Potential estimates for solutions to quasilinear elliptic problems with general growth. Scalar and vectorial case

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Nonlinear Potential Theoretic Methods in PDE
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Goals

We study

$$-\mathrm{div}\mathcal{A}(x, Du) = \mu \quad \text{in} \;\; \Omega \subset \mathbb{R}^n$$

with bounded measure μ and Carathéodory's function $\mathcal{A}$ having Orlicz growth (it's family of nonlinear operators including Δ and Δ_p).

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VECTORIAL:

C., Youn, Zatorska–Goldstein, Wolff potentials and measure data vectorial problems with Orlicz growth, arXiv:2102.09313

C., Y., Z.–G., Measure data systems with Orlicz growth, arXiv:2106.11639

Measure data problems with power growth

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$$G(x) = c_{n,p}\left(|x|^{\frac{p-n}{p-1}} - 1\right) \text{ if } 1 < p < n,$$

which does not belong to $W_0^{1,p}(B(0,1))$, for small p, but we like it!

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One may study various kids of very weak solutions:
SOLA (Boccardo&Gallouët '89), renormalized solutions (DiPerna&Lions '89, Boccardo, Giachetti, Diaz, Murat '93), entropy solution (Bénilan, Boccardo, Gallouët, Gariepy, Pierre, Vazquez, Murat '95), or (Kilpeläinen, Kuusi, Tuhola-Kujanpää '11) $\mathcal{A}$-superharmonic functions.

Be careful: if $1 < p < 2 - \frac{1}{n}$, then it is possible that $u \notin W^{1,1}_{loc}$.

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Scalar problem

μ is a bounded measure, $\mathcal{A} : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is a monotone Carathéodory's function, $G \in C^1((0,\infty))$ is a nonnegative, increasing, and convex function such that $G \in \Delta_2 \cap \nabla_2$ and

$$\begin{cases} c_1^{\mathcal{A}} G(|\xi|) \leqslant \mathcal{A}(x,\xi)\cdot\xi, \\ |\mathcal{A}(x,\xi)| \leqslant c_2^{\mathcal{A}} g(|\xi|), \end{cases}$$

where g is the derivative of G.

Who can be called 'a solution'?

$\mathcal{A}$-harmonicity

A <u>continuous</u> function $u \in W^{1,G}_{loc}(\Omega)$ is an $\mathcal{A}$-*harmonic* function in an open set Ω if it is a (weak) solution to $-\mathrm{div}\mathcal{A}(x, Du) = 0$.

$\mathcal{A}$-super/subharmonicity

We say that a lower semicontinuous function u is $\mathcal{A}$-superharmonic if for any $K \Subset \Omega$ and any $\mathcal{A}$-harmonic $h \in C(\overline{K})$ in K, $u \geqslant h$ on ∂K implies $u \geqslant h$ in K (u is $\mathcal{A}$-subharmonic if $(-u)$ is $\mathcal{A}$-superharmonic).

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- is defined everywhere,
- can be unbounded,
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This guy we want to 'control by a potential' and prove its regularity.

Potential estimate in the linear case 1/2

Global case

If u solves $-\Delta u = \mu$ in $\mathbb{R}^n$, then

$$u(x) = \int_{\mathbb{R}^n} G(x,y)\, d\mu(y)$$

with Green's function

$$G(x) = \frac{c_n}{|x-y|^{n-2}} \quad \text{if } n > 2,$$

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so it can be estimated as follows

$$|u(x)| \lesssim \int_{\mathbb{R}^n} \frac{d|\mu|(y)}{|x-y|^{n-2}} =: \mathrm{I}_2(|\mu|)(x) \quad \Leftarrow \text{Riesz potential}$$

Potential estimate in the linear case 2/2

Local behaviour of solutions to $-\Delta u = \mu$

Localized/trucated Riesz potential of a nonnegative measure

$$\mathrm{I}_2^{\mu}(x,R) := \int_0^R \frac{|\mu|(B_\varrho(x))}{\varrho^{n-2}}\frac{d\varrho}{\varrho} \lesssim_n \int_{B_R(x)} \frac{d|\mu|(y)}{|x-y|^{n-2}}$$

$$\leqslant \int_{\mathbb{R}^n} \frac{d|\mu|(y)}{|x-y|^{n-2}} = \mathrm{I}_2(|\mu|)(x) \quad \Leftarrow \text{Riesz potential}$$

