Homology of Arithmetic Groups and Galois Representations

Banff Workshop on cohomology of arithmetic groups: duality, stability, and computations

Avner Ash

Boston College

October 11, 2021

Basic objects 1

k a field, either characteristic 0 or finite of characteristic p.

Basic objects 1

k a field, either characteristic 0 or finite of characteristic p.

Homology:

- $G = \mathrm{GL}_n / \mathbb{Q}$ (a "reductive algebraic group")

Basic objects 1

k a field, either characteristic 0 or finite of characteristic p.

Homology:

- $G = \mathrm{GL}_n/\mathbb{Q}$ (a "reductive algebraic group")
- $\Gamma \subset G(\mathbb{Z})$ of finite index (an "arithmetic group")

Basic objects 1

k a field, either characteristic 0 or finite of characteristic p.

Homology:

- $G = \mathrm{GL}_n / \mathbb{Q}$ (a "reductive algebraic group")
- $\Gamma \subset G(\mathbb{Z})$ of finite index (an "arithmetic group")
- Example: $\Gamma_0(N) = \{g \in G(\mathbb{Z}) \mid \det(g) = 1,\ ge_1 \equiv g_{11}e_1 \mod N\}$
  N is the "level".

Basic objects 1

k a field, either characteristic 0 or finite of characteristic p.

Homology:

- $G = \mathrm{GL}_n/\mathbb{Q}$ (a "reductive algebraic group")
- $\Gamma \subset G(\mathbb{Z})$ of finite index (an "arithmetic group")
- Example: $\Gamma_0(N) = \{g \in G(\mathbb{Z}) \mid \det(g) = 1,\ ge_1 \equiv g_{11}e_1 \mod N\}$
  N is the "level".
- $\epsilon : (\mathbb{Z}/N\mathbb{Z}^\times) \to k^\times$ is a nebentype character.
  $\epsilon(g) = \epsilon(g_{11})$

Basic objects 2

Galois:

- $G_{\mathbb{Q}}$ = Galois group of $\bar{\mathbb{Q}}/\mathbb{Q}$.

Basic objects 2

Galois:

- $G_{\mathbb{Q}}$ = Galois group of $\bar{\mathbb{Q}}/\mathbb{Q}$.
- A "Galois representation" $\rho : G_{\mathbb{Q}} \to GL_n(k)$ is a continuous semisimple homomorphism unramified outside a finite set of primes.

Basic objects 2

Galois:

- $G_{\mathbb{Q}} =$ Galois group of $\bar{\mathbb{Q}}/\mathbb{Q}$.
- A "Galois representation" $\rho : G_{\mathbb{Q}} \to GL_n(k)$ is a continuous semisimple homomorphism unramified outside a finite set of primes.
- $c=$ complex conjugation in $G_{\mathbb{Q}}$.

Basic objects 2

Galois:

- $G_{\mathbb{Q}}$ = Galois group of $\bar{\mathbb{Q}}/\mathbb{Q}$.
- A "Galois representation" $\rho : G_{\mathbb{Q}} \to GL_n(k)$ is a continuous semisimple homomorphism unramified outside a finite set of primes.
- $c=$ complex conjugation in $G_{\mathbb{Q}}$.
- ρ is "odd" if the eigenvalues of $\rho(c)$ are $\pm(1, -1, 1, -1, \dots)$.
  So if the characteristic of k is 2, all Galois representations are odd.

Hecke algebra and weights

Hecke algebra:

- $\Gamma \subset S \subset G(\mathbb{Q})$ (S need only be a semigroup.)

Hecke algebra and weights

Hecke algebra:

- $\Gamma \subset S \subset G(\mathbb{Q})$ (S need only be a semigroup.)
- $\mathcal{H} = k[\Gamma \backslash S / \Gamma]$ algebra of double cosets, "Hecke algebra"

Hecke algebra and weights

Hecke algebra:

- $\Gamma \subset S \subset G(\mathbb{Q})$ (S need only be a semigroup.)
- $\mathcal{H} = k[\Gamma \backslash S / \Gamma]$ algebra of double cosets, "Hecke algebra"
- $T_{\ell,k} = \Gamma \, \mathrm{diag}(\ell, \ldots, \ell, 1, \ldots, 1)\Gamma$ (with k ℓ's)

Hecke algebra and weights

Hecke algebra:

- $\Gamma \subset S \subset G(\mathbb{Q})$ (S need only be a semigroup.)
- $\mathcal{H} = k[\Gamma\backslash S/\Gamma]$ algebra of double cosets, "Hecke algebra"
- $T_{\ell,k} = \Gamma \operatorname{diag}(\ell, \ldots, \ell, 1, \ldots, 1)\Gamma$ (with k ℓ's)

Weights:

- W an irreducible kS-module, finite-dimensional over k, a "weight"

Hecke algebra and weights

Hecke algebra:

- $\Gamma \subset S \subset G(\mathbb{Q})$ (S need only be a semigroup.)
- $\mathcal{H} = k[\Gamma \backslash S / \Gamma]$ algebra of double cosets, "Hecke algebra"
- $T_{\ell,k} = \Gamma \operatorname{diag}(\ell, \ldots, \ell, 1, \ldots, 1)\Gamma$ (with k ℓ's)

Weights:

- W an irreducible kS-module, finite-dimensional over k, a "weight"
- Example: $W = F(a_1, \ldots, a_n)_\epsilon$ for $\Gamma_0(N)$.

Hecke algebra and weights

Hecke algebra:

- $\Gamma \subset S \subset G(\mathbb{Q})$ (S need only be a semigroup.)
- $\mathcal{H} = k[\Gamma \backslash S / \Gamma]$ algebra of double cosets, "Hecke algebra"
- $T_{\ell,k} = \Gamma \operatorname{diag}(\ell, \ldots, \ell, 1, \ldots, 1)\Gamma$ (with k ℓ's)

Weights:

- W an irreducible kS-module, finite-dimensional over k, a "weight"
- Example: $W = F(a_1, \ldots, a_n)_\epsilon$ for $\Gamma_0(N)$.

