SKT structures and a conformal generalization

Anna Fino

Dipartimento di Matematica
Universitá di Torino

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SKT metrics

Definition

A Hermitian metric g on a complex manifold (M^{2n}, J) is called SKT (or pluriclosed) if

$$i\partial\overline{\partial}\omega = dd^c\omega = 0,$$

where $d^c = -J^{-1}dJ = -i(\overline{\partial} - \partial)$.

Remark

The SKT condition is essentially the only weakening of the Kähler condition which is linear in the fundamental form!

Theorem (Gauduchon)

(M^{2n}, g, J) *compact Hermitian. Then* $\exists!\ u \in \mathcal{C}^\infty(M^{2n})$ *such that*

$$\partial\overline{\partial}(e^{2u}\omega)^{n-1} = 0, \quad \int_{M^{2n}} u\, dV_g = 0.$$

$\hookrightarrow$ Every conformal hermitian structure on a compact complex (M^{2n}, J) contains a hermitian metric $\tilde{\omega}$ such that $\partial\overline{\partial}\tilde{\omega}^{n-1} = 0$
$\Rightarrow$ every compact complex surface admits SKT metrics!

Theorem (Gauduchon)

On any Hermitian manifold (M^{2n}, J, g) there exists an affine line of canonical Hermitian connections ∇^t ($\nabla^t J = 0$, $\nabla^t g = 0$), completely determined by their torsion

$$T(X, Y, Z) := g(T(X, Y), Z).$$

The family includes:

- the Chern connection ∇^C (T^C has trivial $(1,1)$-component)
- the Bismut (or Strominger) connection ∇^B (T^B is a 3-form)

Bismut and Chern connections

Remark

∇^B and ∇^C are related to the Levi-Civita connection ∇^{LC} by

$$g(\nabla^B_X Y, Z) = g(\nabla^{LC}_X Y, Z) + \frac{1}{2} d^c\omega(X, Y, Z),$$
$$g(\nabla^C_X Y, Z) = g(\nabla^{LC}_X Y, Z) + \frac{1}{2} d\omega(JX, Y, Z).$$

Remark

- g is SKT if and only if $dT^B = 0$.
- The trace of the torsion of ∇^C is equal to the Lee form

$\theta := Jd^*\omega$, which is the unique 1-form satisfying

$$d\omega^{n-1} = \theta \wedge \omega^{n-1}.$$

A conformal generalization

$\mathfrak{t}^{\mathbb{C}}$:= Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$

• Left-invariant cpx structures J on $G \iff$ pairs $(J_{\mathfrak{t}}, P)$, with $J_{\mathfrak{t}}$ any cpx structure on $\mathfrak{t}$ and $P \subseteq \Delta$ is a system of positive roots:

$$\mathfrak{g}^{1,0} = \mathfrak{t}^{1,0} \oplus \bigoplus_{\alpha \in P} \mathfrak{g}^{\mathbb{C}}_{\alpha}$$

• Left-invariant pluriclosed metrics g on G are obtained by extending the negative of the Killing form on $[\mathfrak{g}, \mathfrak{g}]$ to a J-compatible positive definite inner product:

$$\nabla^{LC}_X Y = \frac{1}{2}[X, Y], \quad \nabla^B_X Y = 0, \quad X, Y \in \mathfrak{g},$$

with $T^B(X, Y, Z) = g([X, Y], Z)$ a closed 3-form!

Non-Kähler C-spaces

A non-Kähler C-space M is a compact complex manifold with a transitive action by a compact Lie group of biholomorphisms and finite fundamental group.

M admits a transitive action of a cpt semisimple Lie group [Wang].

Theorem (F, Grantcharov, Vezzoni)

Every non-Kähler C-space M admitting a pluriclosed metric is (up to a finite cover) the product of a compact Lie group and a generalized flag manifold.

For the proof we use

- $h^{3,0}(M) = 0$
- $h^{2,1}(M) \neq 0$ only if M has a finite cover which is biholomorphic to a product of a compact Lie group and another C-space.

Compact locally homogeneous spaces

Compact $(\Gamma\backslash G, J)$ with J invariant complex structure

• Classification results for the existence of pluriclosed metrics on nilmanifolds [F, Parton, Salamon; Enrietti, F, Vezzoni]

Conjecture: Every nilmanifold admitting a pluriclosed metric has to be 2-step and the total space of a holomorphic torus bundle over a torus!

• Classification results for the existence of pluriclosed metrics on solvmanifolds [F, Otal, Ugarte; F, Paradiso; Freibert, Swann].