Then locally

$$|u(x)| \leqslant C\left(\mathrm{I}_2^{\mu}(x,R) + \text{`}\textit{sth not that much important}\text{'}\right).$$

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$-\Delta_p u = -\mathrm{div}(|Du|^{p-2}Du) = \mu$ **for** $1 < p < \infty$

Expecting

$$|u(x)| \leqslant C\left(\mathcal{W}_p^{\mu}(x,R) + \text{`}sth(u,R)\ not\ that\ much\ important\text{'}\right),$$

we have to employ another potential

$$\mathcal{W}_p^{\mu}(x,R) = \int_0^R \left(\frac{|\mu|(B_\varrho(x))}{\varrho^{n-1}}\right)^{\frac{1}{p-1}} d\varrho$$

called Wolff potential (similar ones were considered by Havin & Maz'ya).

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For $p = 2$ we are back with Riesz potential.

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Kilpeläinen & **Malý** ['92,'94] proven that for $\mu \geqslant 0$ we actually have

$$\mathcal{W}_p^{\mu}(x,R) \lesssim u(x) \lesssim \mathcal{W}_p^{\mu}(x,2R) + \text{`}sth(u,R)'$$

next proofs: Trudinger & Wang [2002] and Korte & Kuusi [2010]

Estimates for scalar $\mathcal{A}$-superharmonic functions

Theorem by C, Giannetti, Zatorska-Goldstein, arXiv:2006.02172

Assume that u is a nonnegative function being $\mathcal{A}$-superharmonic and finite a.e. in $B(x_0, R_{\mathcal{W}}) \Subset \Omega$ for some $R_{\mathcal{W}}$, μ_u is generated by u and $g = G'$. Let (Havin-Mazy'a-)Wolff potential be given by

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Then for $R \in (0, R_{\mathcal{W}}/2)$ we have

$$C_L\left(\mathcal{W}_G^{\mu_u}(x_0, R) - R\right) \leqslant u(x_0) \leqslant C_U\left(\inf_{B(x_0,R)} u(x) + \mathcal{W}_G^{\mu_u}(x_0, R) + R\right)$$

with $C_L, C_U > 0$ depending only on parameters $i_G, s_G, c_1^{\mathcal{A}}, c_2^{\mathcal{A}}, n$.

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* Similar upper bound was proven by **Malý** in 2003 for $\mathcal{A}$-superminimizer.

Consequences

Quick remarks

- The result is sharp as the same potential controls bounds from above and from below.
- Let $u \geqslant 0$ be $\mathcal{A}$-superharmonic, finite a.e., $\mu_u := -\mathrm{div}\mathcal{A}(x, Du)$. Then u is continuous in $x_0 \iff \mathcal{W}_G^{\mu_u}(x, r)$ is small for $x \in B_{x_0}(r)$.

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Orlicz version of Hedberg–Wolff Theorem

Let μ be a nonnegative bounded measure compactly supported in bounded open set $\Omega \subset \mathbb{R}^n$. Then

$$\mu \in (W_0^{1,G}(\Omega))' \quad \iff \quad \int_\Omega \mathcal{W}_G^{\mu}(x, R)\, d\mu(x) < \infty \text{ for some } R > 0.$$

Fundamental solution

for operators of Zygmund growth

Suppose that $1 < p < n$, $\alpha \in \mathbb{R}$, $0 < a \in L^\infty(\Omega)$ separated from zero, and u is a nonnegative $\mathcal{A}$-superharmonic function in Ω, such that

$$-\mathrm{div}\mathcal{A}(x, Du) = -\mathrm{div}\left(a(x)|Du|^{p-2}\log^\alpha(\mathrm{e} + |Du|)Du\right) = \delta_0$$

in the sense of distributions. Then

$$c^{-1}|x|^{-\frac{n-p}{p-1}}\log^{-\frac{\alpha}{p-1}}(\mathrm{e}+|x|) \leqslant u(x) \leqslant c\left(|x|^{-\frac{n-p}{p-1}}\log^{-\frac{\alpha}{p-1}}(\mathrm{e}+|x|) + \inf_{B(x,2|x|)} u\right).$$

Lorentz data $\implies$ continuity of solutions

Let u be a nonnegative $\mathcal{A}$-superharmonic function in Ω and $F_u := -\mathrm{div}\mathcal{A}(x, Du)$ in the sense of distributions. If F_u satisfies

$$\int_0^\infty t^{\frac{1}{n}} g^{-1}\left(t^{\frac{1}{n}} F_u{}^{**}(t)\right) \frac{dt}{t} < \infty$$

for $\Omega_0 \Subset \Omega$, then $\underline{u \in C(\Omega_0)}$.

