Novikov Fundmental group and symplectic isotopies

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Floer + Novikov + Fundamental group

Topology $\longleftrightarrow$ Geometry/Dynamics

$H_*(M) \xleftrightarrow{\text{Morse Theory}}$ Gradient flow

$HF_*(M) \xleftrightarrow{\text{Floer Theory}}$ Hamiltonian isotopy

$HN_*(M, \alpha) \xleftrightarrow{\text{Morse Novikov Theory}}$ Closed 1-form flow

$HNF_*(M, \alpha) \xleftrightarrow[\text{[Le-Ono]}]{\text{Floer Novikov Theory}}$ Symplectic isotopy

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$\pi_1(M, \alpha)$? $\xleftrightarrow{\text{Floer Novikov Theory}}$ Symplectic isotopy

— *Morse fundamental group* —

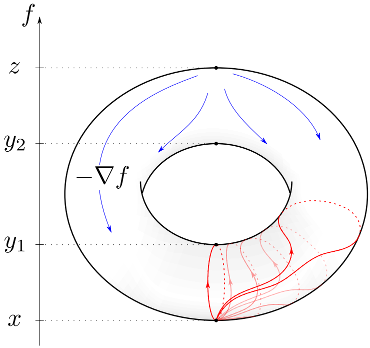
(generators)

Morse fundamental group 1

- M closed manifold,
- $M \xrightarrow{f} \mathbb{R}$ Morse function,
- g metric s.t. (f, g) is Morse-Smale.

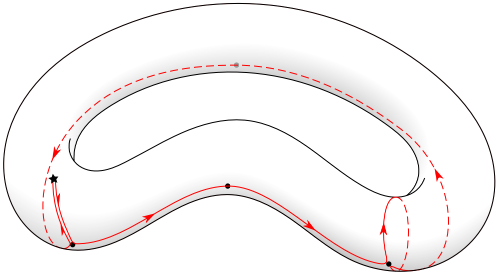
Suppose f has a single minium which is the base point. Then for each $y \in \mathrm{Crit}_1(f)$, $W^u(y)$ is a based loop (up to a choice of orientation) and :

$$\pi_1(M, f) = < y^{\pm 1}, y \in \mathrm{Crit}_1(f) >_{/\sim}$$

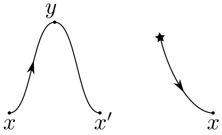


Morse fundamental group 2

In general, $\pi_1(M)$ is generated by concatenation of dim 1 unstable manifolds + base point flow line :

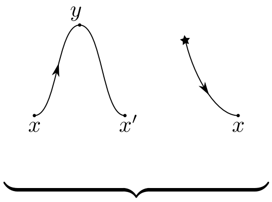


Morse steps :



Morse fundamental group 3

Morse loops : concatenation of Morse steps



$$\mathcal{L}(f,g) \longrightarrow \pi_1(M)$$

Proposition

The evaluation map $\mathcal{L}(f,g) \longrightarrow \pi_1(M)$ is onto.

Morse fundamental group 3

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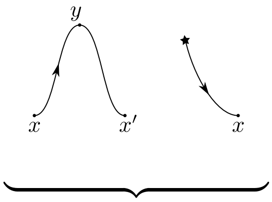

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— Floer fundamental group —

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- (M, ω) is a closed monotone symplectic manifold
- (H, J) generic Floer data.

Morse

$\mathcal{M}(y, \varnothing)$

$$\dot{\gamma}(s) = -\chi(s)\nabla f(\gamma(s))$$

Floer

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$$\frac{\partial u}{\partial s} + J(u)\left(\frac{\partial u}{\partial t} - \chi(s)X_H(u)\right) = 0$$

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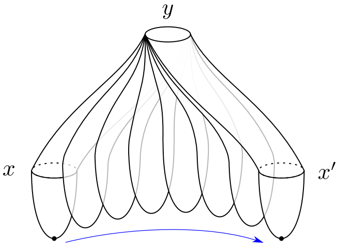

