Khovanov homology and the search for exotic 4-spheres

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Over time, many potential counterexamples have been proposed (manifolds that are homeomorphic to S^4, but not known to be diffeomorphic to it). Many of them were later shown to be standard S^4's.

One strategy for disproof

Find a knot $K \subset S^3$ such that K is not slice (does not bound a smooth disk in B^4) but K bounds a smooth disk in some homotopy ball Z.
Therefore, $Z \ncong B^4$ and $Z \cup B^4$ would be a nontrivial homotopy 4-sphere.

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Note: Gauge theoretic invariants cannot distinguish between sliceness in B^4 and in a homotopy 4-ball. It is unclear whether s can do so.

Outline of the talk

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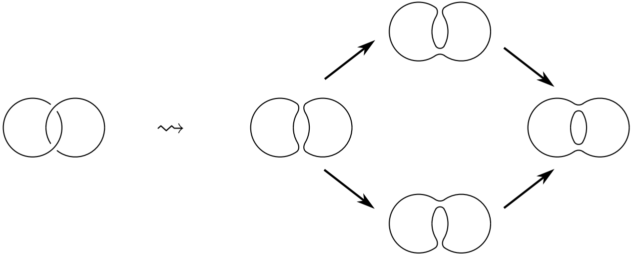
4. A new attempt to pursue the FGMW strategy, using 0-surgery homeomorphisms (**M.-Piccirillo, 2021**).

Khovanov homology

For links $K \subset S^3$, **Khovanov** (1999) defined a homology theory

$$Kh(K) = \bigoplus_{i,j} Kh_{i,j}(K).$$

Its construction involves taking all possible "resolutions" of a link diagram, associating a two-dimensional vector space V to each circle in a resolution, and defining a chain complex using an algebraically-defined differential d:



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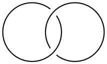

$\rightsquigarrow$

$$\begin{array}{ccccc} & & V\{1\} & & \\ & \overset{d}{\nearrow} & & \overset{d}{\searrow} & \\ V^{\otimes 2} & & & & V^{\otimes 2}\{2\} \\ & \underset{d}{\searrow} & & \underset{d}{\nearrow} & \\ & & V\{1\} & & \end{array}$$

More on Khovanov homology

Its Euler characteristic is the famous Jones polynomial:

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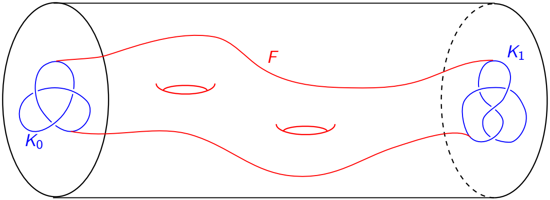
A surface (knot cobordism) $F \subset S^3 \times [0,1]$ from K_0 to K_1 induces a map on Khovanov homology: $Kh(F) : Kh(K_0) \to Kh(K_1)$.



The Rasmussen invariant

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When K is a knot, we have $Kh_{\mathrm{Lee}}(K) \cong \mathbb{Q} \oplus \mathbb{Q}$ in degrees $(0, s-1)$ and $(0, s+1)$, where $s = s(K)$ is *Rasmussen's invariant*.

Using the cobordism maps on spectral sequences, **Rasmussen (2004)** showed that s gives a lower bound for the slice genus

$$\frac{|s(K)|}{2} \leq g_s(K) = \min\{g(\Sigma) \mid \Sigma \subset B^4 \text{ orientable, } \partial\Sigma = K\}$$

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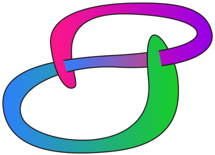

Topological applications

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- the Milnor Conjecture: for a torus knot $T_{p,q}$, we have $g_s(T_{p,q}) = (p-1)(q-1)/2$ (original proof: **Kronheimer-Mrowka, 1993**; new proof: **Rasmussen, 2004**)

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- the existence of topologically slice knots that are not smoothly slice, and hence the existence of exotic smooth structures on $\mathbb{R}^4$ (cf. **Freedman, Donaldson, Casson, Gompf** in the 1980s; new proof: **Rasmussen, 2004**)

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- the *Thom conjecture*: the minimal genus of a surface in the class $d[\mathbb{CP}^1] \in H_2(\mathbb{CP}^2; \mathbb{Z})$ is $(d-1)(d-2)/2$ (original proof: **Kronheimer-Mrowka, 1994**; new proof: **Lambert-Cole, 2018**);

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Question

Can Khovanov homology say something new about 4-manifolds?