Homology is a Hecke module:

$\mathcal{H}$ acts on $H_*(\Gamma, W)$.

Attachment

Let be $z \in H_*(\Gamma, W)$ be a Hecke eigenclass with $T_{\ell,k}(z) = a_{\ell,k} z$ and let $\mathbb{F}$ be an extension of k.

Definition

The Galois representation $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_n(\mathbb{F})$ is "attached" to z if

$$\det(I - \rho(\mathrm{Frob}_\ell)X) = \sum_{k=0}^{n} (-1)^k \ell^{k(k-1)/2} a(\ell, k) X^k$$

for almost all unramified primes ℓ.

Frob = arithmetic Frobenius: $\omega(\mathrm{Frob}_\ell) = \ell$, where ω = the cycl. char.

Attachment

Let be $z \in H_*(\Gamma, W)$ be a Hecke eigenclass with $T_{\ell,k}(z) = a_{\ell,k} z$ and let $\mathbb{F}$ be an extension of k.

Definition

The Galois representation $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_n(\mathbb{F})$ is "attached" to z if

$$\det(I - \rho(\mathrm{Frob}_\ell)X) = \sum_{k=0}^{n} (-1)^k \ell^{k(k-1)/2} a(\ell, k) X^k$$

for almost all unramified primes ℓ.

Frob = arithmetic Frobenius: $\omega(\mathrm{Frob}_\ell) = \ell$, where ω = the cycl. char.

If ρ is attached to z, the characteristic polynomials of $\rho(\mathrm{Frob}_\ell)$ for almost all prime ℓ are determined by z and hence ρ is determined up to isomorphism, since we are assuming ρ is semisimple. But z is not determined by ρ: many z's can have same ρ attached.

Scholze's Theorem

Theorem

- *Let k be a finite field of characteristic p.*
- *Let W be an irreducible finite-dimensional $k[\mathrm{GL}_n(\mathbb{F}_p)]$-module on which S acts via reduction mod p tensored with a nebentype character.*
- *Let $z \in H_i(\Gamma_0(N), W)$ be a Hecke eigenclass.*

Then there exists a Galois representation ρ attached to z.

Scholze's Theorem

Theorem

- *Let k be a finite field of characteristic p.*
- *Let W be an irreducible finite-dimensional $k[\mathrm{GL}_n(\mathbb{F}_p)]$-module on which S acts via reduction mod p tensored with a nebentype character.*
- *Let $z \in H_i(\Gamma_0(N), W)$ be a Hecke eigenclass.*

Then there exists a Galois representation ρ attached to z.

- It is known by a theorem of Caraiani and Le Hung that ρ is odd.

Scholze's Theorem

Theorem

- *Let k be a finite field of characteristic p.*
- *Let W be an irreducible finite-dimensional $k[\mathrm{GL}_n(\mathbb{F}_p)]$-module on which S acts via reduction mod p tensored with a nebentype character.*
- *Let $z \in H_i(\Gamma_0(N), W)$ be a Hecke eigenclass.*

Then there exists a Galois representation ρ attached to z.

- It is known by a theorem of Caraiani and Le Hung that ρ is odd.
- As far as I know, it is not proved what the "smallest" N and W can be for a given ρ nor what i ought to be.

Some uses of Galois representations

- Proving Fermat's Last Theorem etc.

Some uses of Galois representations

- Proving Fermat's Last Theorem etc.
- Providing a check on homology computations.

Some uses of Galois representations

- Proving Fermat's Last Theorem etc.
- Providing a check on homology computations.
- Describing Hecke eigenclasses.
  - Naming them, e.g. z has $1 \oplus \omega \oplus \omega^2$ attached.
  - Seeing what (geometrically determined) part of the homology they come from. e.g. z looks like it comes from the lowest dimensional stratum of the Borel-Serre boundary.

Some uses of Galois representations

- Proving Fermat's Last Theorem etc.
- Providing a check on homology computations.
- Describing Hecke eigenclasses.
  - Naming them, e.g. z has $1 \oplus \omega \oplus \omega^2$ attached.
  - Seeing what (geometrically determined) part of the homology they come from. e.g. z looks like it comes from the lowest dimensional stratum of the Borel-Serre boundary.
- Proving differentials in homology spectral sequences vanish.

Some uses of Galois representations

- Proving Fermat's Last Theorem etc.
- Providing a check on homology computations.
- Describing Hecke eigenclasses.
  - Naming them, e.g. z has $1 \oplus \omega \oplus \omega^2$ attached.
  - Seeing what (geometrically determined) part of the homology they come from. e.g. z looks like it comes from the lowest dimensional stratum of the Borel-Serre boundary.
- Proving differentials in homology spectral sequences vanish.
- Predicting the existence of homology classes (Serre-type conjecture).

Some uses of Galois representations

- Proving Fermat's Last Theorem etc.
- Providing a check on homology computations.
- Describing Hecke eigenclasses.
  - Naming them, e.g. z has $1 \oplus \omega \oplus \omega^2$ attached.
  - Seeing what (geometrically determined) part of the homology they come from. e.g. z looks like it comes from the lowest dimensional stratum of the Borel-Serre boundary.
- Proving differentials in homology spectral sequences vanish.
- Predicting the existence of homology classes (Serre-type conjecture).
- Predicting the asymptotic growth of the size of the torsion part of the homology of a family of Γ's that shrink to 1.

Example 1

Example

(work with Gunnells and McConnell) $k = \mathbb{C}$. We compute for $\Gamma_0(N) \subset \mathrm{SL}_4(\mathbb{Z})$ and various W's the Hecke module $H_5(\Gamma, W)$. The dimensions of the Hecke eigenspaces are interesting, and we find the Galois representations that appear to be attached to them.