Theorem (F, Tardini, Vezzoni)

The existence of a left-invariant pluriclosed metric on a unimodular Lie group G with a left-invariant abelian complex structure J forces the group G to be 2-step nilpotent.

Sketch of the proof

$$dT^B = 0 \Leftrightarrow g([y,z],[w,x]) - g([x,z,][w,y]) + g([x,y],[w,z]) = 0$$

- $\|[x,y]\|^2 + \|[x,Jy]\|^2 = g([x,Jx],[y,Jy]) \hookrightarrow$

$$x \in \mathfrak{z} \Leftrightarrow [x,Jx] = 0$$

- $\mathfrak{g}$ is unimodular $\Rightarrow \mathfrak{g}^1_J := \mathfrak{g}^1 + J\mathfrak{g}^1 \subset \mathfrak{g}$
- If $\mathfrak{g}^1_J$ is 2-step nilpotent, then $\mathfrak{g}$ is 2-step nilpotent.
- By induction on the dimension, $\mathfrak{g}^1_J$ is 2-step nilpotent $\hookrightarrow$
$\mathfrak{g}$ is 2-step nilpotent!

Other examples which are not Bismut flat

• Characterization of the existence of pluriclosed metrics on Oeljkelaus-Toma (OT) manifolds $X(K, U) := \mathbb{H}^s \times \mathbb{C}^t / U \times \mathcal{O}_K$, where $\mathbb{Q} \subseteq K$ is an algebraic number field, $\mathcal{O}_K$ is the ring of algebraic integers of K and U is an admissible subgroup of the group of totally positive units $\mathcal{O}^{*,+}$ [Otiman].

• For any positive integer $k \geq 1$, $(k-1)(S^2 \times S^4)\#_k(S^3 \times S^3)$ has a pluriclosed metric [D. Grantcharov, G. Grantcharov, Y. Poon].

• Total spaces E of principal bundles over a projective manifold M with structure group an even dimensional unitary, special orthogonal or compact symplectic Lie group [Poddar, Takhur].

An extension result

The complex blow-up preserves the existence of pluriclosed metrics [F, Tomassini] $\hookrightarrow$ resolutions of orbifolds with pluriclosed metrics.

Let (M^{2n}, J) cpx, if $M^{2n} \setminus \{p\}$ admits a Kähler metric, then there exists a Kähler metric on M^{2n} [Miyaoka].

Theorem (F, Tomassini)

Let (M^{2n}, J), $n \geq 2$. If $M^{2n} \setminus \{p\}$ admits a pluriclosed metric, then there exists a pluriclosed metric on M^{2n}.

Remark

If ω is the fundamental form of a pluriclosed metric g on (M^{2n}, g), then ω corresponds to a real strictly positive current of bi-degree $(1,1)$ which is $\partial\overline{\partial}$-closed.

A Hermitian metric which is pluriclosed and balanced is Kähler [Alexandrov, Ivanov; Popovici].

Conjecture

Every compact complex manifold admitting a pluriclosed and a balanced metric is Kähler.

The conjecture is true for

- the twistor space of a compact anti-self-dual 4-dim Riemannian manifold [Verbitsky]
- compact complex manifolds in the Fujiki class $\mathcal{C}$ [Chiose]

• The non-Kähler balanced manifolds constructed by Li, Fu and Yau using conifold transictions. In particular, $\#_k(S^3 \times S^3)$ $k \geq 2$, since they have no pluriclosed metrics.

• 2-step nilmanifolds with invariant complex structures [F, Vezzoni].

• Special classes of solvmanifolds [F, Vezzoni; F. Paradiso; Otiman].

• Compact real semisimple Lie groups [F, Grantcharov, Vezzoni].

• Non-compact real simple Lie groups of inner type [Giusti, Podestà]

Behaviour under small deformations

The pluriclosed condition is in general not stable under small deformations of the complex structure.

Example (F, Tomassini)

The Iwasawa manifold $\Gamma \backslash H_3^{\mathbb{C}}$, where

$$H_3^{\mathbb{C}} = \left\{ \begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} \mid z_j \in \mathbb{C} \right\}$$

has family $J_{t,s}$, $t, s \in \mathbb{R}$, $s \neq 0$, such that $J_{1,1}$ has a compatible pluriclosed metric, but for $t \neq s \neq 1$ there exist no compatible pluriclosed metrics!

Link with symplectic geometry

Definition (Gromov)

An almost cpx structure J on a symplectic manifold (M^{2n}, Ω) is tamed by Ω if $\Omega(X, JX) > 0$, $\forall X \neq 0$.

