb{R}^3/\Gamma$ with $\sum dx_i dx_i$.
In this talk, we are interested in monopoles on $\mathscr{M}_\Gamma \setminus Z$ (Z: finite subset).

- *Periodic monopole* $\Longleftrightarrow \Gamma \simeq \mathbb{Z}$
- *Doubly periodic monopole* $\Longleftrightarrow \Gamma \simeq \mathbb{Z}^2$
- *Triply periodic monopole* $\Longleftrightarrow \Gamma \simeq \mathbb{Z}^3$.

Difference modules

Let R be a commutative algebra over $\mathbb{C}$. Let Φ^* be an automorphism of R, i.e., $\Phi^* : R \longrightarrow R$, $\mathbb{C}$-linear isomorphism, $\Phi^*(f_1 f_2) = \Phi^*(f_1)\Phi^*(f_2)$ $(\forall f_i \in R)$.

Definition *A difference module over (R, Φ^*)* **is an R-module $\boldsymbol{V}$ equipped with a $\mathbb{C}$-linear isomorphism $\Phi^*_{\boldsymbol{V}} : \boldsymbol{V} \longrightarrow \boldsymbol{V}$ such that**

$$\Phi^*_{\boldsymbol{V}}(fs) = \Phi^*(f)\Phi^*_{\boldsymbol{V}}(s) \quad (\forall f \in R, \, \forall s \in \boldsymbol{V}).$$

- *additive difference modules* $\Longleftrightarrow R = \mathbb{C}(y)$, $\Phi^*(f)(y) = f(y+\alpha)$ $(\alpha \in \mathbb{C})$

  Φ^* *is induced by the automorphism* $\Phi : \mathbb{C} \longrightarrow \mathbb{C}$, $\Phi(y) = y + \alpha$.

- *multiplicative difference modules* $\Longleftrightarrow R = \mathbb{C}(y)$, $\Phi^*(f)(y) = f(qy)$ $(q \in \mathbb{C}^*)$

  Φ^* *is induced by the automorphism* $\Phi : \mathbb{C}^* \longrightarrow \mathbb{C}^*$, $\Phi(y) = qy$.

- *elliptic difference modules* $\Longleftrightarrow R$ *is the field of meromorphic functions on an elliptic curve* C*, and* Φ^* *is induced by* $\Phi : C \longrightarrow C$, $\Phi(y) = y + \alpha$ $(\alpha \in C)$.

Theorem (rough statement)

Differential Geometry		*Algebraic Geometry*
Periodic monopoles	$\longleftrightarrow$	**Additive difference modules**
Doubly periodic monopoles	$\longleftrightarrow$	**Multiplicative difference modules**
Triply periodic monopoles	$\longleftrightarrow$	**Elliptic difference modules**

We need to impose the asymptotic condition to monopoles, and we should enhance difference modules with parabolic structure and stability condition.

- **Non-abelian Hodge theory for harmonic bundles on Riemann surfaces.**
  (Higgs bundles $\longleftrightarrow$ harmonic bundles $\longleftrightarrow$ flat bundles)
- **Classification of monopoles by algebraic data.**

Previous works on classification of monopoles

Donaldson, Hitchin

$SU(2)$-**monopoles on** $\mathbb{R}^3$ (**L^2-curvature**)	$\longleftrightarrow$	$\mathbb{P}^1 \longrightarrow \mathbb{P}^1$ **holomorphic**

Hurtubise, Murray, Jarvis

G-**monopoles on** $\mathbb{R}^3$	$\longleftrightarrow$	$\mathbb{P}^1 \longrightarrow$ flag varieties **holomorphic**

Let Σ be a compact Riemann surface.

Norbury

Singular monopoles on $\{0 \leq t \leq 1\} \times \Sigma$ (boundary condition)	$\longleftrightarrow$	**Holomorphic bundles on Σ with Hecke modifications**

(Recently, it was generalized to *the Higgs case* **by** *He-Walpuski***.)**

Charbonneau-Hurtubise

Singular monopoles on $S^1 \times \Sigma$	$\longleftrightarrow$	**Holomorphic bundles on Σ with a meromorphic auto. and Hecke modifications (stability condition)**

We recall more details of the theorem of Charbonneau-Hurtubise.

Review of the theorem of Charbonneau-Hurtubise

- $S^1 := \mathbb{R}/\mathbb{Z}$ with the standard metric $dt\,dt$.
- Σ: a compact Riemann surface with a Kähler metric.
- Z: a finite subset of $S^1 \times \Sigma$. (Assume $Z \cap (\{0\} \times \Sigma) = \emptyset$ for simplicity.)

We consider a monopole (E, h, ∇, ϕ) on $(S^1 \times \Sigma) \setminus Z$.

Condition Each $P \in Z$ is *Dirac type singularity* of (E, h, ∇, ϕ),
i.e., for a neighbourhood U_P of P in $S^1 \times \Sigma$,

$$(E, h, \nabla, \phi)_{|U_P \setminus \{P\}} \sim \left(\begin{array}{c} \text{a direct sum of} \\ \text{Dirac monopoles} \end{array} \right)$$

The induced differential operators

We obtain $\nabla^{0,1}_{|\Sigma} : E \longrightarrow E \otimes \Omega^{0,1}_{\Sigma}$ **induced by**

$$\nabla : E \longrightarrow E \otimes \left(\Omega^1_{S^1} \otimes \mathbb{C} \oplus \Omega^{0,1}_{\Sigma} \oplus \Omega^{1,0}_{\Sigma}\right).$$

We also set $\partial_t := \nabla_t - \sqrt{-1}\phi$**.**

Key lemma $[\partial_t, \nabla^{0,1}_{|\Sigma}] = 0$ *($\because$ Bogomolny equation)*

The induced holomorphic vector bundles

- We obtain the vector bundle $E^0 := E_{|\{0\}\times\Sigma}$ on Σ with the holomorphic structure $\nabla^{0,1}_{|\Sigma}$.
- More generally, for any $0 \le t \le 1$, we obtain the vector bundle $E^t := E_{|(\{t\}\times\Sigma)\setminus Z}$ with the holomorphic structure $\nabla^{0,1}_{|\Sigma}$ on $(\{t\}\times\Sigma)\setminus Z$.
- $E^1 = E^0$. (Recall $S^1 = \mathbb{R}/\mathbb{Z}$.)

Notation

- Let $\mathscr{E}^t$ denote the sheaf of holomorphic sections of $(E^t, \nabla^{0,1}_{|\Sigma})$.
- For a finite subset $S \subset \Sigma$, let $\mathscr{E}^t(*S)$ denote the sheaf of meromorphic sections of $\mathscr{E}^t$, which may have poles along S.

Scattering map (1)

Take $0 \leq t_1 < t_2 \leq 1$.

If $Z \cap (\{t_1 \leq t \leq t_2\} \times \Sigma) = \emptyset$, we obtain the isomorphism $F^{t_2,t_1} : E^{t_1} \simeq E^{t_2}$ as the parallel transport with respect to ∂_t.