- a check on correctness of the computations
- insight into the Borel-Serre boundary
- if we can't find a reducible Galois representation attached, then we appear to have a cuspform

Example 2

Example

(work with Yasaki) We compute for $\Gamma_0(N) \subset \mathrm{SL}(3, \mathbb{Z})$ and $W = \mathbb{Q}$ the Hecke module $H^3(\Gamma, W)$ and a certain Hecke stable subspace $K(\Gamma, E)$ of it defined using the units of a real quadratic field E. We compute Hecke on it and find Galois representations that appear to be attached to the Hecke eigenvalues.

- a check on correctness of the computations
- refine our conjecture as to what $K(\Gamma, E)$ is in terms of the Borel-Serre boundary

ADPS conjecture

Conjecture (Ash-Doud-Pollack-Sinnott) (modified): Let k be a finite field and ρ an odd Galois representation. Then ρ is attached to a Hecke eigenclass in $H_i(\Gamma_0^{\pm}(N), F(a_1, \ldots, a_n)_\epsilon)$ for some i.

ADPS conjecture

Conjecture (Ash-Doud-Pollack-Sinnott) (modified): Let k be a finite field and ρ an odd Galois representation. Then ρ is attached to a Hecke eigenclass in $H_i(\Gamma_0^{\pm}(N), F(a_1, \ldots, a_n)_\epsilon)$ for some i.

- The level N, nebentype ϵ and set of weights F are predicted by certain formulas from ρ. The plus-minus means we allow elements of determinant -1. $\Gamma_0^{\pm}(N)$ is better suited for induction than $\Gamma_0(N)$.

ADPS conjecture

Conjecture (Ash-Doud-Pollack-Sinnott) (modified): Let k be a finite field and ρ an odd Galois representation. Then ρ is attached to a Hecke eigenclass in $H_i(\Gamma_0^{\pm}(N), F(a_1, \ldots, a_n)_\epsilon)$ for some i.

- The level N, nebentype ϵ and set of weights F are predicted by certain formulas from ρ. The plus-minus means we allow elements of determinant -1. $\Gamma_0^{\pm}(N)$ is better suited for induction than $\Gamma_0(N)$.
- Herzig has enlarged the set of weights for "generic" ρ, but the complete set for all ρ is not known.

ADPS conjecture

Conjecture (Ash-Doud-Pollack-Sinnott) (modified): Let k be a finite field and ρ an odd Galois representation. Then ρ is attached to a Hecke eigenclass in $H_i(\Gamma_0^{\pm}(N), F(a_1, \ldots, a_n)_\epsilon)$ for some i.

- The level N, nebentype ϵ and set of weights F are predicted by certain formulas from ρ. The plus-minus means we allow elements of determinant -1. $\Gamma_0^{\pm}(N)$ is better suited for induction than $\Gamma_0(N)$.
- Herzig has enlarged the set of weights for "generic" ρ, but the complete set for all ρ is not known.
- This generalizes Serre's conjecture for $n = 2$ which is now a theorem, putting together work of Khare, Wintenberger and Kisin.

ADPS conjecture

Conjecture (Ash-Doud-Pollack-Sinnott) (modified): Let k be a finite field and ρ an odd Galois representation. Then ρ is attached to a Hecke eigenclass in $H_i(\Gamma_0^{\pm}(N), F(a_1, \ldots, a_n)_\epsilon)$ for some i.

- The level N, nebentype ϵ and set of weights F are predicted by certain formulas from ρ. The plus-minus means we allow elements of determinant -1. $\Gamma_0^{\pm}(N)$ is better suited for induction than $\Gamma_0(N)$.
- Herzig has enlarged the set of weights for "generic" ρ, but the complete set for all ρ is not known.
- This generalizes Serre's conjecture for $n = 2$ which is now a theorem, putting together work of Khare, Wintenberger and Kisin.
- It is wide open for $n > 2$.

ADPS conjecture

Conjecture (Ash-Doud-Pollack-Sinnott) (modified): Let k be a finite field and ρ an odd Galois representation. Then ρ is attached to a Hecke eigenclass in $H_i(\Gamma_0^{\pm}(N), F(a_1, \ldots, a_n)_\epsilon)$ for some i.

- The level N, nebentype ϵ and set of weights F are predicted by certain formulas from ρ. The plus-minus means we allow elements of determinant -1. $\Gamma_0^{\pm}(N)$ is better suited for induction than $\Gamma_0(N)$.
- Herzig has enlarged the set of weights for "generic" ρ, but the complete set for all ρ is not known.
- This generalizes Serre's conjecture for $n = 2$ which is now a theorem, putting together work of Khare, Wintenberger and Kisin.
- It is wide open for $n > 2$.
- In general we do not know the range of possible i's. Such knowledge could help in our project.

What the rest of this talk is about

Joint work with Darren Doud.

What the rest of this talk is about

Joint work with Darren Doud. Our ultimate goal is to prove the ADPS conjecture for reducible $\rho = \sigma_1 \oplus \cdots \oplus \sigma_m$, assuming that all the σ_i are odd and that the conjecture holds for them.

What the rest of this talk is about

Joint work with Darren Doud. Our ultimate goal is to prove the ADPS conjecture for reducible $\rho = \sigma_1 \oplus \cdots \oplus \sigma_m$, assuming that all the σ_i are odd and that the conjecture holds for them.

- From now on N is square-free, char $k = p > n + 1$, and $m = 2$.

What the rest of this talk is about

Joint work with Darren Doud. Our ultimate goal is to prove the ADPS conjecture for reducible $\rho = \sigma_1 \oplus \cdots \oplus \sigma_m$, assuming that all the σ_i are odd and that the conjecture holds for them.

- From now on N is square-free, char $k = p > n + 1$, and $m = 2$.
- Idea:

What the rest of this talk is about

Joint work with Darren Doud. Our ultimate goal is to prove the ADPS conjecture for reducible $\rho = \sigma_1 \oplus \cdots \oplus \sigma_m$, assuming that all the σ_i are odd and that the conjecture holds for them.

- From now on N is square-free, char $k = p > n + 1$, and $m = 2$.
- Idea:
  - First show ρ is attached to the homology of a "parabolic" subgroup of Γ.