If J is tamed by Ω, then $g(X, Y) = \frac{1}{2}(\Omega(X, JY) - \Omega(JX, Y))$ is a J-Hermitian metric.

If a compact complex (M^4, J) admits a symplectic structure taming J, then (M^4, J) has a Kähler metric [Streets, Tian; Li, Zhang].

Problem

Does there exist an example of a compact complex (M^{2n}, J), with $n > 2$, admitting a symplectic form Ω taming J, but no Kähler structures?

Negative answers can be given using that Ω tames $J \iff \partial\Omega^{1,1} = \overline{\partial}\beta$, for some ∂-closed $(2,0)$-form β.

$\hookrightarrow$ in particular $\omega = \Omega^{1,1}$ defines a pluriclosed metric.

Theorem (Enrietti, F, Vezzoni)

A nilmanifold M with invariant J has a symplectic form taming $J \iff M$ is a torus.

The same result holds for solvmanifolds of completely solvable type [F, Kasuya].

On a compact Kähler manifold (M, J, g) the Ricci flow

$$\partial_t g(t) = -Ric(g(t)), \quad g(0) = g,$$

preserves the Kähler condition ($\hookrightarrow$ Kähler Ricci flow) and reduces to a parabolic Monge-Ampére equation (Cao, Tian....).

Remark

For a non-Kähler manifold (M, J, g)

• the Levi-Civita connection does not not preserve the complex structure and the Ricci flow does not preserve the Hermitian condition!

• One may consider other connections preserving both the complex structure and the metric (e.g. the Bismut connection).

Let $(M^{2n}, J, g_0, \omega_0)$ be a Hermitian manifold. Streets and Tian introduced the geometric flow

$$\partial_t \omega(t) = -(\rho^B)^{1,1}(\omega(t)), \quad \omega(0) = \omega_0.$$

$\omega \mapsto -(\rho^B)^{1,1}(\omega)$ is a real quasi-linear second-order elliptic operator when restricted to pluriclosed J-Hermitian metrics $\hookrightarrow$

Theorem (Streets, Tian)

Let (M^{2n}, J) be a compact complex manifold. If ω_0 is pluriclosed, then $\exists \epsilon > 0$ and a unique solution $\omega(t)$ to the pluriclosed flow with initial condition ω_0.
If ω_0 is Kähler, then $\omega(t)$ is the unique solution to the Kähler-Ricci flow with initial datum ω_0.

Remark

In local cpx coordinates the pluriclosed flow can be written as:

$$\partial_t \omega(t) = \partial \partial^* \omega(t) + \overline{\partial \partial}^* \omega(t) + i \partial \overline{\partial} \log \det g(t).$$

Proposition (Streets, Tian)

If a pluriclosed metric ω on (M^{2n}, J) satisfies $(\rho^B)^{1,1} = \lambda \omega$, for a constant $\lambda \neq 0$, then $\omega = \Omega^{1,1}$ with Ω a symplectic form Ω taming the complex structure J.

Problem

- *Describe the maximal smooth existence time T.*
- *Study the limiting behavior at the time T.*

Consider the real $(1,1)$ Aeppli cohomology:

$$H^{1,1}_{\mathcal{A},\mathbb{R}} := \frac{\{\operatorname{Ker} i\partial\overline{\partial} : \Lambda^{1,1} \to \Lambda^{2,2}\}}{\{\partial\overline{\eta} + \overline{\partial}\eta \mid \eta \in \Lambda^{1,0}\}}.$$

$\hookrightarrow$ the (1, 1) Aeppli positive cone

$$\mathcal{P} := \{[\psi] \in H^{1,1}_{\mathcal{A},\mathbb{R}} \mid \exists \omega \in [\psi], \omega > 0\}.$$

consists precisely of the (1, 1) Aeppli classes represented by pluriclosed metrics.

Remark

For a general complex manifold (M^{2n}, J)

$$c_1(M^{2n}) \in H^{1,1}_{BC,\mathbb{R}} := \frac{\{\operatorname{Ker} d : \Lambda^{1,1} \to \Lambda^{2,2}\}}{\{i\partial\overline{\partial} f \mid f \in \mathcal{C}^\infty\}} \hookrightarrow H^{1,1}_{\mathcal{A},\mathbb{R}}.$$

As in the Kähler-Ricci flow case for the real $(1,1)$ Aeppli class: $[\omega(t)] = [\omega_0] - t\, c_1(M^{2n})$.