> *Proposition* F^{t_2,t_1} **is** *holomorphic* **($\because [\partial_t, \nabla^{0,1}_{|\Sigma}] = 0$), i.e., $F^{t_2,t_1} : \mathscr{E}^{t_1} \simeq \mathscr{E}^{t_2}$.**

Scattering map (2)

Suppose $Z \cap (\{t_1 \leq t \leq t_2\} \times \Sigma) = Z \cap (\{t_0\} \times \Sigma) =: D_{t_0} \neq \emptyset$ $(t_1 < t_0 < t_2)$. **We obtain the holomorphic isomorphism** $F^{t_2,t_1} : E^{t_1}_{|\Sigma \setminus D_{t_0}} \simeq E^{t_2}_{|\Sigma \setminus D_{t_0}}$.

Proposition

F^{t_2,t_1} **is** *meromorphic* **at** D_{t_0}**, i.e.,** $F^{t_2,t_1} : \mathscr{E}^{t_1}(*D_{t_0}) \simeq \mathscr{E}^{t_2}(*D_{t_0})$.

(∵ Dirac type singularity)

For any $Q \in D_{t_0}$**, we obtain a** *Hecke modification*,

i.e., there are two lattices of the stalk $\mathscr{E}^{t_1}(*D)_Q \simeq \mathscr{E}^{t_2}(*D)_Q$

$$\mathscr{E}^{t_1}_Q \subset \mathscr{E}^{t_1}(*D)_Q \simeq \mathscr{E}^{t_2}(*D)_Q \supset \mathscr{E}^{t_2}_Q.$$

Algebraic data associated to monopoles on $S^1 \times \Sigma$

From (E, h, ∇, ϕ), we obtain $(\mathscr{E}, F, \{t_{Q,i}\}, \{\mathscr{L}_{Q,i}\})$.

- *a holomorphic vector bundle $\mathscr{E} := \mathscr{E}^0$ on Σ*
- *an automorphism F of $\mathscr{E}(*D)$* **by setting D as the image of Z by $S^1 \times \Sigma \longrightarrow \Sigma$:**

$$\mathscr{E}(*D) = \mathscr{E}^0(*D) \overset{F^{1,0}}{\simeq} \mathscr{E}^1(*D) = \mathscr{E}^0(*D) = \mathscr{E}(*D).$$

- *a sequence $0 \leq t_{Q,1} < \cdots < t_{Q,m(Q)} < 1$* **for $Q \in D$ by**

$$Z \cap (S^1 \times \{Q\}) = \{(t_{Q,i}, Q)\}.$$

- *lattices $\mathscr{L}_{Q,i}$ $(i = 0, \ldots, m(Q))$ of $\mathscr{E}(*D)_Q$*:
  We set $\mathscr{L}_{Q,0} = \mathscr{L}_{Q,m(Q)} := \mathscr{E}_Q$, and

$$\mathscr{L}_{Q,i} := \mathscr{E}^t_Q \subset \mathscr{E}^t(*D)_Q \simeq \mathscr{E}^0(*D)_Q = \mathscr{E}(*D)_Q \quad (t_{Q,i} < t < t_{Q,i+1})$$

Degree of subobjects of algebraic data

Suppose that $(\mathscr{E}, F, \{t_{Q,i}\}, \{\mathscr{L}_{Q,i}\})$ is given *(not necessarily induced by a monopole)*.
Let $\mathscr{E}' \subset \mathscr{E}$ be a non-zero holomorphic subbundle such that $F(\mathscr{E}'(*D)) = \mathscr{E}'(*D)$.
We obtain lattices $\mathscr{L}'_{Q,i}$ $(i = 0, \ldots, m(Q))$ of $\mathscr{E}'(*D)_Q$ by setting

$$\mathscr{L}'_{Q,i} := \mathscr{L}_{Q,i} \cap \mathscr{E}'(*D)_Q \quad \text{in } \mathscr{E}(*D)_Q.$$

Definition (degree)

$$\deg(\mathscr{E}'; F, \{t_{Q,i}\}, \{\mathscr{L}_{Q,i}\}) := \deg(\mathscr{E}') + \sum_{Q \in D} \sum_{i=1}^{m(Q)} (1 - t_{Q,i}) \deg\left(\mathscr{L}'_{Q,i}, \mathscr{L}'_{Q,i-1}\right)$$

Here, we put

$$\deg(\mathscr{L}'_{Q,i}, \mathscr{L}'_{Q,i-1}) := \dim_{\mathbb{C}}\left(\mathscr{L}'_{Q,i}/(\mathscr{L}'_{Q,i} \cap \mathscr{L}'_{Q,i-1})\right) - \dim_{\mathbb{C}}\left(\mathscr{L}'_{Q,i-1}/(\mathscr{L}'_{Q,i} \cap \mathscr{L}'_{Q,i-1})\right).$$

Remark **$\exists$ a naturally induced family of holomorphic vector bundles $(\mathscr{E}')^t$, and**

$$\deg(\mathscr{E}', F, \{t_{Q,i}\}, \{\mathscr{L}_{Q,i}\}) = \int_0^1 \deg(\mathscr{E}')^t \, dt.$$

Stability condition

Definition **Suppose that** $\deg(\mathscr{E};F,\{t_{Q,i}\},\{\mathscr{L}_{Q,i}\})=0$ **(for simplicity).**

- $(\mathscr{E},F,\{t_{Q,i}\},\{\mathscr{L}_{Q,i}\})$ **is** ***stable*** **if**

$$\deg(\mathscr{E}';F,\{t_{Q,i}\},\{\mathscr{L}_{Q,i}\})<0$$

for any non-zero subbundle $\mathscr{E}'\subsetneq\mathscr{E}$ **such that** $F(\mathscr{E}'(*D))=\mathscr{E}'(*D)$**.**

- $(\mathscr{E},F,\{t_{Q,i}\},\{\mathscr{L}_{Q,i}\})$ **is** ***polystable*** **if it is a direct sum of stable objects of degree** 0**, i.e.,**

$$(\mathscr{E},F,\{t_{Q,i}\},\{\mathscr{L}_{Q,i}\})=\bigoplus_j(\mathscr{E}_j,F_j,\{t_{Q,i}\},\{\mathscr{L}_{j,Q,i}\})$$

such that $(\mathscr{E}_j,F_j,\{t_{Q,i}\},\{\mathscr{L}_{j,Q,i}\})$ **are stable of degree** 0**.**

Theorem (Charbonneau-Hurtubise)

- **If $(\mathscr{E}, F, \{t_{Q,i}\}, \{\mathscr{L}_{Q,i}\})$ is induced by a monopole with Dirac singularity on $(S^1 \times \Sigma) \setminus Z$, then $(\mathscr{E}, F, \{t_{Q,i}\}, \{\mathscr{L}_{Q,i}\})$ is polystable of degree 0.**
- **The above correspondence induces an equivalence**

$$\left(\begin{array}{c} \textit{monopoles on } (S^1 \times \Sigma) \setminus Z \\ \textit{(Dirac type singularity)} \end{array} \right) \longleftrightarrow \left(\begin{array}{c} \textit{holomorphic vector bundles } \mathscr{E} \textit{ on } \Sigma \\ \textit{with an automorphism } F \textit{ at } D \\ \textit{and lattices } \{\mathscr{L}_{Q,i}\} \\ \textit{(polystable w.r.t. } \{t_{Q,i}\}_{Q \in D}\textit{)} \end{array} \right)$$

(D and $\{t_{Q,i}\}$ are determined by Z.)