What the rest of this talk is about

Joint work with Darren Doud. Our ultimate goal is to prove the ADPS conjecture for reducible $\rho = \sigma_1 \oplus \cdots \oplus \sigma_m$, assuming that all the σ_i are odd and that the conjecture holds for them.

- From now on N is square-free, char $k = p > n + 1$, and $m = 2$.
- Idea:
  - First show ρ is attached to the homology of a "parabolic" subgroup of Γ.
  - Then "lift" to the homology of Γ

What the rest of this talk is about

Joint work with Darren Doud. Our ultimate goal is to prove the ADPS conjecture for reducible $\rho = \sigma_1 \oplus \cdots \oplus \sigma_m$, assuming that all the σ_i are odd and that the conjecture holds for them.

- From now on N is square-free, char $k = p > n + 1$, and $m = 2$.
- Idea:
  - First show ρ is attached to the homology of a "parabolic" subgroup of Γ.
  - Then "lift" to the homology of Γ

On the next few slides I introduce the tools we use for this proof.

Parabolic subgroups

- P a parabolic subgroup of $\mathrm{GL}_n(\mathbb{Q})$.

Parabolic subgroups

- P a parabolic subgroup of $\mathrm{GL}_n(\mathbb{Q})$.
- P is the stabilizer of a flag of subvector spaces of $\mathbb{Q}^n$:

$$0 \subset V_1 \subset \cdots \subset V_k \subset \mathbb{Q}^n.$$

Parabolic subgroups

- P a parabolic subgroup of $\mathrm{GL}_n(\mathbb{Q})$.
- P is the stabilizer of a flag of subvector spaces of $\mathbb{Q}^n$:

$$0 \subset V_1 \subset \cdots \subset V_k \subset \mathbb{Q}^n.$$

- P is conjugate to a subgroup of $\mathrm{GL}_n(\mathbb{Q})$ (in block form) looking like:

$$\begin{bmatrix} * & * & * & \cdots & * \\ 0 & * & * & \cdots & * \\ 0 & 0 & * & \cdots & * \\ & & \vdots & & \\ 0 & 0 & 0 & \cdots & * \end{bmatrix}$$

More about parabolic subgroups

- $P = LU$ where

$$L \text{ is conjugate to } \begin{bmatrix} * & 0 & 0 & \cdots & 0 \\ 0 & * & 0 & \cdots & 0 \\ 0 & 0 & * & \cdots & 0 \\ & & \vdots & & \\ 0 & 0 & 0 & \cdots & * \end{bmatrix}$$

  and

More about parabolic subgroups

- $P = LU$ where

$$L \text{ is conjugate to } \begin{bmatrix} * & 0 & 0 & \cdots & 0 \\ 0 & * & 0 & \cdots & 0 \\ 0 & 0 & * & \cdots & 0 \\ & & \vdots & & \\ 0 & 0 & 0 & \cdots & * \end{bmatrix}$$

and

$$U \text{ is conjugate to } \begin{bmatrix} I & * & * & \cdots & * \\ 0 & I & * & \cdots & * \\ 0 & 0 & I & \cdots & * \\ & & \vdots & & \\ 0 & 0 & 0 & \cdots & I \end{bmatrix}$$

P and Γ

Let Γ be any congruence subgroup of $\mathrm{SL}_n(\mathbb{Z})$.

Definition

$\Gamma_P = \Gamma \cap P; \quad \Gamma_U = \Gamma \cap U; \quad \Gamma_L = \Gamma_P / \Gamma_U.$

P and Γ

Let Γ be any congruence subgroup of $\mathrm{SL}_n(\mathbb{Z})$.

Definition

$\Gamma_P = \Gamma \cap P; \quad \Gamma_U = \Gamma \cap U; \quad \Gamma_L = \Gamma_P / \Gamma_U.$

- Exact sequence:

$$1 \to \Gamma_U \to \Gamma_P \to \Gamma_L \to 1.$$

P and Γ

Let Γ be any congruence subgroup of $\mathrm{SL}_n(\mathbb{Z})$.

Definition

$\Gamma_P = \Gamma \cap P; \quad \Gamma_U = \Gamma \cap U; \quad \Gamma_L = \Gamma_P / \Gamma_U.$

- Exact sequence:

$$1 \to \Gamma_U \to \Gamma_P \to \Gamma_L \to 1.$$

Warning: in general, this is not split exact.

P and Γ

Let Γ be any congruence subgroup of $\mathrm{SL}_n(\mathbb{Z})$.

Definition

$\Gamma_P = \Gamma \cap P; \quad \Gamma_U = \Gamma \cap U; \quad \Gamma_L = \Gamma_P / \Gamma_U.$

- Exact sequence:

$$1 \to \Gamma_U \to \Gamma_P \to \Gamma_L \to 1.$$

Warning: in general, this is not split exact.

- Lyndon-Hochschild-Serre (LHS) spectral sequence:

$$E^2_{ij} = H_i(\Gamma_L, H_j(\Gamma_U, W)) \Rightarrow H_{i+j}(\Gamma_P, W).$$

P and Γ

Let Γ be any congruence subgroup of $\mathrm{SL}_n(\mathbb{Z})$.

Definition

$\Gamma_P = \Gamma \cap P; \quad \Gamma_U = \Gamma \cap U; \quad \Gamma_L = \Gamma_P / \Gamma_U.$

- Exact sequence:
$$1 \to \Gamma_U \to \Gamma_P \to \Gamma_L \to 1.$$
Warning: in general, this is not split exact.

- Lyndon-Hochschild-Serre (LHS) spectral sequence:
$$E^2_{ij} = H_i(\Gamma_L, H_j(\Gamma_U, W)) \Rightarrow H_{i+j}(\Gamma_P, W).$$

- It is Hecke equivariant.

Tits building and Steinberg module

Let $n \geq 3$ to obtain simplicity of statements.

- The Tits building T of $\mathrm{GL}_n(\mathbb{Q})$ is the $(n-2)$-dimensional simplicial complex whose vertices are the maximal parabolic subgroups of $\mathrm{GL}_n(\mathbb{Q})$. An i-simplex has the vertices $P_0, \ldots, P_i$ if $\cap P_\alpha$ is a parabolic subgroup.