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p-Laplace case
If u is nonnegative & p-superharmonic, $p > 1$, and $F_u \in L(\frac{n}{p}, \frac{1}{p-1})(\Omega)$, then u is continuous.

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Zygmund-growth operator case

If $u \geqslant 0$, $-\mathrm{div}\left(a(x)|Du|^{p-2}\log^{\alpha}(\mathrm{e} + |Du|)Du\right) = F_u \geqslant 0$, $p > 1$, $\alpha \in \mathbb{R}$, and F_u is as above with $g^{-1}(\lambda) \simeq \lambda^{\frac{1}{p-1}} \log^{-\frac{\alpha}{p-1}}(\mathrm{e} + \lambda)$, then u is continuous.

Morrey data $\iff$ Hölder continuity of solutions

Consider the density condition

$$\mu_\theta(B(x,r)) \leqslant c r^{n-1} g(r^{\theta-1}) \simeq r^{n-\theta} G(r^{\theta-1}). \qquad \text{(M)}$$

Suppose $u \geqslant 0$ is $\mathcal{A}$-superharmonic and $\mu_u := -\mathrm{div}\mathcal{A}(x, Du)$.

- If $u \in C^{0,\theta}_{loc}(\Omega)$ with certain $\theta \in (0,1)$, then μ satisfies (M).
- If μ_θ satisfies (M) for $\theta \in (0,1)$, then u is locally Hölder continuous.

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(M) reads $\mu(B(x,r)) \leqslant cr^{n-p+\theta(p-1)}$

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* we provide natural Marcinkiewicz-type characterization relating to $\mu \in L(\frac{n}{p+\theta(p-1)},\infty)(\Omega)$ for some $\theta \in (0,1)$ implying that μ satisfies (M) and consequently Hölder continuity of a solution.

Methods

for scalar equations

Harmonic analysis

a range of generalized harmonic tools (Maximum principle, Harnack inequality, Poisson modification) prepared for generalized Orlicz framework in *[C, Zatorska-Goldstein, Generalized superharmonic functions with strongly nonlinear operator, Potential Analysis]*

- Björn, Björn, Nonlinear potential theory on metric spaces, 2011

Wolff potential estimates

influential for our proof: Trudinger&Wang 2002, Korte&Kuusi 2010, for regularity consequences: Kuusi&Mingione 2014.

Let's go to systems

Literature on existence and regularity to systems

p-growth

- weak solutions studied from 60's
  (Uraltseva, Uhlenbeck,...)
- measure data systems
  J. M. Rakotoson 1993, Lewis 1993, Fuchs and Reuling 1995,
  Leonetti and Petricca (a few),

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OPEN

uniqueness for measure data problems

Vectorial problem

Notion of solutions * Solutions Obtained as a Limit of Approximation (SOLA)

A map $\boldsymbol{u} \in W_0^{1,1}(\Omega, \mathbb{R}^m)$ such that $\int_\Omega g(|D\boldsymbol{u}|)\,dx < \infty$ is called a SOLA to (S), if there exists a sequence $(\boldsymbol{u}_h) \subset W^{1,G}(\Omega, \mathbb{R}^m)$ of local energy solutions to the systems

$$-\mathbf{div}\mathcal{A}(x, D\boldsymbol{u}_h) = \boldsymbol{\mu}_h$$

such that $\boldsymbol{u}_h \to \boldsymbol{u}$ locally in $W^{1,1}(\Omega, \mathbb{R}^m)$ and $(\boldsymbol{\mu}_h) \subset L^\infty(\Omega, \mathbb{R}^m)$ is a sequence of maps that converges to $\boldsymbol{\mu}$ weakly in the sense of measures and satisfies

$$\limsup |\boldsymbol{\mu}_h|(B) \leqslant |\boldsymbol{\mu}|(B) \qquad \text{for } B \subset \Omega.$$