Remark **Let $\mathfrak{K}(\Sigma)$ denote the field of meromorphic functions on Σ.**

$$\boldsymbol{V} = \{\textbf{meromorphic sections of } \mathscr{E} \textbf{ on } \Sigma\}$$

is naturally a finite dimensional $\mathfrak{K}(\Sigma)$-vector space with an automorphism F.
We may regard $(\boldsymbol{V}, F)$ as a difference module over $(\mathfrak{K}(\Sigma), \mathrm{id})$.
The tuple $(\mathscr{E}, \{t_{Q,i}\}, \{\mathscr{L}_{Q,i}\})$ is regarded as a parabolic structure of $(\boldsymbol{V}, F)$.

Equivalence for periodic monopoles (product case)

Periodic monopoles of GCK-type

Let Γ be a non-trivial discrete subgroup of $\mathbb{R}^3$ with $\Gamma \simeq \mathbb{Z}$. Let Z be a finite subset of $\mathscr{M}_\Gamma = (\mathbb{R}^3/\Gamma)$.

Definition **A monopole (E, h, ∇, ϕ) on $\mathscr{M}_\Gamma \setminus Z$ is called of** *GCK-type (generalized Cherkis-Kapustin type)* **if**

- **each $P \in Z$ is Dirac type singularity of (E, h, ∇, ϕ),**
- **$|\phi_P| = O\big(\log d(P, P_0)\big)$ and $|F(\nabla)_P| \longrightarrow 0$ as P goes to infinity.**

Remark *We can prove that a monopole of GCK type satisfies much stronger condition at infinity.*

Product case

Assume $\Gamma = \{(n,0)\,|\,n \in \mathbb{Z}\} \subset \mathbb{R} \times \mathbb{C} \simeq \mathbb{R}^3$ *(isometry)*.
We obtain *an isometry* $\mathscr{M}_\Gamma \simeq S^1 \times \mathbb{C}$.

First, we shall explain what kind of algebraic objects appear in this product case. For simplicity, we assume $Z \cap (\{0\} \times \mathbb{C}) = \emptyset$.

Remark *There are different isometries* $\mathbb{R}^3 \simeq \mathbb{R}_{t_0} \times \mathbb{C}_{\beta_0}$ *such that* $\Gamma \not\subset \mathbb{R} \times \{0\}$*, from which we obtain different equivalences between monopoles and algebraic objects (explained later).*

Preliminary

Everything goes similarly on $\mathbb{C}$.

- **We obtain the operators** $\partial_{E,t} = \nabla_t - \sqrt{-1}\phi$ **and** $\partial_{E,\overline{w}} = \nabla_{\overline{w}}$ **of** E.
- **For** $0 \le t \le 1$**, we obtain holomorphic vector bundles on** $(\{t\} \times \mathbb{C}) \setminus Z \subset \mathbb{C}$ **:**

$$\mathscr{E}^t = \left(E_{|(\{t\}\times\mathbb{C})\setminus Z}, \nabla_{\overline{w}}\right)$$

  In particular, we set $\mathscr{E} := \mathscr{E}^0 = \mathscr{E}^1$.

- **Let** D **denote the image of** Z **by the projection** $\mathscr{M}_\Gamma = S^1 \times \mathbb{C} \longrightarrow \mathbb{C}$. **Then,** $\partial_{E,t}$ **induces**

$$F : \mathscr{E}(*D) = \mathscr{E}^0(*D) \simeq \mathscr{E}^1(*D) = \mathscr{E}(*D)$$

- **We also obtain tuples of numbers** $0 \le t_{Q,1} < \cdots < t_{Q,m(Q)} < 1$ **and lattices** $\mathscr{L}_{Q,i}$ $(i = 0, \ldots, m(Q))$ **for** $Q \in D$.

Remark **However,** $(\mathscr{E}, F, \{t_{Q,i}\}, \{\mathscr{L}_{Q,i}\})$ **is** *transcendental* **object on** $\mathbb{C}$. **We would like to extend it to an algebraic object on** $\mathbb{P}^1$ **by using** h.

Acceptability

Theorem $(\mathscr{E}^t, h^t := h_{|\{t\}\times\mathbb{C}})$ **is** *acceptable***, i.e.,**

$$\Big|F(\nabla_{h^t})\Big|_{h^t} = O\Big(\frac{dw\,d\overline{w}}{|w|^2(\log|w|)^2}\Big)$$

Here, ∇_{h^t} **denotes the Chern connection of** $(\mathscr{E}^t, h^t)$**, and** $F(\nabla_{h^t})$ **denotes the curvature.**

Remark **We may apply a general theory to extend acceptable bundles on** $\mathbb{C}$ **to a filtered bundle on** $(\mathbb{P}, \infty)$ *(Cornalba-Griffiths, Simpson)***.**

Extension of acceptable bundles to filtered bundles

For any $a \in \mathbb{R}$, $\mathscr{E}^t$ on $\mathbb{C} \setminus D_t$ extends to the sheaf $\mathscr{P}_a\mathscr{E}^t$ on $\mathbb{P}^1 \setminus D_t$ as follows.

- For any neighbourhood $U \subset \mathbb{P}^1$ of ∞,

$$\mathscr{P}_a\mathscr{E}^t(U) = \left\{ s \in \mathscr{E}^t(U \setminus \{\infty\}) \;\middle|\; |s|_h = O(|w|^{a+\varepsilon}) \; \forall \varepsilon > 0 \right\}.$$

We obtain an increasing sequence of $\mathscr{O}_{\mathbb{P}^1 \setminus D_t}$-modules $\mathscr{P}_*\mathscr{E}^t = \left(\mathscr{P}_a\mathscr{E}^t \,|\, a \in \mathbb{R}\right)$.
We also set $\mathscr{P}\mathscr{E}^t = \bigcup_{a\in\mathbb{R}} \mathscr{P}_a\mathscr{E}^t$.

Theorem (Cornalba-Griffiths, Simpson) $\mathscr{P}_a\mathscr{E}^t$ are locally free $\mathscr{O}_{\mathbb{P}^1 \setminus D_t}$-modules.
($\because$ $(\mathscr{E}^t, h^t)$ is acceptable.)
Hence, $\mathscr{P}\mathscr{E}^t$ is a locally free $\mathscr{O}_{\mathbb{P}^1 \setminus D_t}(*\infty)$-module.

Remark This kind of increasing sequence $\mathscr{P}_*\mathscr{E}^t$ is called a filtered bundle on $(\mathbb{P}^1 \setminus D_t, \infty)$.

Lemma The automorphism F of $\mathscr{E}^0(*D)$ induces an automorphism F of $\mathscr{P}\mathscr{E}^0(*D)$. (But, not necessarily, $F(\mathscr{P}_a\mathscr{E}^0(*D)) \subset \mathscr{P}_a\mathscr{E}^0(*D)$.)

