Entropy quantities associated to Gauss curvature type flows

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Workshop on ”Nonlinear Potential Theoretic Methods in Partial Differential Equations”
BIRS, 6-10 September, 2021

$\Omega \subset \mathbb{R}^{n+1}$ bounded convex domain, $M = \partial\Omega$.
$X : M \to \mathbf{R^{n+1}}$ position vector, $K(x)$ the Gauss curvature.

The Gauss curvature flow:

$$\frac{\partial X(x,t)}{\partial t} = -K(x,t)\nu \tag{1.1}$$

Introduced and studied under symmetry by W. Firey;

Existence, convergence to a point z_∞ after finite time. (K.S. Chou).

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The **shape** of z_∞?

The normalization (volume preserved to be $|B_1|$):

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For flow (1.2),

- convergent to sphere when $n = 2$, (Andrews).
- convergent to a soliton for $n \geq 3$: $K = u$, (Guan-Ni).
- Soliton is the unit sphere! (Brendle-Choi-Daskopolous)

Entropy functionals

$\Omega \subset \mathbb{R}^{n+1}$ bounded closed convex body, $\forall z_0 \in \Omega$, the support function with respect to z_0

$$u_{z_0}(x) \doteqdot \sup_{z \in \Omega} \langle x, z - z_0 \rangle.$$

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Define two *entropy* functional $\mathscr{E}(\Omega)$, $\mathscr{C}(\Omega)$:

$$\mathscr{E}(\Omega) \doteqdot \sup_{u_{z_0} > 0} \fint_{\mathbb{S}^n} \log u_{z_0}(x)\, d\theta(x). \quad \text{(Firey, Andrews)}$$

$$\mathscr{C}(\Omega) \doteqdot \fint_{\mathbb{S}^n} K \log K\, d\theta(x). \quad \text{(Chow)}$$

Guan-Ni:

- $$\mathscr{E}(\Omega) \geq \frac{\log V(\Omega) - \log V(B_1)}{n+1}.$$
- $$\mathscr{E}(\Omega) \leq \mathscr{C}(\Omega), \quad \text{"="} \text{ if } u \equiv K.$$

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- $\rho_-(\Omega)$ inner radius, $\rho_+(\Omega)$ outer radius,

$$\rho_+(\Omega) \leq C_n e^{\mathscr{E}(\Omega)}, \;\; \rho_-(\Omega) \geq C_n' V(\Omega) e^{-n\mathscr{E}(\Omega)}.$$

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- The entropy point: there exists a unique $z_e(\Omega) \in \mathrm{Int}(\Omega)$
$$\mathscr{E}(\Omega) = ⨏_{\mathbf{S^n}} \log u_{z_e}(x), \; \int_{\mathbb{S}^n} \frac{x_j}{u_e(x)} d\theta(x) = 0.$$

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- $\exists \delta(\rho_+(\Omega), V(\Omega), n) > 0, d(z_e(\Omega), \partial\Omega) \geq \delta.$

Along flow (1.2),

- $\mathscr{E}(\Omega_t), \mathscr{C}(\Omega_t)$ monotone non-increasing.
- $\rho_+(\Omega(t)) \le C_1, \;\; \rho_-(\Omega(t)) \ge C_2 > 0.$

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- $0 < C_3 \le K_{\Omega(t)} \le C_4$.
- (1.2) is a quasi-gradient flow.
- Flow (1.2) converges to a soliton $u = K$.
- The soliton is a critical point of $\mathscr{E}(\Omega)$ under the constraint $|\Omega)| = |B_1|$.

Flow by power of Gauss curvature

Consider

$$X_t = -K^{\alpha}\nu, \quad \alpha > 0. \tag{1.3}$$

$$\begin{aligned} \mathscr{C}_\alpha(\Omega) : &= \frac{\alpha}{\alpha-1}\log\left(\oint_{\mathbb{S}^n} K^{\alpha-1}\, d\theta(x)\right) \\ \mathscr{E}_\alpha(\Omega) : &= \sup_{z_0\in\Omega} \mathscr{E}_\alpha(\Omega, z_0), \end{aligned}$$

where

$$\mathscr{E}_\alpha(\Omega, z_0) := \frac{\alpha}{\alpha-1}\log\left(\oint_{\mathbb{S}^n} u_{z_0}(x)^{1-\frac{1}{\alpha}}\, d\theta(x)\right).$$

Andrews-Guan-Ni:

- $\forall z_0 \in \mathrm{Int}(\Omega)$, $\mathscr{E}_\alpha(\Omega, z_0)$ is continuous and increasing in α.
- If $|\Omega| = |B(1)|$, $\forall \alpha > \frac{1}{n+2}$,

$$\mathscr{E}_\alpha(\Omega) \geq 0, \quad \text{equality iff } \Omega \text{ is a ball..}$$

