LEGENDRE SYMBOLS

and

SECONDARY STABILITY

(a question)

(joint with Mark Shusterman

... and you?)

LEGENDRE SYMBOL:

$$\left(\frac{p}{q}\right) = \begin{cases} 1 & \text{if } p \text{ is } \square \bmod q \\ -1 & \text{if } p \text{ not a square} \end{cases}$$

$(0 \text{ if } p = q)$

If a, b coprime squarefree,

$a = p_1, \ldots, p_r$

$b = q_1 \cdots q_s$

$$\left(\frac{a}{b}\right) = \prod_{i,j} \left(\frac{p_i}{q_j}\right) \in \{\pm 1\}$$

Arithmetic functions should be equidistributed:

$$\sum_{\substack{a<M,\, b<N \\ \text{coprime} \\ \text{squarefree}}} 1 \sim cMN$$

$$\sum_{\substack{a<M,\, b<N \\ \text{coprime} \\ \text{squarefree}}} \left(\frac{a}{b}\right) = o(MN)$$

$$\sum_{\substack{a<M,\ b<N \\ \text{coprime} \\ \text{squarefree}}} \left(\frac{a}{b}\right) = ?$$

Same question over $\mathbb{F}_q[T]$.

vanishes iff f and g have a root in common

$$\left(\frac{f}{g}\right) = \mathrm{Res}(f,g)^{\frac{q-1}{2}}$$

$= 1$ $\mathrm{Res}(f,g)$ $\square$

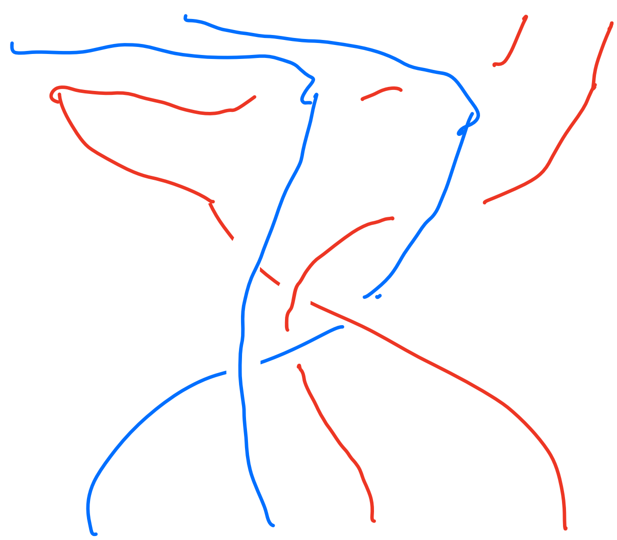
-1 $\mathrm{Res}(f,g)$ non-$\square$

Q: What is

$$\sum_{\substack{\deg f = m \\ \deg g = n \\ \text{coprime} \\ \text{squarefree}}} \operatorname{Res}(f,g)^{\frac{q-1}{2}} \ ?$$

This can be expressed as a question about $\mathbb{F}_q$-points on a certain moduli space. What is its cohomology?

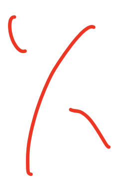
$Br_{m,n}$: <u>mixed</u> braid group with m red strands and n blue strands

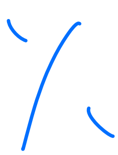


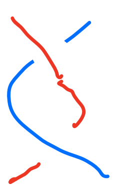
$$P_{m+n} \to Br_{m,n} \to S_{m+n}$$
$$\cup$$
$$S_m \times S_n$$

$$B_{m,n} \to S_m \times S_n \to (\mathbb{Z}/2\mathbb{Z})^2$$

But this is not the only homomorphism to $\mathbb{Z}/2\mathbb{Z}$!

 $\longmapsto 0$

 $\longmapsto 0$

 $\longmapsto 1$

(does not factor through pure braid group!)

Let $\phi : B_{m,n} \to \pm 1$ a homomorphism, and V_ϕ the corresponding 1-dimensional representation of $B_{m,n}$.

(interesting cases: ϕ doesn't factor through pure braid group)

Q: What can we say about

$$H^i(B_{m,n}, V_\phi)$$

as $m, n \to \infty$?

(Algebraically: study double cover of moduli space of coprime sqfree pairs (f, g) obtained by adjoining $\sqrt{\mathrm{Res}(f,g)}$)

Q: What can we say about

$$H^i(B_{m,n}, V_\phi)$$

as $m, n \to \infty$?

GUESS: (based on computations of $\sum \mathrm{Res}(f,g)^{\frac{q-1}{2}}$)

when $m = n$,

- 0 for $i \leq \frac{n}{2}$
- (maaaaybe) stable secondary class at $i = \frac{n}{2}$? (maybe only for some ϕ)

$$H^i(B_{m,n}, V_\phi) \to H^{i+1}(B_{m+2,n+2}, V_\phi)$$

($m \sim \frac{n}{2}$ might be most interesting case)

Size of stable range – may be hard to beat upper bounds from analytic number theory.

Secondary stability – phenomena which (we think) analytic number theory doesn't see. A new kind of application to number theory over function fields!

FI × FI ??

Thanks to the organizers!