The associated difference module with parabolic structure in the product case

We obtain *a finite dimensional $\mathbb{C}(w)$-vector space $\boldsymbol{V}$*:

$$\boldsymbol{V} = H^0\big(\mathbb{P}^1, \mathscr{P}\mathscr{E}^0\big) \otimes_{\mathbb{C}[w]} \mathbb{C}(w).$$

It is equipped with *the $\mathbb{C}(w)$-linear automorphism F*. We regard $(\boldsymbol{V}, F)$ as a difference module on $(\mathbb{C}(w), \mathrm{id}_{\mathbb{C}(w)})$. It is equipped with *the parabolic structure*

- *a filtered bundle $\mathscr{P}_*\mathscr{E} = \mathscr{P}_*\mathscr{E}^0$ on $(\mathbb{P}^1, \infty)$*
- *a sequence $\{t_{Q,i}\}_{Q \in D}$*
- *lattices $\mathscr{L}_{Q,i}$ of $\mathscr{P}\mathscr{E}(*D)_Q$.*

Remark *We need to clarify the compatibility condition of F and $\mathscr{P}_*\mathscr{E}$ (similar to the case of wild harmonic bundles).*

Eigenvalues of F at ∞

We may regard the stalk $\mathscr{P}\mathscr{E}_\infty$ of the sheaf $\mathscr{P}\mathscr{E}$ at ∞ as a finite dimensional vector space over $\mathbb{C}(\{w^{-1}\})$.

$$\mathbb{C}(\{w^{-1}\}) = \{\text{convergent Laurent power series of } w^{-1}\} = \mathscr{O}_{\mathbb{P}^1}(*\infty)_\infty.$$

The vector space $\mathscr{P}\mathscr{E}_\infty$ is equipped with the $\mathbb{C}(\{w^{-1}\})$-linear automorphism F.

$$\mathrm{Sp}(F) := \{\text{eigenvalue of } F\}$$

Unramified case **If $\mathrm{Sp}(F) \subset \mathbb{C}(\{w^{-1}\})$, $\exists$ the generalized eigen decomposition:**

$$\mathscr{P}\mathscr{E}_\infty = \bigoplus_{\alpha \in \mathrm{Sp}(F)} \mathbb{E}_\alpha.$$

Each $\alpha \in \mathrm{Sp}(F)$ is expressed as

$$\alpha = w^{-\omega(\alpha)}\beta(\alpha)\Big(1 + \sum_{j=1}^{\infty} \gamma_j(\alpha) w^{-j}\Big) \quad (\omega(\alpha) \in \mathbb{Z},\ \beta(\alpha) \in \mathbb{C}^*,\ \gamma_j(\alpha) \in \mathbb{C}.)$$

The equivalence relation $\sim$ on $\mathrm{Sp}(F)$: $\alpha_1 \sim \alpha_2 \Leftrightarrow \omega(\alpha_1) = \omega(\alpha_2),\ \beta(\alpha_1) = \beta(\alpha_2)$.

For $[\alpha] \in \mathrm{Sp}(F)/\sim$, we define $\omega([\alpha]) := \omega(\alpha)$ and $\beta([\alpha]) := \beta(\alpha)$. We also set $\mathbb{E}_{[\alpha]} = \bigoplus_{\alpha_1 \sim \alpha} \mathbb{E}_{\alpha_1}$. We obtain the decomposition

$$\mathscr{P}\mathscr{E}_\infty = \bigoplus_{\mathrm{Sp}(F)/\sim} \mathbb{E}_{[\alpha]}.$$

Compatibility condition

- $\mathscr{P}_a\mathscr{E}_\infty = \bigoplus\Big(\mathscr{P}_a\mathscr{E}_\infty \cap \mathbb{E}_{[\alpha]}\Big)$ **for any $a \in \mathbb{R}$.**
- $\Big(w^{\omega([\alpha])}\beta([\alpha])^{-1}F - \mathrm{id}_{\mathbb{E}_{[\alpha]}}\Big)(\mathscr{P}_a\mathscr{E}_\infty \cap \mathbb{E}_{[\alpha]}) \subset w^{-1}\mathscr{P}_a\mathscr{E}_\infty \cap \mathbb{E}_{[\alpha]}$ **for any $a \in \mathbb{R}$.**

Ramified case

$\exists \ell$ such that

$$\mathrm{Sp}(F) \subset \mathbb{C}(\{w^{-1/\ell}\})$$

$\exists$ the generalized eigen decomposition:

$$\mathscr{P}^{(\ell)}\mathscr{E}_\infty := \mathscr{P}\mathscr{E}_\infty \otimes_{\mathbb{C}(\{w^{-1}\})} \mathbb{C}(\{w^{-1/\ell}\}) = \bigoplus_{\alpha \in \mathrm{Sp}(F)} \mathbb{E}_\alpha.$$

Each $\alpha \in \mathrm{Sp}(F)$ is expressed as

$$\alpha = w^{-\omega(\alpha)} \cdot \beta(\alpha) \cdot \left(1 + \sum_{j=1}^{\infty} \gamma_{j/\ell}(\alpha) w^{-j/\ell}\right) \quad (\omega(\alpha) \in \mathbb{Q},\ \beta(\alpha) \in \mathbb{C}^*,\ \gamma_{j/\ell}(\alpha) \in \mathbb{C}.)$$

We define the equivalence relation on $\mathrm{Sp}(F)$ by

$$\alpha_1 \sim \alpha_2 \Longleftrightarrow \omega(\alpha_1) = \omega(\alpha_2),\ \beta(\alpha_1) = \beta(\alpha_2),\ \gamma_{j/\ell}(\alpha_1) = \gamma_{j/\ell}(\alpha_2)\ (1 \leq j < \ell).$$

For $[\alpha] \in \mathrm{Sp}(F)/\sim$, we define $\omega([\alpha]) := \omega(\alpha)$, $\beta([\alpha]) := \beta(\alpha)$ and $\gamma_{j/\ell}([\alpha]) := \gamma_{j/\ell}(\alpha)$ $(1 \leq j < \ell)$. We set $\mathbb{E}_{[\alpha]} = \bigoplus_{\alpha_1 \sim \alpha} \mathbb{E}_\alpha$. We obtain the decomposition

$$\mathscr{P}^{(\ell)}\mathscr{E}_\infty = \bigoplus_{\mathrm{Sp}(F)/\sim} \mathbb{E}_{[\alpha]}.$$

There exists the natural filtration of $\mathscr{P}^{(\ell)}\mathscr{E}_\infty$:

$$\mathscr{P}^{(\ell)}_a\mathscr{E}_\infty := \sum_{\ell b+n\leq a} w^{-n/\ell}\mathscr{P}_b\mathscr{E}_\infty\otimes_{\mathbb{C}\{w^{-1}\}}\mathbb{C}\{w^{-1/\ell}\}$$

Here, $\mathbb{C}\{w^{-1}\}$ denotes the ring of the convergent power series of w^{-1}.