- $\exists!\ z_e \in \mathrm{Int}(\Omega)$ such that $\mathscr{E}_\alpha(\Omega) = \mathscr{E}_\alpha(\Omega, z_e)$,

$$\int_{\mathbb{S}^n} \frac{x_j}{u_{z_e}^{\frac{1}{\alpha}}(x)} d\theta(x) = 0.$$

- $\forall \alpha > \frac{1}{n+2}$, $\exists \beta(\alpha, n) > 0, C(\alpha, n) > 0$, $\forall \Omega$ with $|\Omega| = |B(1)|$,

$$\rho_-(\Omega) \geq C^{-1} e^{-\beta \mathscr{E}_\alpha(\Omega)}, \quad \rho_+(\Omega) \leq C e^{n\beta \mathscr{E}_\alpha(\Omega)}.$$

- $\exists \delta(\rho_+(\Omega), V(\Omega), \alpha, n) > 0$,

$$d(z_e(\Omega), \partial\Omega) \geq \delta.$$

Normalized flow

$$\frac{\partial}{\partial t}X(x,t) = -\frac{K^{\alpha}(x,t)}{\oint_{\mathbb{S}^n} K^{\alpha-1}}\nu(x,t) + X(x,t). \tag{1.4}$$

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- Under the normalized flow (1.4), $\mathscr{C}_\alpha(\Omega(t))$ and $\mathscr{E}_\alpha(\Omega(t))$ are non-increasing.
- $\forall \alpha \geq \frac{1}{n+2}$, $\mathscr{E}_\alpha^\infty := \lim_{t\to\infty} \mathscr{E}_\alpha(\Omega_t)$ exists,
- $$\mathscr{E}_\alpha^\infty - \mathscr{E}_\alpha(\Omega(t_0)) \leq -\int_{t_0}^\infty \left[\frac{\int_{\mathbb{S}^n} f^{1+\frac{1}{\alpha}}\, d\sigma_t \cdot \int_{\mathbb{S}^n} d\sigma_t}{\int_{\mathbb{S}^n} f^{\frac{1}{\alpha}}\, d\sigma_t \cdot \int_{\mathbb{S}^n} f\, d\sigma_t} - 1\right] dt \leq 0.$$

  Here $f(x,t) = \frac{K^\alpha(x,t)}{u(x,t)}$, $d\sigma_t(x) = \frac{u(x,t)}{K(x,t)}\, d\theta(x)$.

Convergence of (1.4)

$\forall \alpha \geq \frac{1}{n+2}$, along flow (1.4),

- $u \geq c_0$ independent of t.
- $0 < C_3 \leq K_{\Omega(t)} \leq C_4$.
- (1.4) is a quasi-gradient flow.
- Flow (1.4) converges to a soliton $\eta u = K^{\alpha}, \ \eta = \int_{\mathbb{S}^n} K^{\alpha-1}$.
- The soliton is a critical point of $\mathscr{E}_{\alpha}(\Omega)$ under the constraint $|\Omega)| = |B_1|$.

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Classification of solitons:

- $\alpha = \frac{1}{n+2}$, solitons are ellipsoids. (Andrews)
- $\forall\alpha > \frac{1}{n+2}$, soliton is the sphere. (Brendle-Choi-Daskopolous)

Anisotropic flows

Joint work of Andrews-Boroczky-Guan-Ni.

$$X_t = -f^{\alpha}(\nu)K^{\alpha}\nu, \ \alpha > 0, \ \ 0 < f \in C^2(\mathbb{S}^n). \tag{2.1}$$

The normalized flow

$$X_t = -\frac{f^{\alpha}(\nu)K^{\alpha}}{\oint_{\mathbb{S}^n} f^{\alpha}K^{\alpha-1}}\nu + X. \tag{2.2}$$

Entropy functionals

$$\mathscr{C}_{\alpha,f}(\Omega) := \frac{\alpha}{\alpha-1}\log\left(\oint_{\mathbb{S}^n} K^{\alpha-1}f(x)d\theta(x)\right).$$

$$\mathscr{E}_{\alpha,f}(\Omega) := \sup_{z_0\in\Omega}\frac{\alpha}{\alpha-1}\log\left(\oint_{\mathbb{S}^n} u_{z_0}(x)^{1-\frac{1}{\alpha}}f(x)d\theta(x)\right).$$

If $V(\Omega)=V(B(1))$, $\forall\alpha>\frac{1}{n+2}$, $\exists\beta(\alpha,n,f)>0,\ C(\alpha,n,f)>0$, $\forall\Omega$,

$$\rho_-(\Omega)\geq C^{-1}e^{-\beta\mathscr{E}_{\alpha,f}(\Omega)},\ \ \rho_+(\Omega)\leq Ce^{n\beta\mathscr{E}_{\alpha,f}(\Omega)}.$$

$$\mathscr{E}_{\alpha,f}(\Omega_{t_2},z)-\mathscr{E}_{\alpha,f}(\Omega_{t_1},z=\int_{t_1}^{t_2}\left(\frac{\oint_{\mathbb{S}^n}h^{\alpha+1}(x,t)\,d\sigma_t}{\oint_{\mathbb{S}^n}h(x,t)\,d\sigma_t\cdot\oint_{\mathbb{S}^n}h^{\alpha}(x,t)\,d\sigma_t}-1\right)dt\leq 0,$$

with equality if and only if $h(x,t)=const$.
$h(x,t)\doteqdot f(x)u_z^{-\frac{1}{\alpha}}(x,t)K(x,t)$, $d\sigma_t(x)=\frac{u_z(x,t)}{K(x,t)}d\theta(x)$ which satisfies that $\oint_{\mathbb{S}^n}d\sigma_t(x)=\oint_{\mathbb{S}^n}d\theta(x)=1$.