Khovanov homology and 4-manifolds

Ideally, we would like to use Khovanov homology to construct 4-manifold invariants. **Morrison-Walker-Wedrich (2019)** proposed a candidate, the *skein lasagna algebra*. So far it can only be computed in simple examples like S^4, disk bundles over S^2, $\mathbb{CP}^2$, $\overline{\mathbb{CP}^2}$; see **M.-Neithalath (2020)**.

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Next: three recent results about the FGMW strategy.

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Remove N and glue it back:

$$X = (S^4 \setminus N) \cup_f N$$

where

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The result is a homotopy 4-sphere X. For many families of 2-knots this is known to be diffeomorphic to S^4, but it is not known in general.

I. A negative result

Theorem (M.-Marengon-Sarkar-Willis, 2019)

If K bounds a smooth disk in a homotopy 4-ball Z obtained from B^4 by a Gluck twist, then $s(K) = 0$. Thus, the FGMW strategy fails for Gluck twists.

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Sketch of proof: We show that if K bounds a null-homologous disk in $\mathbb{CP}^2 \# B^4 = \mathbb{CP}^2 \setminus B^4$, then $s(K) \geq 0$. Similarly, if it bounds a null-homologous disk in $\overline{\mathbb{CP}^2} \setminus B^4$, then $s(K) \leq 0$.

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On the other hand, it was known that if Z is obtained from B^4 by a Gluck twist, then

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Thus, for K as in the hypothesis, we have $s(K) \geq 0$ and $s(K) \leq 0$. □

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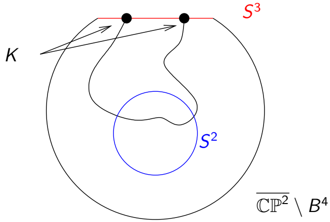

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Sketch of proof, continued: The usual cobordism inequalities in $S^3 \times [0,1]$ (cf. Rasmussen, Beliakova-Wehrli) give

$$s(K) \leq s(F_p) - \chi(C) = s(F_p) + 2g(\Sigma) + 2p - 1.$$

We compute $s(F_p) = 1 - 2p$ (using Hochschild homology), and conclude that when $g(\Sigma) = 0$, we have $s(K) \leq 0$. □

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We showed that if K is H-slice in $\overline{\mathbb{CP}^2}$, then $s(K) \leq 0$. Applying this to the mirror of K, we get that if K is H-slice in $\mathbb{CP}^2$, then $s(K) \geq 0$.

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The argument also extends to connected sums, e.g.: If K is H-slice in $\#^n\mathbb{CP}^2$ for some n, we still have $s(K) \geq 0$.

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Theorem (M.-Marengon-Piccirillo, 2020)

There exist smooth, closed, homeomorphic four-manifolds X and X' and a knot $K \subset S^3$ that is H-slice in X but not in X'.

For example, one can take

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The proof uses gauge theory (the Seiberg-Witten equations).

Sketch of proof

$X = \#3\mathbb{CP}^2\#20\overline{\mathbb{CP}^2}$ and $X' = K3\#\overline{\mathbb{CP}^2}$ are simply connected smooth four-manifolds with the same intersection form, so they are homeomorphic by **Freedman**'s theorem.

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One can explicitly find:

- a null-homologous disk in $\mathbb{CP}^2 \setminus B^4$ (and hence in $X \setminus B^4$) with boundary $K =$ RH trefoil;
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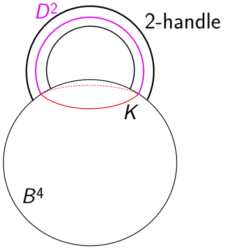
Suppose K bounds a null-homologous disk $\Delta' \subset X' \setminus B^4$. Then $S = \Delta \cup \Delta'$ is an embedded sphere in $K3 \# X' = K3 \# K3 \# \overline{\mathbb{CP}^2}$ with $[S] \neq 0$ but $[S]^2 = 0$. This is impossible by a variant of the adjunction inequality (using the Bauer-Furuta invariants, a stable homotopy refinement of the Seiberg-Witten invariants).

