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10/14/21

A Few additional examples

The purpose of this handout is to give a few more examples of the guiding principle mentioned in the talk.

① Let $\mathcal{M}_{3,3} := \left\{ \begin{array}{l} \text{smooth cubic} \\ \text{surfaces } S \subseteq \mathbb{P}^3 \end{array} \right\} \cong \mathbb{P}^{19} - \Sigma_{sing}$

Let $\widetilde{\mathcal{M}}_{3,3} := \{ (S, L) : S \in \mathcal{M}_{3,3},\ L \subset S \text{ line} \}$

$L \cong \mathbb{P}^1$

Cayley, C. Jordan: $\begin{array}{c} \widetilde{\mathcal{M}}_{3,3} \\ \downarrow \pi \\ \mathcal{M}_{3,3} \end{array}$ $\pi(S, L) := S$

"∃ 27 lines on any smooth cubic surface"

is a 27-sheeted cover, with monodromy

$$\rho : \pi_1 \mathcal{M}_{3,3} \twoheadrightarrow W(E_6) \subset S_{27}$$

Thm (P. Huxford 2021, -Magma conj. by Farb): The homomorphism ρ is unique up to conjugacy.

Let $\mathbb{G}(1,3) :=$ Projective Grassmannian of lines in $\mathbb{P}^3$

$\mathrm{UConf}_n(X) := \{$ unordered n-tuples of distinct points in $X \}$

The above gives a morphism

$$\Psi : \mathcal{M}_{3,3} \longrightarrow \mathrm{UConf}_{27}(\mathbb{G}(1,3))$$

Guiding Principle gives: $\forall m \geq 1$

If $F : \mathcal{M}_{3,3} \longrightarrow \mathrm{UConf}_m(\mathbb{G}(1,3))$

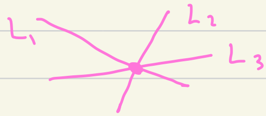
is "nontrivial" then $m = 27$ and $F \sim \Psi$.

If in addition F is holomorphic then $F = \Psi$.

For <u>generic</u> $S \subset \mathcal{M}_{3,3}$,

$$\left| \{ L_i \cap L_j : L_i \neq L_j \text{ lines in } S \} \right| = 135$$

(only generic since $\exists$ "Eckhardt Points")



$\rightsquigarrow$ $\exists$ rational section σ of

$$\begin{tikzcd} \mathrm{Uconf}_{135}(S) \arrow[r] & E_{135} \arrow[d, "\pi"'] \\ & \mathcal{M}_{3,3} \arrow[u, dashed, bend right, "\sigma"'] \end{tikzcd}$$

Guiding principle: σ should be only rational n-multisection.

② Guiding principle #2: Any "exceptional homomorphism" of finite groups is <u>modular</u>: it can be explained by a map of moduli spaces.

<u>Example #1</u>: $S_4 \twoheadrightarrow S_3$ explained by

$$\begin{tikzcd} \widetilde{\mathrm{Poly}}_4 \arrow[r, "\widetilde{\mathrm{Fer}}"] \arrow[d, "S_4"'] & \widetilde{\mathrm{Poly}}_3 \arrow[d, "S_3"] \\ \mathrm{Poly}_4 \arrow[r, "\mathrm{Fer}"'] & \mathrm{Poly}_3 \end{tikzcd}$$

Example #2: $S_6 \cong Sp_4\mathbb{F}_2$ explained by

$$\begin{tikzcd}
\mathcal{M}_{0,6} \arrow[r, "\cong"] \arrow[d, "S_6"'] & \mathcal{M}_2[2] \arrow[r, dashed, "\cong"] \arrow[d] & \mathcal{A}_2[2] \arrow[d, "Sp_4\mathbb{F}_2"] \\
\mathcal{M}_{0,6}/S_6 \arrow[r, "\cong", "R"'] & \mathcal{M}_2 \arrow[r, dashed, "\cong", "J"'] & \mathcal{A}_2
\end{tikzcd}$$

J period mapping

$R(\{p_1,\ldots,p_6\}) :=$ 2-sheeted cover $X \to \mathbb{P}^1$ branched over $\{p_1,\ldots,p_6\}$

$\mathcal{M}_2[2] = \{(X,\mathcal{B}) : X \in \mathcal{M}_2,\ \mathcal{B}$ basis for $H_1(X;\mathbb{F}_2)\}$

$\exists$ many more such examples, e.g.

$$W(E_6)^+ \cong PSp_4(\mathbb{F}_3)$$

See Farb-Kisin-Wolfson, "Modular Functions and resolvent Problems" for more examples.

③ A remarkable map:

Given • $S :=$ smooth cubic surface in $\mathbb{P}^3$

• $p \in S \setminus \{L_1, \dots, L_{27}\}$

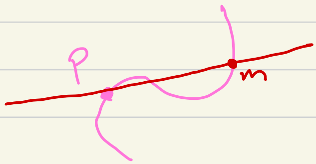
$\rightsquigarrow$ $\exists$ 2-sheeted cover

$$\begin{array}{c} Bl_p(S) \\ \pi \downarrow \\ \mathbb{P}^2 \supset Q \end{array}$$

exc. divisor e

branched over a smooth quartic

π is:



$$\pi(m) = \begin{cases} \text{line } \overline{pm} & m \notin e \\ [m] & m \in e \end{cases}$$

<u>Fact</u>: $\{28$ bitangents to $Q\} =$

$\pi(\{L_1, \dots, L_{27}\}$ on $S) \cup \pi(e)$!

Let $S \to E_{3,3} := \{(S,p) : S \in \mathcal{M}_{3,3}, p \in S\}$

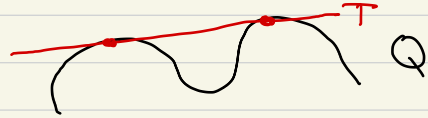
$$\downarrow$$

$$\mathcal{M}_{3,3}$$

Let $\mathcal{M}_{4,2} = \{$smooth quartic curves in $\mathbb{P}^2\}$

The above gives a dominant rational map

$$\begin{array}{c} \Psi : \mathcal{U}_{3,3} \dashrightarrow \{ (Q, T) : Q \in \mathcal{M}_{4,2},\ T \text{ is bitangent to } Q \} \\ \big\downarrow 28 \\ \mathcal{M}_{4,2} \end{array}$$



__Exercise:__ Apply Guiding Principle to Ψ to give conjectures on holomorphic and topological rigidity.

④ Many classical moduli spaces $\mathcal{M}$ locally symmetric varieties, i.e.

$$\mathcal{M} = {}_{\Gamma}\backslash G/K$$

G = real semisimple Lie group of Hermitian type

K = max compact subgroup of G

Γ = arithmetic lattice in G

and the type of rigidity conjectures I've been giving can be proved using Margulis Superrigidity, Congruence Subgroup Property, etc.