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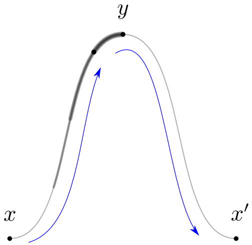

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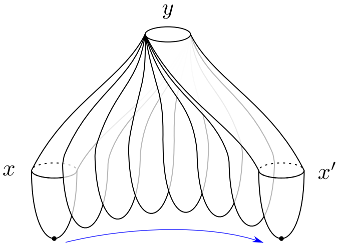

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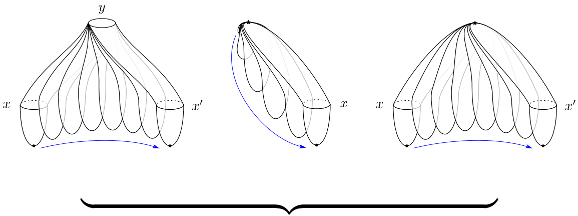

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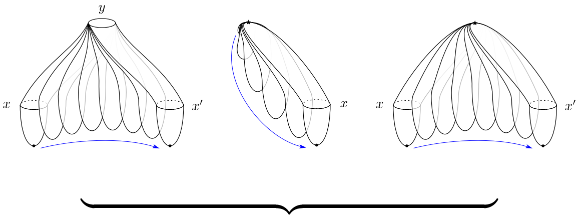

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Novikov replay...

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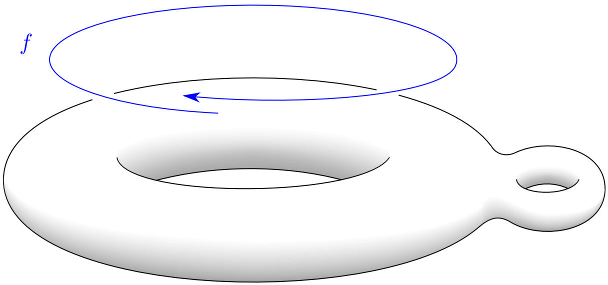
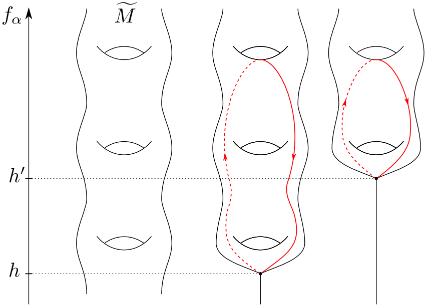


Morse Novikov fundamental group 1

— *Novikov fundamental group* —

- M is a closed manifold,
- α a closed 1-form,
- $\widetilde{M}$ an integration cover for α.



Definition (B,G,G,L)

$$\pi_1(\widetilde{M}, \alpha) = \varprojlim_h \pi_1(\widetilde{M}/\widetilde{M}^{\leq h})$$

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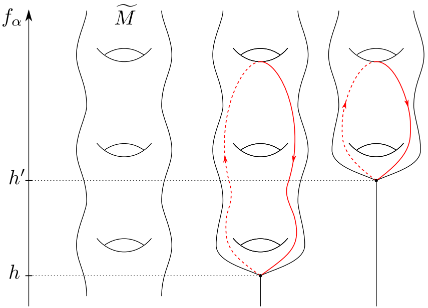

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Morse Novikov fundamental group 2

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$\pi_1(\widetilde{M}, [\alpha])$ only depends on the cohomology class $[\alpha] \in H^1(M)$ and the choice of integration cover $\widetilde{M}$.

If (α, g) is Morse-Smale, $\pi_1(\widetilde{M}, [\alpha])$ "is generated" by the (unstable manifolds of the) index 1 critical points of α.

Main ingredients :

- Difference of primitives of cohomologous forms are bounded.
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— Floer Novikov fundamental group for small flux —

(generators)

Floer Novikov fundamental group 1

- Pick a (non degenerate) symplectic isotopy $\phi = (\phi_t)_{t\in[0,1]}$.
- Let X_t be the vector field generating ϕ.
- Then $\alpha = \int_0^1 \omega(X_t, \cdot)dt$ is a closed 1-form.