Compatibility condition

- $\mathscr{P}^{(\ell)}_a\mathscr{E}_\infty = \bigoplus\Big(\mathscr{P}^{(\ell)}_a\mathscr{E}_\infty\cap\mathbb{E}_{[\alpha]}\Big)$ **for any** $a\in\mathbb{R}$.
- $\Big(w^{\omega([\alpha])}\beta([\alpha])^{-1}F-(1+\sum_{j=1}^{\ell-1}\gamma_{j/\ell}([\alpha])w^{-j/\ell})\,\mathrm{id}_{\mathbb{E}_{[\alpha]}}\Big)\mathscr{P}^{(\ell)}_a\mathscr{E}_\infty\cap\mathbb{E}_{[\alpha]}\subset w^{-1}\mathscr{P}^{(\ell)}_a\mathscr{E}_\infty\cap\mathbb{E}_{[\alpha]}$ **for any** $a\in\mathbb{R}$.

Remark *This type of compatibility condition is standard in the study of wild harmonic bundles, and it should be useful for the classification.*

Degree and stability condition

Let $0 \neq \boldsymbol{V}' \subset \boldsymbol{V}$ be a $\mathbb{C}(w)$-subspace such that $F(\boldsymbol{V}') = \boldsymbol{V}'$.

$\mathscr{O}_{\mathbb{P}^1}(*\infty)$-submodule $\mathscr{P}\mathscr{E}' \subset \mathscr{P}\mathscr{E}$ **such that** $H^0(\mathbb{P}^1, \mathscr{P}\mathscr{E}') = \boldsymbol{V}' \cap H^0(\mathbb{P}^1, \mathscr{P}\mathscr{E})$.

lattices $\mathscr{L}'_{Q,i} = \mathscr{P}\mathscr{E}'(*D)_Q \cap \mathscr{L}_{Q,i} \quad (Q \in D,\ 0 \leq i \leq m(Q))$.

filtration $\mathscr{P}_a\mathscr{E}' = \mathscr{P}_a\mathscr{E} \cap \mathscr{P}\mathscr{E}'$.

decomposition $\mathscr{P}^{(\ell)}\mathscr{E}'_\infty = \bigoplus_{[\alpha]\in \mathrm{Sp}(F)} \Big(\mathbb{E}_{[\alpha]} \cap \mathscr{P}^{(\ell)}\mathscr{E}'_\infty\Big)$.

Definition

$$\deg\big(\boldsymbol{V}'; \mathscr{P}_*\mathscr{E}, F, \{t_{Q,i}\}, \{\mathscr{L}_{Q,i}\}\big) := \deg(\mathscr{P}_0\mathscr{E}') - \sum_{-1<a\leq 0} a \dim_{\mathbb{C}}\Big(\mathscr{P}_a\mathscr{E}'/\mathscr{P}_{<a}\mathscr{E}'\Big)$$
$$+ \sum_{Q\in D}\sum_{i=1}^{m(Q)} (1-t_i)\deg(\mathscr{L}'_{Q,i}, \mathscr{L}'_{Q,i-1}) + \sum_{[\alpha]\in \mathrm{Sp}(F)/\sim} \frac{\omega([\alpha])}{2} \mathrm{rank}\big(\mathscr{P}^{(e)}\mathscr{E}' \cap \mathbb{E}_{[\alpha]}\big) \quad (1)$$

We define *stability* **and** *polystability* **conditions for** $(\boldsymbol{V}, F; \mathscr{P}_*\mathscr{E}, \{t_{Q,i}\}, \{\mathscr{L}_{Q,i}\})$ **by using the degree in the standard way.**

Equivalence in the product case

Theorem

- **If $(\boldsymbol{V}, F; \mathscr{P}_*\mathscr{E}, \{t_{Q,i}\}, \{\mathscr{L}_{Q,i}\})$ is induced by a monopole of GCK-type on $\mathscr{M}_\Gamma \setminus Z$, then the compatibility condition is satisfied, and $(\boldsymbol{V}, F; \mathscr{P}_*\mathscr{E}, \{t_{Q,i}\}, \{\mathscr{L}_{Q,i}\})$ is polystable of degree 0.**
- **This correspondence induces an equivalence**

$$\left(\begin{array}{c} \textbf{Singular monopoles} \\ \textbf{on } \mathscr{M}_\Gamma \textbf{ of GCK-type} \end{array} \right) \longleftrightarrow \left(\begin{array}{c} \textbf{Difference modules over } (\mathbb{C}(w), \mathrm{id}) \\ \textbf{with parabolic structure} \\ \textbf{(compatible, polystable, degree } 0\textbf{)} \end{array} \right)$$

Remark *It can be generalized from $S^1 \times \mathbb{C}$ to $S^1 \times (\Sigma \setminus S)$ such that $\Sigma \setminus S$ around Q $(Q \in S)$ are isometric to $\{w \in \mathbb{C} \,\big|\, |w| > R\}$.*

Example 1

Take a finite set $S \subset \mathbb{C}$ and $\ell : S \longrightarrow \mathbb{Z}_{>0}$. Assume $\exists a_0 \in S$ such that $\ell(a_0)$ odd.
Consider $P(y) = \prod_{a \in S}(y-a)^{\ell(a)} \in \mathbb{C}(y)$.
We set $\boldsymbol{V} := \mathbb{C}(y)e_1 \oplus \mathbb{C}(y)e_2$ with a $\mathbb{C}(y)$-linear automorphism $\Phi^*_{\boldsymbol{V}}$:

$$\Phi^*_{\boldsymbol{V}}(e_1,e_2) = (e_1,e_2)\begin{pmatrix} 0 & P(y) \\ 1 & 0 \end{pmatrix}$$

Let $\mathscr{P}\mathscr{E}$ be the locally free $\mathscr{O}_{\mathbb{P}^1}(*\infty)$-module induced by $\mathbb{C}[y]e_1 \oplus \mathbb{C}[y]e_2$.
Take any $(t_a)_{a \in S} \in \{0 \leq x < 1\}^S$. Set $Z := \{(t_a, a) \mid a \in S\} \subset S^1 \times \mathbb{C}$.

Proposition

$\deg_y(P)$ **even:** **Monopoles of GCK-type on $(S^1 \times \mathbb{C}) \setminus Z$ inducing $(\boldsymbol{V}, \Phi^*_{\boldsymbol{V}}, \mathscr{P}\mathscr{E})$ are naturally parameterized by $\mathbb{R}$.**

$\deg_y(P)$ **odd:** **There uniquely exists a monopole of GCK-type on $(S^1 \times \mathbb{C}) \setminus Z$ which induces $(\boldsymbol{V}, \Phi^*_{\boldsymbol{V}}, \mathscr{P}\mathscr{E})$.**

If V' is a $\mathbb{C}(y)$-subspace of V such that $\Phi_V^*(V') = V'$, then V' is V or 0. Hence, the stability condition is trivially satisfied in this case.

It is enough to classify $\mathscr{P}_*\mathscr{E}$ over $\mathscr{P}\mathscr{E}$ satisfying *the compatibility condition* **with Φ_V^* and** *the degree 0 condition* *(an easy algebraic problem)*.