Tits building and Steinberg module

Let $n \geq 3$ to obtain simplicity of statements.

- The Tits building T of $\mathrm{GL}_n(\mathbb{Q})$ is the $(n-2)$-dimensional simplicial complex whose vertices are the maximal parabolic subgroups of $\mathrm{GL}_n(\mathbb{Q})$. An i-simplex has the vertices $P_0, \ldots, P_i$ if $\cap P_\alpha$ is a parabolic subgroup.
- $\mathrm{GL}_n(\mathbb{Q})$ acts on T by conjugation.

Tits building and Steinberg module

Let $n \geq 3$ to obtain simplicity of statements.

- The Tits building T of $\mathrm{GL}_n(\mathbb{Q})$ is the $(n-2)$-dimensional simplicial complex whose vertices are the maximal parabolic subgroups of $\mathrm{GL}_n(\mathbb{Q})$. An i-simplex has the vertices $P_0, \ldots, P_i$ if $\cap P_\alpha$ is a parabolic subgroup.
- $\mathrm{GL}_n(\mathbb{Q})$ acts on T by conjugation.
- The stabilizer of an i-simplex under this action is the parabolic subgroup $\cap P_\alpha$.

Tits building and Steinberg module

Let $n \geq 3$ to obtain simplicity of statements.

- The Tits building T of $\mathrm{GL}_n(\mathbb{Q})$ is the $(n-2)$-dimensional simplicial complex whose vertices are the maximal parabolic subgroups of $\mathrm{GL}_n(\mathbb{Q})$. An i-simplex has the vertices $P_0, \ldots, P_i$ if $\cap P_\alpha$ is a parabolic subgroup.
- $\mathrm{GL}_n(\mathbb{Q})$ acts on T by conjugation.
- The stabilizer of an i-simplex under this action is the parabolic subgroup $\cap P_\alpha$.

Theorem (Solomon-Tits)

The reduced homology of T is trivial in all dimensions except dimension $n-2$.

Tits building and Steinberg module

Let $n \geq 3$ to obtain simplicity of statements.

- The Tits building T of $\mathrm{GL}_n(\mathbb{Q})$ is the $(n-2)$-dimensional simplicial complex whose vertices are the maximal parabolic subgroups of $\mathrm{GL}_n(\mathbb{Q})$. An i-simplex has the vertices $P_0, \ldots, P_i$ if $\cap P_\alpha$ is a parabolic subgroup.
- $\mathrm{GL}_n(\mathbb{Q})$ acts on T by conjugation.
- The stabilizer of an i-simplex under this action is the parabolic subgroup $\cap P_\alpha$.

Theorem (Solomon-Tits)

The reduced homology of T is trivial in all dimensions except dimension $n-2$.

- Define the Steinberg module (a module for $\mathrm{GL}_n(\mathbb{Q})$) by:

$$\mathrm{St}(\mathbb{Q}^n) = H_{n-2}(T, \mathbb{Z}).$$

Tits spectral sequence

Tits spectral sequence

Consider the complex C of $\mathrm{GL}_n(\mathbb{Q})$-modules

$$0 \to T_{n-2} \to T_{n-1} \to \cdots \to T_0 \to k \to 0,$$

where T_i denotes the k-vector space with basis the i-simplices of T, and k is acted on trivially by $\mathrm{GL}_n(\mathbb{Q})$.

$$C_i = \begin{cases} T_{i-1} & \text{for } i > 0 \\ k & \text{for } i = 0. \end{cases}.$$

Tits spectral sequence

Consider the complex C of $\mathrm{GL}_n(\mathbb{Q})$-modules

$$0 \to T_{n-2} \to T_{n-1} \to \cdots \to T_0 \to k \to 0,$$

where T_i denotes the k-vector space with basis the i-simplices of T, and k is acted on trivially by $\mathrm{GL}_n(\mathbb{Q})$.

$$C_i = \begin{cases} T_{i-1} & \text{for } i > 0 \\ k & \text{for } i = 0. \end{cases}.$$

Then

$$H_i(C) = \begin{cases} St(\mathbb{Q}^n) \otimes_{\mathbb{Z}} k & \text{for } i = n-1 \\ 0 & \text{for } i \neq n-1. \end{cases}$$

Tits spectral sequence continued

Let W be a $k[S]$ module.
Studying $H_*(\Gamma, C \otimes_k W)$ we obtain a Hecke equivariant (!) spectral sequence:

Tits spectral sequence continued

Let W be a $k[S]$ module.
Studying $H_*(\Gamma, C \otimes_k W)$ we obtain a Hecke equivariant (!) spectral sequence:

- Let $\mathcal{P}_i$ denote a set of representatives of Γ-orbits of parabolic subgroups that are stabilizers of $i-1$-cells in T and $\mathcal{P}_0 = \{\mathrm{GL}_n(\mathbb{Q})\}$ (an honorary parabolic subgroup.)

Tits spectral sequence continued

Let W be a $k[S]$ module.
Studying $H_*(\Gamma, C \otimes_k W)$ we obtain a Hecke equivariant (!) spectral sequence:

- Let $\mathcal{P}_i$ denote a set of representatives of Γ-orbits of parabolic subgroups that are stabilizers of $i-1$-cells in T and $\mathcal{P}_0 = \{\mathrm{GL}_n(\mathbb{Q})\}$ (an honorary parabolic subgroup.)
- $$E^1_{ij} = \bigoplus_{P \in \mathcal{P}_i} H_j(\Gamma \cap P, W) \Rightarrow H_*(\Gamma, \mathrm{St}(\mathbb{Q}^n) \otimes_k W).$$

Tits spectral sequence continued

Let W be a $k[S]$ module.
Studying $H_*(\Gamma, C \otimes_k W)$ we obtain a Hecke equivariant (!) spectral sequence:

- Let $\mathcal{P}_i$ denote a set of representatives of Γ-orbits of parabolic subgroups that are stabilizers of $i-1$-cells in T and $\mathcal{P}_0 = \{\mathrm{GL}_n(\mathbb{Q})\}$ (an honorary parabolic subgroup.)
- $$E^1_{ij} = \bigoplus_{P \in \mathcal{P}_i} H_j(\Gamma \cap P, W) \Rightarrow H_*(\Gamma, \mathrm{St}(\mathbb{Q}^n) \otimes_k W).$$
- The proof uses Shapiro's lemma which is how the parabolic subgroups appear.