III. A new attempt at pursuing the FGMW strategy

Given a knot K, we can construct a four-manifold $X(K)$ (with boundary), the *trace* of K, by attaching a 2-handle (a neighborhood of a disk) to B^4 along K:



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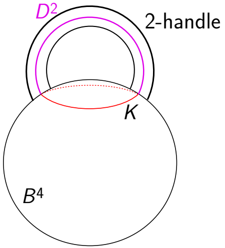

The boundary of $X(K)$ is the 0-surgery on K:

$$S^3_0(K) = (S^3 - \text{nbhd}(K)) \cup (S^1 \times D^2),$$

where the gluing reverses the meridian and longitude of the torus $\partial(\text{nbhd}(K)) = S^1 \times S^1$.

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Caveat: We would like to avoid the case when ϕ extends to a trace diffeomorphism

$$X(K) \xrightarrow{\cong} X(K').$$

Trace diffeomorphisms

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However, for our strategy, this is no good: If $X(K) = X(K')$, then

$$W = V \cup_{S^3_0(K)} (-X(K')) = V \cup_{S^3_0(K)} (-X(K)) = S^4$$

so we do not produce an exotic 4-sphere.

Knots with the same 0-surgeries

Constructions in the literature:

- blowing down two-component links (**Lickorish; 1976**);
- dualizable patterns (**Akbulut, Lickorish, Brakes; 1977-80**);
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In some cases these produce knots with the same traces.

M.-Piccirillo (2021) give a general construction of *all* zero-surgery homeomorphisms $\phi : S^3_0(K) \to S^3_0(K')$ based on certain 3-component links called *RBG links*.

RBG links

An *RBG link* $L = R \cup B \cup G \subset S^3$ is a 3-component rationally framed link, with framings r, b, g respectively, such that there exist homeomorphisms $\psi_B : S^3_{r,g}(R \cup G) \to S^3$ and $\psi_G : S^3_{r,b}(R \cup B) \to S^3$ and such that $H_1(S^3_{r,b,g}(R \cup B \cup G); \mathbb{Z}) = \mathbb{Z}$.

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Theorem (M.-Piccirillo, 2021)

Any RBG link L has a pair of associated knots K_B and K_G and homeomorphism $\phi_L : S^3_0(K_B) \to S^3_0(K_G)$. Conversely, for any 0-surgery homeomorphism $\phi : S^3_0(K) \to S^3_0(K')$ there is an associated RBG link L_ϕ with $K_B = K'$ and $K_G = K$.

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Given L, define K_B to be the image of B under ψ_B, and K_G the image of G under ψ_G. Then $S^3_0(K_B) = S^3_{r,b,g}(L) = S^3_0(K_G)$.

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Given ϕ, let $B = K'$ and $b = 0$. Let μ_K be the meridian for K, and let (R, r) be the framed curve given as the image of $(\mu_K, 0)$ under the homeomorphism ϕ. Finally, let G be the 0-framed meridian for R.

Special RBG links

Definition

A *special RBG link* is a framed 3-component link $L = R \cup B \cup G$ with $b = g = 0$, $r \in \mathbb{Z}$, such that there are isotopies

$$R \cup B \cong R \cup \mu_R \cong R \cup G$$

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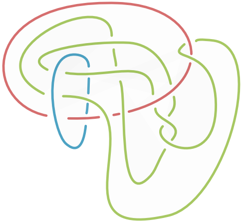
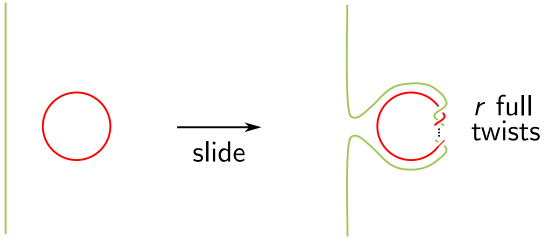


Slides

From a special RBG link L we obtain a knot K_G by sliding G over R until no geometric linking of B and G remains. Similarly, we obtain a knot K_B by sliding B over R until no geometric linking of B and G remains.