- **If $\deg(P)$ is even, $(\mathscr{P}\mathscr{E}_\infty, \Phi_V^*)$ is unramified.** *The compatibility condition* **implies**

$$\mathscr{P}_*\mathscr{E}_\infty = (\mathscr{P}_*\mathscr{E}_\infty \cap \mathbb{E}_{[\alpha_1]}) \oplus (\mathscr{P}_*\mathscr{E}_\infty \cap \mathbb{E}_{[\alpha_2]}).$$

  The filtrations $(\mathscr{P}_*\mathscr{E}_\infty \cap \mathbb{E}_{[\alpha_i]})$ are determined by numbers d_i $(i = 1, 2)$. *The degree 0 condition* **implies $d_1 + d_2 + \sum_{a \in S}(1 - t_a)\ell(a)$. (We choose appropriate frames of $\mathbb{E}_{[\alpha_i]}$.)**

- **If $\deg(P)$ is odd, $(\mathscr{P}\mathscr{E}_\infty, \Phi_V^*)$ is ramified at infinity.** *The compatibility condition* **implies**

$$\mathscr{P}_*^{(2)}\mathscr{E}_\infty = (\mathscr{P}_*^{(2)}\mathscr{E}_\infty \cap \mathbb{E}_{[\alpha]}) \oplus (\mathscr{P}_*^{(2)}\mathscr{E}_\infty \cap \mathbb{E}_{[-\alpha]}).$$

  By the Galois action, the filtrations $\mathscr{P}_*^{(2)}\mathscr{E}_\infty \cap \mathbb{E}_{[\alpha]}$ and $\mathscr{P}_*^{(2)}\mathscr{E}_\infty \cap \mathbb{E}_{[-\alpha]}$ are determined by a number d. By *the degree 0 condition*, **d is uniquely determined.**

Example 2

Take a polynomial $Q(y) \in \mathbb{C}[y]$. Consider $\boldsymbol{V} = \mathbb{C}(y)e_1 \oplus \mathbb{C}(y)e_2$ with the automorphism

$$\Phi^*(e_1, e_2) = (e_1, e_2) \begin{pmatrix} 0 & 1 \\ -1 & Q \end{pmatrix}.$$

Let $\mathscr{P}\mathscr{E}$ be the $\mathscr{O}_{\mathbb{P}^1}(*\infty)$-module induced by $\mathbb{C}[y]e_1 \oplus \mathbb{C}[e_2]$.

Proposition

Monopoles of GCK-type on $S^1 \times \mathbb{C}$ inducing $(\boldsymbol{V}, \Phi^*, \mathscr{P}\mathscr{E})$ are naturally parameterized by $\mathbb{R}$.

Equivalence for periodic monopoles (non-product case)

We explained the case $\Gamma \subset \mathbb{R} \times \{0\} \subset \mathbb{R} \times \mathbb{C}$ under $\mathbb{R}^3 \simeq \mathbb{R} \times \mathbb{C}$. There are many isometry $\mathbb{R}^3 \simeq \mathbb{R} \times \mathbb{C}$ (parameterized by $\mathbb{P}^1$).
It is natural to expect to obtain additive difference modules in the case $\Gamma \not\subset \mathbb{R} \times \{0\}$.

A coordinate system

Let λ be any complex number.

We introduce a coordinate system (t_0,β_0) on $\mathbb{R}_t\times\mathbb{C}_w$:

$$(t_0,\beta_0)=\frac{1}{1+|\lambda|^2}\Big((1-|\lambda|^2)t+2\,\mathrm{Im}(\lambda\overline{w}),\; w+2\sqrt{-1}\lambda t+\lambda^2\overline{w}\Big)\in\mathbb{R}\times\mathbb{C}.$$

- $dt_0\,dt_0+d\beta_0\,d\overline{\beta}_0=dt\,dt+dw\,d\overline{w}$.
- Γ **is described as**

$$\Gamma=\left\{\frac{n}{1+|\lambda|^2}\Big(1-|\lambda|^2,2\sqrt{-1}\lambda\Big)\,\Big|\,n\in\mathbb{Z}\right\}$$

- **We set $\partial_{E,t_0}:=\nabla_{t_0}-\sqrt{-1}\phi$ and $\partial_{E,\overline{\beta}_0}:=\nabla_{\overline{\beta}_0}$. Then, $[\partial_{E,t_0},\partial_{E,\overline{\beta}_0}]=0$.**
- **We obtain the holomorphic vector bundles $(E_{|(\{t_0\}\times\mathbb{C}_{\beta_0})\setminus Z},\nabla_{\overline{\beta}_0})$. There exist meromorphic isomorphisms**

$$(E_{|\{t_0\}\times(\mathbb{C}_{\beta_0}\setminus D(t_0,t_0'))},\nabla_{\overline{\beta}_0})\simeq(E_{|\{t_0'\}\times(\mathbb{C}_{\beta_0}\setminus D(t_0,t_0'))},\nabla_{\overline{\beta}_0})\quad(\exists D(t_0,t_0')\subset\mathbb{C},\textbf{finite})$$

If $|\lambda| \neq 1$, we set

$$T(\lambda) := \frac{1-|\lambda|^2}{1+|\lambda|^2}.$$

∂_{E,t_0} **induces a meromorphic isomorphism**

$$E_{|\{0\}\times\mathbb{C}_{\beta_0}}(*D) \simeq E_{|\{T(\lambda)\}\times\mathbb{C}_{\beta_0}}(*D). \quad (\exists D \subset \mathbb{C}, \textbf{ finite})$$

For the automorphism $\Phi_0 : \mathbb{C} \longrightarrow \mathbb{C}$ defined by

$$\Phi_0(\beta_0) = \beta_0 + \frac{2\sqrt{-1}\lambda}{1+|\lambda|^2},$$

we have the natural identification $\Phi_0^* E_{|\{T(\lambda)\}\times\mathbb{C}} = E_{|\{0\}\times\mathbb{C}}$.

Remark **It is natural to expect to obtain difference modules by using these isomorphisms (it could be done in some cases), but....**

- *we do not obtain a difference module in the case $|\lambda| = 1$, i.e., $T(\lambda) = 0$.*
- *in general, $(E_{|\{t_0\}\times\mathbb{C}_{\beta_0}}, \nabla_{\overline{\beta}_0})$ with $h_{|\{t_0\}\times\mathbb{C}_{\beta_0}}$ is not acceptable. It is not clear how to extend $(E_{|\{t_0\}\times\mathbb{C}_{\beta_0}}, \nabla_{\overline{\beta}_0})$ to a meromorphic object on $\mathbb{P}^1$.*

Another coordinate system

We introduce another coordinate system (t_1, β_1):

$$(t_1, \beta_1) = \left(t_0 + \mathrm{Im}(\overline{\lambda}\beta_0),\ (1 + |\lambda|^2)\beta_0\right) = \left(t + \mathrm{Im}(\lambda\overline{w}),\ w + 2\sqrt{-1}\lambda t + \lambda^2\overline{w}\right).$$

Γ is described as $\Gamma = \left\{ n \cdot (1,\ 2\sqrt{-1}\lambda) \,\middle|\, n \in \mathbb{Z} \right\}$.