Tits spectral sequence continued

Let W be a $k[S]$ module.
Studying $H_*(\Gamma, C \otimes_k W)$ we obtain a Hecke equivariant (!) spectral sequence:

- Let $\mathcal{P}_i$ denote a set of representatives of Γ-orbits of parabolic subgroups that are stabilizers of $i-1$-cells in T and $\mathcal{P}_0 = \{\mathrm{GL}_n(\mathbb{Q})\}$ (an honorary parabolic subgroup.)
- $$E^1_{ij} = \bigoplus_{P \in \mathcal{P}_i} H_j(\Gamma \cap P, W) \Rightarrow H_*(\Gamma, \mathrm{St}(\mathbb{Q}^n) \otimes_k W).$$
- The proof uses Shapiro's lemma which is how the parabolic subgroups appear.
- The left most column consists of $H_j(\Gamma, W)$.

Tits spectral sequence continued

Let W be a $k[S]$ module.
Studying $H_*(\Gamma, C \otimes_k W)$ we obtain a Hecke equivariant (!) spectral sequence:

- Let $\mathcal{P}_i$ denote a set of representatives of Γ-orbits of parabolic subgroups that are stabilizers of $i-1$-cells in T and $\mathcal{P}_0 = \{\mathrm{GL}_n(\mathbb{Q})\}$ (an honorary parabolic subgroup.)
- $$E^1_{ij} = \bigoplus_{P \in \mathcal{P}_i} H_j(\Gamma \cap P, W) \Rightarrow H_*(\Gamma, \mathrm{St}(\mathbb{Q}^n) \otimes_k W).$$
- The proof uses Shapiro's lemma which is how the parabolic subgroups appear.
- The left most column consists of $H_j(\Gamma, W)$.
- The next column consists of $H_j(\Gamma_P, W)$, for maximal parabolic subgroups P. Etc.

Use of Tits spectral sequence

We want to show that ρ is attached to a Hecke eigenclass in $H_*(\Gamma, W)$.

Use of Tits spectral sequence

We want to show that ρ is attached to a Hecke eigenclass in $H_*(\Gamma, W)$.

- If ρ is attached to z for a maximal parabolic subgroup P,

$$E^1_{ij}: \begin{array}{ccc} \vdots & \vdots & \vdots \\ H_j(\Gamma, W) & {\color{red} z \in} \bigoplus_{P \in \mathcal{P}_1} H_j(\Gamma_P, W) & \bigoplus_{P \in \mathcal{P}_2} H_j(\Gamma_P, W) \\ H_{j-1}(\Gamma, W) & \bigoplus_{P \in \mathcal{P}_1} H_{j-1}(\Gamma_P, W) & \bigoplus_{P \in \mathcal{P}_2} H_{j-1}(\Gamma_P, W) \\ \vdots & \vdots & \vdots \end{array}$$

$$\Rightarrow H_*(\Gamma, \mathrm{St}(\mathbb{Q}^n) \otimes_k W).$$

Use of Tits spectral sequence

We want to show that ρ is attached to a Hecke eigenclass in $H_*(\Gamma, W)$.

- If ρ is attached to z for a maximal parabolic subgroup P,

$$
E^1_{ij}: \begin{array}{ccc}
\vdots & \vdots & \vdots \\
H_j(\Gamma, W) & z \in \bigoplus_{P \in \mathcal{P}_1} H_j(\Gamma_P, W) & \bigoplus_{P \in \mathcal{P}_2} H_j(\Gamma_P, W) \\
H_{j-1}(\Gamma, W) & \bigoplus_{P \in \mathcal{P}_1} H_{j-1}(\Gamma_P, W) & \bigoplus_{P \in \mathcal{P}_2} H_{j-1}(\Gamma_P, W) \\
\vdots & \vdots & \vdots
\end{array}
$$

$$\Rightarrow H_*(\Gamma, \mathrm{St}(\mathbb{Q}^n) \otimes_k W).$$

- either z maps nonzero under d_1 in which case ρ is attached to $H_j(\Gamma, W)$, or else z survives to E^2. If for some reason z can't be "hit" from the right, it survives to E^∞ and ρ is attached to $H_*(\Gamma, \mathrm{St}(\mathbb{Q}^n) \otimes_k W) \approx H^{n(n-1)/2-*}(\Gamma, W)$.

The two parts of the proof

The two parts of the proof

- Show (*) ρ is attached to $z \in H_j(\Gamma_P, W)$ for a maximal parabolic subgroup P.

The two parts of the proof

- Show (*) ρ is attached to $z \in H_j(\Gamma_P, W)$ for a maximal parabolic subgroup P.
- Show (**) if a Galois representation τ is attached to a parabolic subgroup with k blocks, then τ is the direct sum of at least k irreducible pieces.