Non-collapsing estimates

If $V(\Omega) = V(B(1))$, $\forall \alpha > \frac{1}{n+2}$, $\exists \beta(\alpha, n, f) > 0$, $C(\alpha, n, f) > 0$, $\forall \Omega$,

$$\rho_-(\Omega) \geq C^{-1} e^{-\beta \mathscr{E}_{\alpha,f}(\Omega)}, \;\; \rho_+(\Omega) \leq C e^{n\beta \mathscr{E}_{\alpha,f}(\Omega)}.$$

For solution to (2.2) with $V(\Omega(t)) = V(B(1))$ and $\alpha \geq \frac{1}{n+2}$,

$$\rho_+(\Omega_t) \leq C, \quad \rho_-(\Omega_t) \geq \frac{1}{C}, \quad \forall t > 0.$$

$\Omega(t)$ solution to the un-normalized flow (2.1),

$$\frac{\rho_+(\Omega_t)}{\rho_-(\Omega_t)} \leq C, \quad 0 < K(t) \leq \frac{C}{\rho_+^n(\Omega_t)}.$$

As a consequence, solution to (2.2) sequentially converges to

$$\eta u = f^{\alpha}(\nu)K^{\alpha}, \quad \text{on } \mathbb{S}^n.$$

u is a solution of L^p-Minkowski problem ($p = \frac{1}{\alpha}$)

$$\sigma_n(u_{ij} + u\delta_{ij}) = fu^{-p}, \text{on } \mathbb{S}^n. \tag{2.3}$$

Regularity on f can be weakened.

Normalize $\int_{\mathbb{S}^n} f = \omega_n$, $V(\Omega) = V(B(1))$. Set $\mathrm{diam}\Omega = D$.

$\forall \delta > 0, z \in \mathbb{S}^n$, set $\Xi_{\delta,z} = \{x \in \mathbb{S}^n | \quad | < x, z > | \leq \delta\}$. Let σ be the centroid of Ω.

(i) $0 < p < 1$, $\exists \varepsilon > 0$, $\delta \in (0, \frac{1}{2})$, $D \geq 2$ and

$$\int_{\Xi_{\delta,z}} f \leq (1-\varepsilon)\omega_n, \quad \text{for any } z \in S^n, \text{ then} \quad \int_{S^n} f u_\sigma^{1-p} \geq \gamma_1 (\delta D)^{1-p}.$$

(ii) $p = 1$, $\delta \in (0, \frac{1}{2})$, $D \geq 2$ and

$$\int_{\Xi_{\delta,z}} f \leq \frac{\omega_n}{2n} \text{ for any } z \in S^n, \text{ then, } \frac{1}{\omega_n} \int_{S^n} f \log u_\sigma \geq \frac{1}{2} \log D + \log \delta - \gamma_2.$$

(iii) $1 < p < n+2$, $\zeta \in (0, \frac{1}{2})$, $f \in L^{\frac{n+1}{n+2-p}}(\mathbb{S}^n)$, $D = \mathrm{diam}\Omega$ is large enough to satisfy $D^{\frac{(1-p)(n+1)}{2(n+2-p)}} \leq \zeta$ and

$$\int_{\Xi_{\frac{16n}{\sqrt{D}},z}} f^{\frac{n+1}{n+2-p}} \leq \zeta \text{ for any } z \in S^n, \text{ then, } \int_{S^n} u_\sigma^{1-p} \leq \gamma_3 \zeta^{\frac{n+2-p}{n+1}}.$$

For each $\frac{1}{n+2} < \alpha < \infty$, $p = \frac{1}{\alpha}$. **Under the corresponding assumptions on f as in the previous page**, there is a solution $u \in C^{0,1}(\mathbb{S}^n)$ of (2.3) with $\Omega_u \subset \mathbb{R}^{n+1}$ bounded and non-degenerate.

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If $0 < f \in L^\infty(\mathbb{S}^n)$, then the Gauss curvature of the boundary of Ω is bounded from below and above by two positive constants, and $\partial\Omega \in C^{1,\beta}$ for $0 < \beta < 1$. If $0 < f \in C^\beta(\mathbb{S}^n)$, then $\partial\Omega \in C^{2,\beta}$.

A parabolic proof of Chou-Wang's result with weakened assumption.

Thank you