Definition

$[\alpha] \in H^1(M)$ is the flux of ϕ.

Theorem

If $[\alpha]$ is small enough, then the components of the moduli spaces $\mathcal{M}(y, \varnothing)$ for $|y| = 1$ "generate" $\pi_1(M, [\alpha])$.

Floer Novikov fundamental group 2

Required ingredients for the construction :

- Genericity ?
- Gluing ?
- Compactness ?

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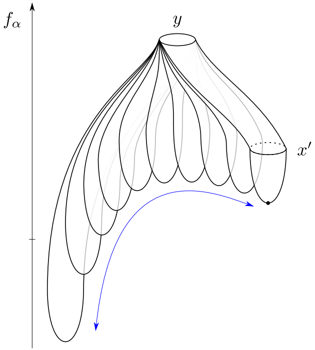

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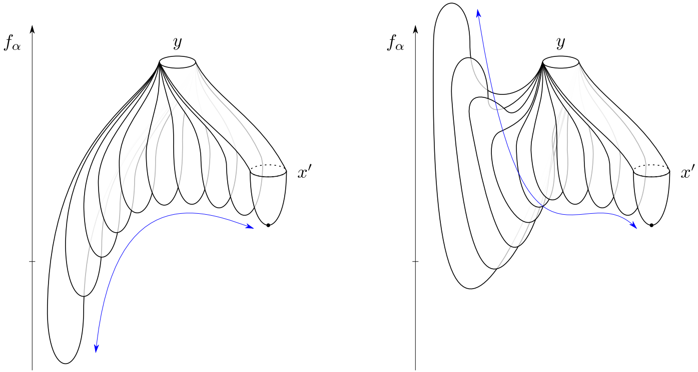

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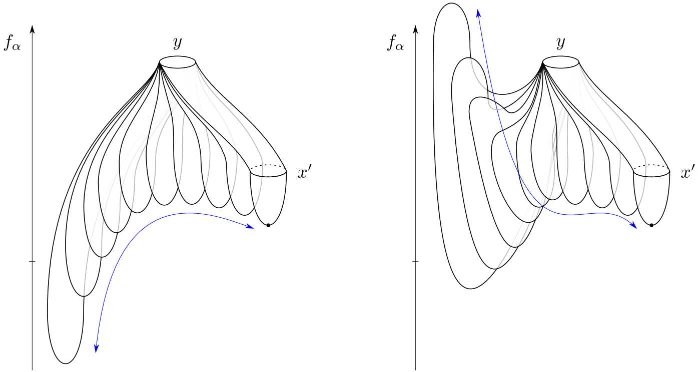


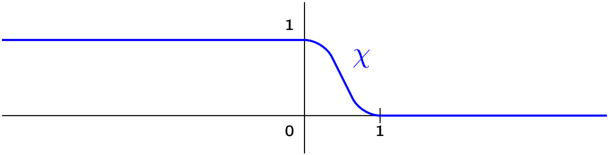
Floer Novikov fundamental group 3

Let $H_{\alpha,t} : \widetilde{M} \to \mathbb{R}$ be a Hamiltonian function on $\widetilde{M}$ generating the isotopty

$$dH_{\alpha,t} = -\omega(X_t, \cdot).$$

Action : $\mathcal{A}(\tilde{\gamma}) = -\iint \tilde{\gamma}^*\omega + \int_0^1 H_t(\gamma(t))dt$
Floer equation for $\mathcal{M}(y, \varnothing)$:

$$\frac{\partial u}{\partial s} + J(u)\left(\frac{\partial u}{\partial t} - \chi(s)X_t(u)\right) = 0$$



$$E(u) = \iint \|\frac{\partial u}{\partial s}\|^2 dsdt$$

Floer Novikov fundamental Group 4

$$E(u) = \mathcal{A}(y) - \iint_{[-1,1]\times[0,1]} H_t(u(s,t))\ |\chi'(s)| dsdt$$