The two parts of the proof

- Show (*) ρ is attached to $z \in H_j(\Gamma_P, W)$ for a maximal parabolic subgroup P.
- Show (**) if a Galois representation τ is attached to a parabolic subgroup with k blocks, then τ is the direct sum of at least k irreducible pieces.
- Then (**) implies that z cannot be "hit" from the right:

$$E^1_{ij} = \begin{array}{ccc} \vdots & \vdots & \vdots \\ H_j(\Gamma, W) & {\color{red} z \in} \bigoplus_{P \in \mathcal{P}_1} H_j(\Gamma_P, W) & \bigoplus_{P \in \mathcal{P}_2} H_j(\Gamma_P, W) \\ H_{j-1}(\Gamma, W) & \bigoplus_{P \in \mathcal{P}_1} H_{j-1}(\Gamma_P, W) & \bigoplus_{P \in \mathcal{P}_2} H_{j-1}(\Gamma_P, W) \\ \vdots & \vdots & \vdots \end{array}$$

The two parts of the proof

- Show (*) ρ is attached to $z \in H_j(\Gamma_P, W)$ for a maximal parabolic subgroup P.
- Show (**) if a Galois representation τ is attached to a parabolic subgroup with k blocks, then τ is the direct sum of at least k irreducible pieces.
- Then (**) implies that z cannot be "hit" from the right:

$$E^1_{ij} = \begin{array}{ccc} \vdots & \vdots & \vdots \\ H_j(\Gamma, W) & {\color{red} z \in} \bigoplus_{P\in\mathcal{P}_1} H_j(\Gamma_P, W) & \bigoplus_{P\in\mathcal{P}_2} H_j(\Gamma_P, W) \\ H_{j-1}(\Gamma, W) & \bigoplus_{P\in\mathcal{P}_1} H_{j-1}(\Gamma_P, W) & \bigoplus_{P\in\mathcal{P}_2} H_{j-1}(\Gamma_P, W) \\ \vdots & \vdots & \vdots \end{array}$$

- The proof of (**) is not too hard. It remains to prove (*).

Kunneth

Given the Galois representations σ_1 and σ_2 attached to Hecke eigenclasses.

Kunneth

Given the Galois representations σ_1 and σ_2 attached to Hecke eigenclasses.

To show that $\rho = \sigma_1 \oplus \sigma_2$ is attached to a Hecke eigenclass in $H_*(\Gamma_P, W)$ where W is a predicted weight.

Kunneth

Given the Galois representations σ_1 and σ_2 attached to Hecke eigenclasses.

To show that $\rho = \sigma_1 \oplus \sigma_2$ is attached to a Hecke eigenclass in $H_*(\Gamma_P, W)$ where W is a predicted weight.

Step 1: Get $\sigma_1 \oplus \sigma_2$ attached to a Hecke eigenclass of Γ_L. Use:

Theorem (Ash-Doud)

Let $(\Gamma^{\pm}, S^{\pm}) = (\Gamma_0^{\pm}(n, N), S_0^{\pm}(n, N))$. Let P be a maximal parabolic subgroup of $\mathrm{GL}_n(\mathbb{Q})$ of type (n_1, n_2), with unipotent radical U and Levi quotient L, and denote the two components of the Levi quotient by L^1 and L^2. For $i = 1, 2$, let M_i be an L^i-module and set $M = M_1 \otimes M_2$. Let $f_i \in H_{s_i}(\Gamma^{\pm}_{L^i}, M_i)$ be an eigenclass of all the Hecke operators $T_{n_i}(\ell, j)$. Then $f_1 \otimes f_2$ may be considered as an element of $H_{s_1+s_2}(\Gamma^{\pm}_L, M)$, and if each f_i is attached to a Galois representation Σ_i, then $f_1 \otimes f_2$ is attached to $\Sigma_1 \oplus \omega^{k_1}\Sigma_2$.

E^2 of LHS spectral sequence

Step 2: Get $\sigma_1 \oplus \sigma_2$ attached to a Hecke eigenclass in

$$E^2_{ij} = H_i(\Gamma_L, H_j(\Gamma_U, W)).$$

E^2 of LHS spectral sequence

Step 2: Get $\sigma_1 \oplus \sigma_2$ attached to a Hecke eigenclass in

$$E_{ij}^2 = H_i(\Gamma_L, H_j(\Gamma_U, W)).$$

To do this, we have to identify $H_j(\Gamma_U, W)$ as a Γ_L-module.

E^2 of LHS spectral sequence

Step 2: Get $\sigma_1 \oplus \sigma_2$ attached to a Hecke eigenclass in

$$E^2_{ij} = H_i(\Gamma_L, H_j(\Gamma_U, W)).$$

To do this, we have to identify $H_j(\Gamma_U, W)$ as a Γ_L-module.

This Γ_L-module has to be M as in the Kunneth theorem.

E^2 of LHS spectral sequence

Step 2: Get $\sigma_1 \oplus \sigma_2$ attached to a Hecke eigenclass in

$$E^2_{ij} = H_i(\Gamma_L, H_j(\Gamma_U, W)).$$

To do this, we have to identify $H_j(\Gamma_U, W)$ as a Γ_L-module.

This Γ_L-module has to be M as in the Kunneth theorem.

We need
(1) that $H_j(\Gamma_U, W)$ is an "f-admissible" Γ_L-module, (Ash-Doud) and

E^2 of LHS spectral sequence

Step 2: Get $\sigma_1 \oplus \sigma_2$ attached to a Hecke eigenclass in

$$E^2_{ij} = H_i(\Gamma_L, H_j(\Gamma_U, W)).$$

To do this, we have to identify $H_j(\Gamma_U, W)$ as a Γ_L-module.

This Γ_L-module has to be M as in the Kunneth theorem.

We need
(1) that $H_j(\Gamma_U, W)$ is an "f-admissible" Γ_L-module, (Ash-Doud) and

(2) a mod p Kostant theorem, which is not known in general.

Substitute for Kostant

Luckily, we only need $H_j(\Gamma_U, W)$ for $j = n_1 n_2$, the rank of Γ_U.

Substitute for Kostant

Luckily, we only need $H_j(\Gamma_U, W)$ for $j = n_1 n_2$, the rank of Γ_U.

For that j, $H_j(\Gamma_U, W)$ can be computed as the top homology of the torus $\Gamma_U \otimes_{\mathbb{Z}} \mathbb{R}/\Gamma_U$, and we obtain:

Substitute for Kostant

Luckily, we only need $H_j(\Gamma_U, W)$ for $j = n_1 n_2$, the rank of Γ_U.