Measure data systems with Orlicz growth

C., Youn, Zatorska–Goldstein, arXiv:2106.11639

Assume that $\mathcal{A} : \Omega \times \mathbb{R}^{n\times m} \to \mathbb{R}^{n\times m}$ is strictly monotone, $\mathcal{A}(x,0) = 0$, and $\mathcal{A}$ satisfies the following conditions

$$\mathcal{A}(x,\xi) : \xi \geqslant c_1 G(|\xi|), \qquad |\mathcal{A}(x,\xi)| \leqslant c_2 \left(g(|\xi|) + b(x) \right),$$

for some $b \in L^{\widetilde{G}}(\Omega)$. Furthermore, we require $\mathcal{A}$ to satisfy

$$\mathcal{A}(x,\xi) : \big((\mathrm{Id} - w \otimes w)\xi \big) \geqslant 0$$

for a.a. $x \in \Omega$, all $\xi \in \mathbb{R}^{n\times m}$, and every vector $w \in \mathbb{R}^m$ with $|w| \leqslant 1$.

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We show existence for *approximable solution* $\boldsymbol{u}$ and $\boldsymbol{\mu}$ with bounded TV.

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$$\mathcal{A}(x,\xi) : \big((\mathrm{Id} - w \otimes w)\xi \big) \geqslant 0$$

for a.a. $x \in \Omega$, all $\xi \in \mathbb{R}^{n\times m}$, and every vector $w \in \mathbb{R}^m$ with $|w| \leqslant 1$.

We show existence for *approximable solution* $\boldsymbol{u}$ and $\boldsymbol{\mu}$ with bounded TV. If G grows 'slowly' ($\approx p < n$), we provide Marcinkiewicz-type regularity for $|\boldsymbol{u}|$ and $|D\boldsymbol{u}|$.

Measure data systems with Orlicz growth

C., Youn, Zatorska–Goldstein, arXiv:2106.11639

Assume that $\mathcal{A} : \Omega \times \mathbb{R}^{n\times m} \to \mathbb{R}^{n\times m}$ is strictly monotone, $\mathcal{A}(x,0) = 0$, and $\mathcal{A}$ satisfies the following conditions

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[Balci, Cianchi, Diening, Maz'ya, 'A pointwise differential inequality...' Math Ann, to appear]

Assumptions for potential estimates

Vectorial problem

We investigate solutions $\boldsymbol{u} : \Omega \to \mathbb{R}^m$ to the problem

$$\begin{cases} -\mathbf{div}\mathcal{A}(x, D\boldsymbol{u}) = \boldsymbol{\mu} \quad \text{in} \quad \Omega, \\ \boldsymbol{u} = 0 \quad \text{on} \quad \partial\Omega \end{cases} \tag{S}$$

with a datum $\boldsymbol{\mu}$ being a vector-valued bounded Radon measure, $G \in C^2((0,\infty)) \cap C(\mathbb{R}_+)$, $g = G'$ is increasing and $g \in \Delta_2 \cap \nabla_2$, and $\mathcal{A} : \Omega \times \mathbb{R}^{n\times m} \to \mathbb{R}^{n\times m}$ is assumed to admit a form

$$\mathcal{A}(x,\xi) = a(x)\frac{g(|\xi|)}{|\xi|}\,\xi,$$

with continuous weight $a : \Omega \to [c_a, C_a]$, $0 < c_a < C_a$.

Estimates for SOLA to the vectorial problem

Theorem by C, Youn, Zatorska-Goldstein, arXiv:2102.09313

Suppose $\boldsymbol{u} : \Omega \to \mathbb{R}^m$ is a local SOLA to $-\mathbf{div}\mathcal{A}(x, D\boldsymbol{u}) = \boldsymbol{\mu}$ with $\mathcal{A}$ as prescribed, and $\boldsymbol{\mu}$ is bounded. Let $B_r(x_0) \Subset \Omega$ with $r < R_0$ for some $R_0 = R_0(data)$. If $\mathcal{W}^{\boldsymbol{\mu}}_G(x_0, r)$ is finite, then x_0 is a Lebesgue's point of $\boldsymbol{u}$ and

$$|\boldsymbol{u}(x_0) - (\boldsymbol{u})_{B_r(x_0)}| \leqslant C \left(\mathcal{W}^{\boldsymbol{\mu}}_G(x_0, r) + \fint_{B_r(x_0)} |\boldsymbol{u} - (\boldsymbol{u})_{B_r(x_0)}| \, dx \right)$$

holds for $C > 0$ depending only on *data*. In particular, we have the following pointwise estimate

$$|\boldsymbol{u}(x_0)| \leqslant C \left(\mathcal{W}^{\boldsymbol{\mu}}_G(x_0, r) + \fint_{B_r(x_0)} |\boldsymbol{u}(x)| dx \right).$$

p-Laplace problem: [Kuusi&Mingione, JEMS 2018]