$\hookrightarrow$ The maximal smooth existence time T for the pluriclosed flow with initial condition g_0 satisfies:

$$T \leq \tau^*(\omega_0) := \sup\{t \geq 0 \mid [\omega_0] - t\, c_1(M^{2n}) \in \mathcal{P}\}.$$

Conjecture (Streets, Tian)

Let (M^{2n}, J, g_0) be a compact complex manifold with pluriclosed metric. The maximal smooth solution of pluriclosed flow with initial condition g_0 exists on $[0, \tau^*(\omega_0))$.

Nilpotent Lie groups case

On a Lie group G with left-invariant Hermitian structure (J, g), one may deform the Lie bracket instead of the Hermitian metric g

Theorem (Enrietti, F, Vezzoni)

The pluriclosed flow on a 2-step nilpotent simply-connected Lie group (G, J) starting from a left-invariant Hermitian metric g has a long-time solution.

The solutions converge in the Gromov-Hausdorff sense, after a suitable normalization, to self-similar solutions of the flow [Arroyo, Lafuente].

Bismut Kähler-like conditions

Remark

In general ∇^B does not satisfy the first Bianchi identity, since

$$\begin{aligned}\mathfrak{S}_{X,Y,Z}\, R^B(X,Y,Z,U) &= dT^B(X,Y,Z,U) + (\nabla^B_U T^B)(X,Y,Z) \\ &\quad -\sigma_{X,Y,Z}\, g(T^B(X,Y), T^B(Z,U)).\end{aligned}$$

Definition

∇^B is Kähler-like if it satisfies the first Bianchi identity

$$\mathfrak{S}_{X,Y,Z}\, R^B(X,Y,Z) = 0$$

and the type condition

$$R^B(X,Y,Z,W) = R^B(JX,JY,Z,W),\ \forall X,Y,Z,W.$$

Conjecture (Angella, Otal, Ugarte, Villacampa)

If for a Hermitian manifold (M^{2n}, J, g) the Bismut connection ∇^B is Kähler-like, then g is pluriclosed.

Theorem (Zhao, Zheng)

∇^B is Kähler-like $\Longleftrightarrow$ g is pluriclosed and $\nabla^B T^B = 0$.

Problem

Study the behaviour of the Bismut Kähler-like condition along the pluriclosed flow.

Remark

If $n = 2$, then $T^B = - * \theta$.

Definition

A Hermitian metric g on a complex manifold M^{2n} is a Vaisman metric if $d\omega = \theta \wedge \omega$, for some d-closed 1-form θ with $\nabla^{LC}\theta = 0$.

$\hookrightarrow$ Vaisman metrics are Gauduchon and $|\theta|$ is constant.

Theorem (F, Tardini)

Let (M^4, J) be a complex surface.
A Hermitian metric g is Vaisman if and only if g is pluriclosed and ∇^B satisfies the first Bianchi identity.

Compact Vaisman surfaces have been classified by Belgun and they are non-Kähler properly elliptic surfaces, Kodaira surfaces, and Class 1 or elliptic Hopf surfaces.

Theorem (F, Tardini)

If (M^4, J) admits a Vaisman metric g_0 with constant scalar curvature, then the pluriclosed flow starting with ω_0 preserves the Vaisman condition.

We use

- if (M^4, J, g) is a compact Vaisman surface, then $\rho^C = h\, dJ\theta$, for some $h \in \mathcal{C}^\infty(M^4)$.
- $Scal(g)$ is constant if and only if h is constant and, in such a case $c_1(M^4) = 0$.

Nilpotent Lie group case

Remark

If a 6-dimensional nilpotent Lie group (G, J) admits a Bismut Kähler-like metric, then the left-invariant complex structure J has to be abelian.

Theorem (F, Tardini, Vezzoni)

Let (G, J, g_0) be a 2-step nilpotent Lie group with a left-invariant Bismut Kähler-like Hermitian structure and let $g(t)$ be the solution to the pluriclosed flow starting from g_0. Then $g(t)$ is Bismut Kähler-like for every t.

Kähler like conditions for other connections

Theorem (Yang, Zheng)

Let (M^{2n}, J, g) be a cpt Hermitian manifold. If either ∇^{LC} or ∇^{C} is Kähler-like, then g is balanced.

Conjecture (Angella, Otal, Ugarte, Villacampa)

Let (M^{2n}, J, g) be a Hermitian manifold. If a canonical connection in the Gauduchon family ∇^t (different from ∇^B and ∇^C) is Kähler-like, then g is Kähler.

- The conjecture is true for 6-dim compact solvmanifolds with an invariant complex structure having a non-zero invariant closed $(3,0)$-form [Angella, Otal, Ugarte, Villacampa].
- Some recent partial results by Zhao and Zheng.

THANK YOU VERY MUCH FOR THE ATTENTION!!