For that j, $H_j(\Gamma_U, W)$ can be computed as the top homology of the torus $\Gamma_U \otimes_{\mathbb{Z}} \mathbb{R}/\Gamma_U$, and we obtain:

Theorem (Ash-Doud)

Let N be square-free and prime to p, let $\epsilon : (\mathbb{Z}/N\mathbb{Z})^\times \to \overline{\mathbb{F}}_p$. Let $P = LU$ be a maximal parabolic subgroup of type (n_1, n_2). Set $(\Gamma, S) = (\Gamma_0(n, N), S_0(n, N))$. Then

$$H_j(\Gamma_U, F(a_1, \dots, a_n)_\epsilon) \cong$$

$$(F(a_1 + (n-k), \dots, a_k + (n-k)) \otimes F(a_{k+1} - k, \dots, a_n - k))_\epsilon$$

as S_L-modules.

$z \in E^2$ survives to E^∞

Let $j = n_1 n_2$. Knowing $H_j(\Gamma_U, F(a_1, \ldots, a_n)_\epsilon)$, we can prove that $\sigma_1 \oplus \sigma_2$ is attached to a Hecke eigenclass z in

$$E^2_{ij} = H_i(\Gamma_L, H_j(\Gamma_U, F(a_1, \ldots, a_n)_\epsilon)$$

for some i.

$z \in E^2$ survives to E^∞

Let $j = n_1 n_2$. Knowing $H_j(\Gamma_U, F(a_1, \ldots, a_n)_\epsilon)$, we can prove that $\sigma_1 \oplus \sigma_2$ is attached to a Hecke eigenclass z in

$$E^2_{ij} = H_i(\Gamma_L, H_j(\Gamma_U, F(a_1, \ldots, a_n)_\epsilon)$$

for some i.

Step 3: Show that z survives to E^∞_{ij}.

$z \in E^2$ survives to E^∞

Let $j = n_1 n_2$. Knowing $H_j(\Gamma_U, F(a_1, \ldots, a_n)_\epsilon)$, we can prove that $\sigma_1 \oplus \sigma_2$ is attached to a Hecke eigenclass z in

$$E^2_{ij} = H_i(\Gamma_L, H_j(\Gamma_U, F(a_1, \ldots, a_n)_\epsilon)$$

for some i.

Step 3: Show that z survives to E^∞_{ij}.

The degeneration at E^2 of the analogous LHS spectral sequence when $k = \mathbb{C}$ is known. But the methods don't seem to carry over to k of characteristic p except in very special cases.

$z \in E^2$ survives to E^∞

Let $j = n_1 n_2$. Knowing $H_j(\Gamma_U, F(a_1, \ldots, a_n)_\epsilon)$, we can prove that $\sigma_1 \oplus \sigma_2$ is attached to a Hecke eigenclass z in

$$E^2_{ij} = H_i(\Gamma_L, H_j(\Gamma_U, F(a_1, \ldots, a_n)_\epsilon)$$

for some i.

Step 3: Show that z survives to E^∞_{ij}.

The degeneration at E^2 of the analogous LHS spectral sequence when $k = \mathbb{C}$ is known. But the methods don't seem to carry over to k of characteristic p except in very special cases.

Using certain Hecke operators we can do it for this particular E^2_{ij}. (We are still checking this.)

Ash-Doud Theorem (in progress)

In sum, here is the theorem we think we can prove:

Theorem (Ash-Doud)

Let $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_n(\bar{\mathbb{F}}_p)$ be an odd Galois representation with square-free Serre conductor N, $p > n+1$. Assume that $\rho = \sigma_1 \oplus \sigma_2$, with each $\sigma_i : G_{\mathbb{Q}} \to \mathrm{GL}_{n_i}(\bar{\mathbb{F}}_p)$ irreducible, odd, with Serre conductor N_i (so $N = N_1 N_2$.) Assume that the ADPS conjecture holds for σ_1 and σ_2. Then ρ is attached to a Hecke eigenclass in

$$H_*(\Gamma_0^{\pm}(n, N), F_\epsilon)$$

for some weight F and nebentype ϵ predicted for ρ by the conjecture.

When ρ has k irreducible components, $k > 2$

Here are the problems we have in generalizing this to arbitrary non-irreducible odd ρ which is a sum of k irreducible odd components.

When ρ has k irreducible components, $k > 2$

Here are the problems we have in generalizing this to arbitrary non-irreducible odd ρ which is a sum of k irreducible odd components.

- By induction can get ρ attached to a Hecke eigenclass of E^2_{ij} for some MAXIMAL parabolic subgroup.

When ρ has k irreducible components, $k > 2$

Here are the problems we have in generalizing this to arbitrary non-irreducible odd ρ which is a sum of k irreducible odd components.

- By induction can get ρ attached to a Hecke eigenclass of E^2_{ij} for some MAXIMAL parabolic subgroup.
- But now perhaps z can be "hit" from the right in the Tits spectral sequence.

When ρ has k irreducible components, $k > 2$

Here are the problems we have in generalizing this to arbitrary non-irreducible odd ρ which is a sum of k irreducible odd components.

- By induction can get ρ attached to a Hecke eigenclass of E^2_{ij} for some MAXIMAL parabolic subgroup.
- But now perhaps z can be "hit" from the right in the Tits spectral sequence.
- Another approach: get ρ attached to a parabolic subgroup of type $(n_1, \ldots, n_k)$. Then z cannot be "hit" from the right. But now: z may not be in the kernel of higher differentials.

When ρ has k irreducible components, $k > 2$

Here are the problems we have in generalizing this to arbitrary non-irreducible odd ρ which is a sum of k irreducible odd components.

- By induction can get ρ attached to a Hecke eigenclass of E_{ij}^2 for some MAXIMAL parabolic subgroup.
- But now perhaps z can be "hit" from the right in the Tits spectral sequence.
- Another approach: get ρ attached to a parabolic subgroup of type $(n_1, \ldots, n_k)$. Then z cannot be "hit" from the right. But now: z may not be in the kernel of higher differentials.

Independently of all the above, it would be very nice to have a mod p Kostant theorem and a good understanding the LHS spectral sequence.

Thanks to the organizers and thank you for listening.