> *Remark* $\mathbb{R}_{t_1} \times \{0\}$ **and** $\{0\} \times \mathbb{C}_{\beta_1}$ **are** *not orthogonal* **if** $\lambda \neq 0$.

Note that

$$\partial_{t_1} = \partial_{t_0}, \quad \partial_{\overline{\beta}_1} = \frac{\lambda}{1 + |\lambda|^2} \frac{1}{2\sqrt{-1}} \partial_{t_0} + \frac{1}{1 + |\lambda|^2} \partial_{\overline{\beta}_0}.$$

> *Lemma* **We define the differential operators acting on E:**
>
> $$\partial_{E,t_1} := \partial_{E,t_0}, \quad \partial_{E,\overline{\beta}_1} := \frac{\lambda}{1 + |\lambda|^2} \frac{1}{2\sqrt{-1}} \partial_{E,t_0} + \frac{1}{1 + |\lambda|^2} \partial_{E,\overline{\beta}_0}.$$
>
> **Then, ∂_{E,t_1} and $\partial_{E,\overline{\beta}_1}$ are** *commutative*.

Remark **It is more systematic to consider** *mini-holomorphic bundles* **on** *mini-complex manifolds*. **(t_0, β_0) and (t_1, β_1) determines the same mini-complex structure.**

Theorem **The holomorphic bundle** $\mathscr{E}^{t_1} = \left(E_{|\{t_1\}\times\mathbb{C}_{\beta_1}}, \partial_{E,\overline{\beta}_1}\right)$ **with the metric** $h_{|\{t_1\}\times\mathbb{C}_{\beta_1}}$ **is** *acceptable*. **In particular,** *it extends to a filtered bundle* $\mathscr{P}_*\mathscr{E}^{t_1}$ *on* $(\mathbb{P}^1, \{\infty\})$.

We obtain the meromorphic isomorphism induced by ∂_{E,t_1}.

$$\mathscr{P}\mathscr{E}^0(*D) \simeq \mathscr{P}\mathscr{E}^1(*D) \quad (\exists D \subset \mathbb{C} \text{ finite})$$

For the automorphism $\Phi_1 : \mathbb{C} \longrightarrow \mathbb{C}$ **defined by** $\Phi_1(\beta_1) = \beta_1 + 2\sqrt{-1}\lambda$, **we have**

$$\Phi_1^*\left(\mathscr{P}_*\mathscr{E}^1\right) = \mathscr{P}_*\mathscr{E}^0.$$

- $\boldsymbol{V} := H^0(\mathbb{P}^1, \mathscr{P}\mathscr{E}^0) \otimes_{\mathbb{C}[\beta_1]} \mathbb{C}(\beta_1)$.
- **The above two isomorphisms induce a** $\mathbb{C}$**-linear automorphism** $\Phi_{\boldsymbol{V}}^*$ **on** $\boldsymbol{V}$, **and** $(\boldsymbol{V}, \Phi_{\boldsymbol{V}}^*)$ **is a difference module over** $(\mathbb{C}(\beta_1), \Phi_1^*)$.
- **The singularity at** Z **and the filtered bundle** $\mathscr{P}_*\mathscr{E}^0$ **determine a parabolic structure on this difference module** $\boldsymbol{V}$.

Remark **To formulate** *a compatibility condition* **of** $\Phi_{\boldsymbol{V}}^*$ **and the filtration** $\mathscr{P}_*\mathscr{E}^0$, **we can use** *the classification of formal difference modules* **due to** *Turrittin*.

Equivalence in the non-product case

The degree **and** *the stability condition* **for** $(\boldsymbol{V}, F, \mathscr{P}_*\mathscr{E}, \{t_{Q,i}\}, \{\mathscr{L}_{Q,i}\})$ **are defined as before.**

Theorem

$$\left(\begin{array}{c} \textbf{Singular monopoles} \\ \textbf{on } \mathscr{M}_\Gamma \textbf{ of GCK-type} \end{array} \right) \longleftrightarrow \left(\begin{array}{c} \textbf{Difference modules over } (\mathbb{C}(\beta_1), \Phi_1^*) \\ \textbf{with parabolic structure} \\ \textbf{(compatible, polystable, degree } 0\textbf{)} \end{array} \right)$$

Asymptotic behaviour of periodic monopoles of GCK-type

We set $U(R) := \{|w| > R\}$.
Let (E, h, ∇, ϕ) be a monopole on $S^1 \times U(R)$ satisfying the GCK-condition.
For any positive integer ℓ, let $\varphi_\ell : S^1 \times U(R)_\ell \longrightarrow S^1 \times U(R)$ be the covering induced by $w^{1/\ell} \longmapsto (w^{1/\ell})^\ell$.

Theorem **For an appropriate positive integer ℓ,**

$$\varphi_\ell^{-1}(E, h, \nabla, \phi) \sim \bigoplus_i (E_{n_i,\ell}, h_{n_i,\ell}, \nabla_{n_i,\ell}, \phi_{n_i,\ell}) \otimes \mathrm{Hit}_2^3(V_i, \overline{\partial}_{V_i}, \theta_{V_i}, h_{V_i}).$$

Remark $(E_{n_i}, h_{n_i}, \nabla_{n_i}, \phi_{n_i})$ **and** $\mathrm{Hit}_2^3(V_i, \overline{\partial}_{V_i}, \theta_{V_i}, h_{V_i})$ **are almost determined by** $\mathscr{P}_*\mathscr{E}_\infty$ **with the induced difference operator** Φ_1^***.**

Typical examples (1)

By $w = re^{\sqrt{-1}\theta}$, we obtain the isometry (set $S^1_{2\pi} := \mathbb{R}/2\pi\mathbb{Z}$):

$$S^1 \times (\mathbb{C} \setminus \{0\}) \simeq (S^1_{2\pi} \times S^1 \times \mathbb{R}_{>0}, r^2 d\theta\, d\theta + dt\, dt + dr\, dr), \quad (t, w) \longmapsto (\theta, t, r)$$

A line bundle L_n on $S^1_{2\pi} \times S^1$ with $c_1(L) = n$ has a Hermitian metric h_{L_n} and a unitary connection ∇_{L_n} such that $F(\nabla_{L_n}) = -n\sqrt{-1}d\theta\, dt$.

Let $p : S^1_{2\pi} \times S^1 \times \mathbb{R}_{>0} \longrightarrow S^1_{2\pi} \times S^1$ be the projection. We set

$$(E_n, h_n, \nabla_n) := p^*(L_n, h_{L_n}, \nabla_{L_n}).$$

Let ϕ_n be the Higgs field defined by $\phi_n = -n\sqrt{-1}\log r$.

- **$(E_n, h_n, \nabla_n, \phi_n)$ is a monopole on $S^1_{2\pi} \times S^1 \times \mathbb{R}_{>0}$ satisfying the GCK-condition at infinity.**
- **We can compute $(\mathscr{P}_*\mathscr{E}_\infty, \Phi_1^*)$ explicitly.**
  (For example, if $\lambda = 0$, the induced automorphism F is the multiplication of βw^n ($|\beta| = 1$), where β depends on the choice of ∇_{L_n}.)