Consequences 1/2

VMO criterion

Let $\boldsymbol{u}$ be a SOLA to $-\mathbf{div}\mathcal{A}(x, D\boldsymbol{u}) = \boldsymbol{\mu}$ and let $B_r(x_0) \Subset \Omega$. If

$$\lim_{\varrho \to 0} \varrho\, g^{-1}\left(\frac{|\boldsymbol{\mu}|(B_\varrho(x_0))}{\varrho^{n-1}}\right) = 0,$$

then $\boldsymbol{u}$ has vanishing mean oscillations at x_0, i.e.
$\lim_{\varrho \to 0} \fint_{B_\varrho(x_0)} |\boldsymbol{u} - (\boldsymbol{u})_{B_\varrho(x_0)}|\, dx = 0.$

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Continuity criterion

Suppose $\boldsymbol{u}$ be a SOLA to $-\mathbf{div}\mathcal{A}(x, D\boldsymbol{u}) = \boldsymbol{\mu}$ and $B_r(x_0) \Subset \Omega$. If $\lim_{\varrho \to 0} \sup_{x \in B_r(x_0)} \mathcal{W}_G^{\boldsymbol{\mu}}(x, \varrho) = 0$, then $\boldsymbol{u}$ is continuous in $B_r(x_0)$.

Consequences 1/2

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$\Longrightarrow$ any $\mathcal{A}$-harmonic map is continuous

Consequences 2/2

the same what for the scalar equation results from an upper bound

Lorentz data $\Longrightarrow$ continuous solutions

For $-\mathbf{div}\mathcal{A}(x, D\boldsymbol{u}) = \boldsymbol{F}$ let $f = |\boldsymbol{F}|$. If $\int_0^\infty t^{\frac{1}{n}} g^{-1}\big(t^{\frac{1}{n}} f^{**}(t)\big)\,\frac{dt}{t} < \infty$, then a SOLA $\boldsymbol{u}$ is continuous.

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Morrey data $\Longrightarrow$ Hölder continuous solutions

If $\boldsymbol{u}$ is a SOLA to $-\mathbf{div}\mathcal{A}(x, D\boldsymbol{u}) = \boldsymbol{\mu}_\theta$ and $|\boldsymbol{\mu}_\theta|(B(x,r)) \leqslant c r^{n-1} g(r^{\theta-1})$, then $\boldsymbol{u}$ is locally Hölder continuous.

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For $-\mathbf{div}\mathcal{A}(x, D\boldsymbol{u}) = \boldsymbol{F}$ let $f = |\boldsymbol{F}|$. If $\int_0^\infty t^{\frac{1}{n}} g^{-1}\big(t^{\frac{1}{n}} f^{**}(t)\big)\,\frac{dt}{t} < \infty$, then a SOLA $\boldsymbol{u}$ is continuous.

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$+$ natural Marcinkiewicz-type characterization relating to $\mu \in L(\frac{n}{p+\theta(p-1)}, \infty)$, $\theta \in (0,1)$, implying local Hölder continuity of solutions

Methods

for systems

main tool: $\mathcal{A}$-harmonic approximation lemma

the approximation of a $W^{1,G}$-function by an $\mathcal{A}$-harmonic map for weighted operator $\mathcal{A}$ of an Orlicz growth being a generalized version of p-harmonic version from *[Kuusi&Mingione, JEMS 2018]*

OPEN

subquadratic case

more general structure of the operator

Off-topic

Off-topic

Partial Differential Equations in Anisotropic Musielak-Orlicz Spaces
by **C, Gwiazda, Świerczewska-Gwiazda, Wróblewska-Kamińska**,
is to appear in series Springer Monographs in Mathematics very soon

Thank you for your attention!

see https://www.mimuw.edu.pl/~ichlebicka/publications