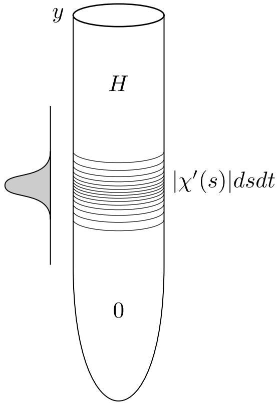
Lemma

If the flux is small enough, then whenever a sequence (u_n) in $\mathcal{M}(y,\emptyset)$ is such that

$$\lim_{n\to+\infty} E(u_n) = +\infty,$$

we have

$$\lim_{n\to+\infty} H(u_n(+\infty)) = -\infty$$

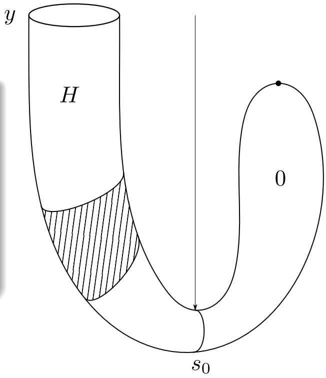


Floer Novikov fundamental Group 5

Lemma

There is a constant K_1, such that for every $u \in \mathcal{M}(y, \varnothing)$, there is at least one $s_0 \geq 1$ for which

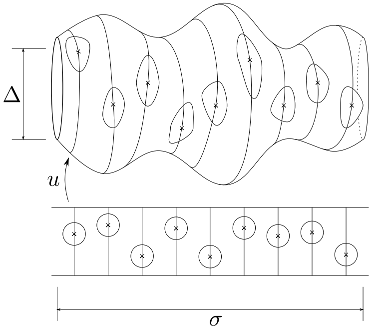
$$\forall t \in \mathbb{S}^1, H_t(u(s_0, t)) \leq \mathcal{A}(y) - E(u)/2 + K_1.$$



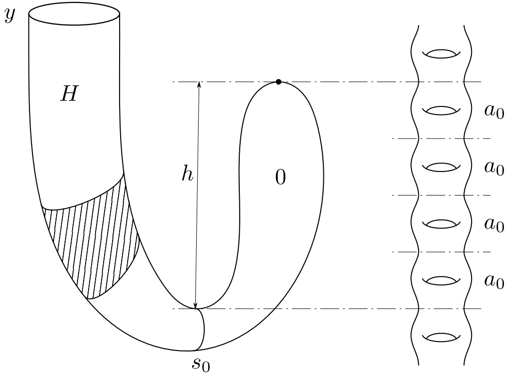
Floer Novikov fundamental group 6

Sketch of proof :

1. $\bar{H}(s) = \int_0^1 H_t(u(s,t))dt$ has to be low on a large annulus (Cauchy-Schwarz).
2. if the magnitude is large for each s, we can find many disjoint discs with large derivative at center, which contain a lot of energy.



Floer Novikov fundamental group 7



Number of "slices" $N \sim \frac{h}{\|[\alpha]\|}$, and $E(u) \geq N a_0$, so $h \lesssim \frac{\|[\alpha]\|}{a_0} E(u)$:

$$\bar{H}(+\infty) \lesssim \mathcal{A}(y) - \left(\frac{1}{2} - \frac{\|\alpha\|}{a_0}\right) E(u)$$

Floer Novikov fundamental group 7

If the flux is small engough :

$$\lim_{n\to+\infty} E(u_n) = +\infty \quad \Longrightarrow \quad \lim_{n\to+\infty} H(u_n(+\infty)) = -\infty$$

i.e. the moduli spaces $\mathcal{M}(y, \varnothing)$ are "compact above every level".

- Can define "Floer loops above level h".
- Compare with Morse loops (requires hybrid moduli spaces $\mathcal{M}(\bullet\!\!\longrightarrow\!\!\!\!-\!\!\!\!-(\,\vdots\;\;\vdots\,))$).
- Floer loops "generate" $\pi_1(\widetilde{M}, \alpha)$.