Similarly, by setting $S_{2\pi\ell} = \mathbb{R}/(2\pi\ell\mathbb{Z})$, let $L_{n,\ell}$ be a line bundle on $S^1_{2\pi\ell} \times S^1$ with a metric $h_{L_{n,\ell}}$ and a unitary connection $\nabla_{L_{n,\ell}}$ such that $F(\nabla_{L_{n,\ell}}) = -\frac{n}{\ell}\sqrt{-1}d\theta\, dt$.

Let $p_\ell : S^1_{2\pi\ell} \times S^1 \times \mathbb{R}_{>0} \longrightarrow S^1_{2\pi\ell} \times S^1$ be the projection. We set

$$(E_{n,\ell}, h_{n,\ell}, \nabla_{n,\ell}) = p_\ell^*(L_{n,\ell}, h_{n,\ell}, \nabla_{n,\ell}).$$

Let $\phi_{n,\ell}$ be the Higgs field defined by $\phi_{n,\ell} = -\frac{n}{\ell}\sqrt{-1}\log r$.

- **$(E_{n,\ell}, h_{n,\ell}, \nabla_{n,\ell}, \phi_{n,\ell})$ is a monopole on $S^1_{2\pi\ell} \times S^1 \times \mathbb{R}_{>0}$ satisfying the GCK condition at infinity.**
- **We can calculate the associated algebraic objects.**

Typical examples (2)

Let $(V, \overline{\partial}_V, \theta, h_V)$ be a harmonic bundle on $U(R)$, i.e., $(V, \overline{\partial}_V)$ is a holomorphic vector bundle, $\theta = f\,dw \in \mathrm{End}(V) \otimes \Omega^1$, and h_V is a Hermitian metric of V, satisfying the Hitchin equation

$$F(\nabla_{h_V}) + [\theta, \theta^\dagger_{h_V}] = 0.$$

Let $p_w : S^1 \times U(R) \longrightarrow U(R)$ be the projection. We obtain the vector bundle with a Hermitian metric $(E, h) = p_w^{-1}(V, h_V)$ with the connection and the Higgs field

$$\nabla = p_w^*(\nabla_h) - \sqrt{-1} p_w^*(f + f_h^\dagger) dt, \quad \phi = p_w^*(f - f_h^\dagger).$$

- $\mathrm{Hit}_2^3(V, \overline{\partial}_V, h_V, \theta) := (E, h, \nabla, \phi)$ **is a monopole on $S^1 \times U(R)$.**
- $\mathrm{Hit}_2^3(V, \overline{\partial}_V, h_V, \theta)$ **satisfies the desired asymptotic condition if and only if the eigenvalues of f are bounded.**
- **We can compute the associated holomorphic objects explicitly. (For example, if $\lambda = 0$, the induced automorphism F is $\exp(2f)$.)**

More generally, let $U(R)_\ell \longrightarrow U(R)$ be the ℓ-th covering map induced by $w^{1/\ell} \longmapsto (w^{1/\ell})^\ell$.

- **A harmonic bundle $(E, \overline{\partial}_E, \theta, h)$ on $U(R)_\ell$ induces a monopole $\mathrm{Hit}_2^3(E, \overline{\partial}_E, \theta, h)$ on $S^1 \times U(R)_\ell$.**
- **Let f be determined by $\theta = f\,dw = f\,d\big((w^{1/\ell})^\ell\big)$.**

  $\mathrm{Hit}_2^3(E, \overline{\partial}_E, \theta, h)$ *satisfies the GCK-condition at infinity if and only if the eigenvalues of f are bounded.*

Approximation

Let (E,h,∇,ϕ) be a monopole on $S^1\times U(R)$ satisfying the GCK-condition. Let $\varphi_\ell : S^1\times U(R)_\ell \longrightarrow S^1\times U(R)$ be the covering induced by $w^{1/\ell}\longmapsto (w^{1/\ell})^\ell$.

Theorem **For an appropriate positive integer ℓ,**

$$\varphi_\ell^{-1}(E,h,\nabla,\phi)\sim\bigoplus_i (E_{n_i,\ell},h_{n_i,\ell},\nabla_{n_i,\ell},\phi_{n_i,\ell})\otimes \mathrm{Hit}_2^3(V_i,\overline{\partial}_{V_i},\theta_{V_i},h_{V_i}).$$

Corollary **For $F(\nabla)=F(\nabla)_{w\overline{w}}dw\,d\overline{w}+F(\nabla)_{w,t}dw\,dt+F(\nabla)_{\overline{w},t}d\overline{w}\,dt$, we obtain the stronger curvature decay**

$$|F(\nabla)_{w\overline{w}}|_h=O\Big(|w|^{-2}(\log|w|)^{-2}\Big),$$

$$|F(\nabla)_{wt}|_h=O\big(|w|^{-1}\big),$$

$$|F(\nabla)_{\overline{w}t}|_h=O\big(|w|^{-1}\big).$$

The doubly periodic case and the triply periodic case

Doubly periodic case $\Gamma \subset \{0\} \times \mathbb{C} \subset \mathbb{R} \times \mathbb{C}$ **such that** $\operatorname{rank}\Gamma = 2$. **Take any complex number** λ. **Take a generator** $\mu_1, \mu_2 \in \Gamma$ **such that (i)** $\lambda \neq \pm\sqrt{-1}\mu_1|\mu_1|^{-1}$, **(ii)** $\operatorname{Im}(\mu_2/\mu_1) > 0$. **We set**

$$\mathbf{q}^{\lambda} := \exp\Big(2\pi\sqrt{-1}\,\frac{\mu_2 + \lambda^2\overline{\mu}_2}{\mu_1 + \lambda^2\overline{\mu}_1}\Big).$$

Theorem **There exists an equivalence between** *monopoles on* $\mathscr{M}_{\Gamma}$ *with finite Dirac type singularity satisfying an asymptotic condition at infinity* **and** *multiplicative difference modules with parabolic structure (compatible, polystable, degree* 0*)*.
(The action $\mathbb{C}^* \longrightarrow \mathbb{C}^*$ **is induced by** $y \longmapsto \mathbf{q}^{\lambda} y$**)**

Triply periodic case **Suppose** $\operatorname{rank}\Gamma = 3$. **We take a generator** $e_i = (a_i, \alpha_i)$ **of** $\Gamma \subset \mathbb{R} \times \mathbb{C}$ **such that (i) the frame** e_1, e_2, e_3 **is compatible with the orientation, (ii)** α_1, α_2 **generates a lattice** $\mathbb{C}$, **(iii)** $\operatorname{Im}(\alpha_2/\alpha_1) > 0$. **We set** $C = \mathbb{C}/\mathbb{Z}\langle\alpha_1, \alpha_2\rangle$.

Theorem (essentially Charbonneau-Hurtubise, Kontsevich-Soibelman) **There exists an equivalence between** *monopoles of* $(\mathbb{R} \times \mathbb{C}/\Gamma)$ *with finite Dirac type singularity* **and** *difference modules on* C *with parabolic structure (polystable, degree* 0*)*.
(The action $C \longrightarrow C$ **is induced by** $z \longmapsto z + \alpha_3$.**)**