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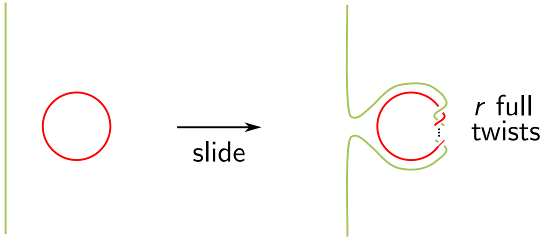
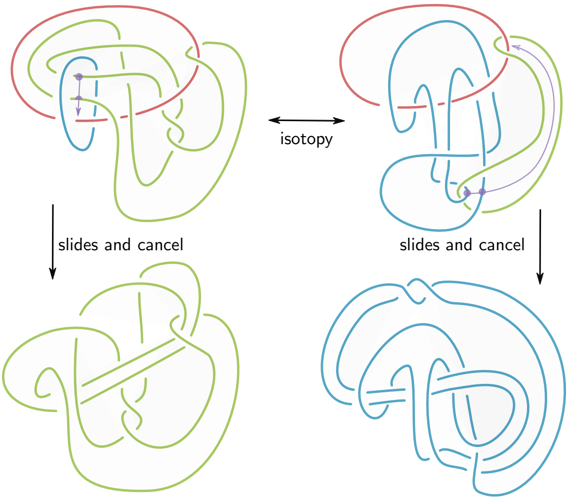


For a special RBG link L, there is an associated homeomorphism

$$\phi_L : S^3_0(K_B) \to S^3_0(K_G).$$

An example



Computer experiments

Goal: Find an example where K_B is slice and $s(K_G) \neq 0$ (or vice versa). If V is the complement of a slice disk for K_B, then the homotopy 4-sphere

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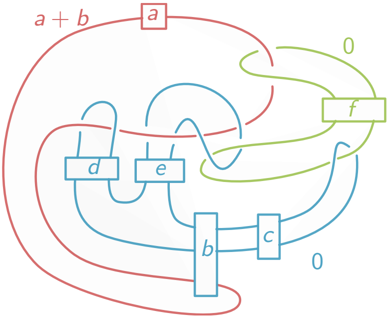
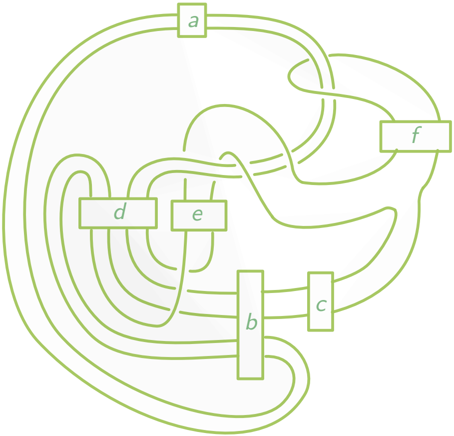
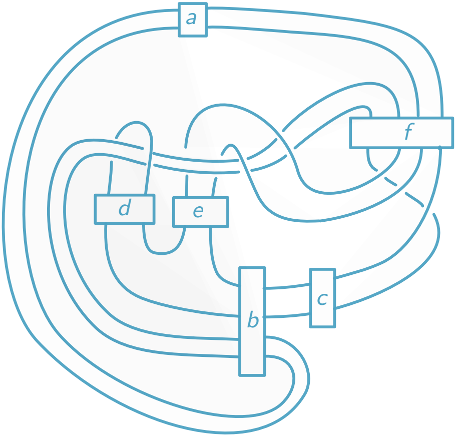
We studied a 6-parameter family consisting of 3375 special RBG links (where boxes indicate the number of full twists):



The resulting knots K_B and K_G





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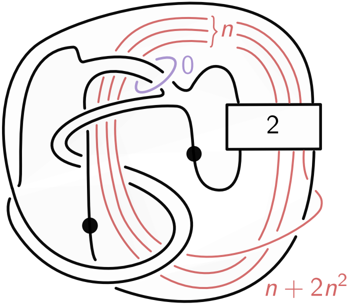
Apart from this RBG family, we also looked at an infinite family of pairs of knots obtained from annulus twisting (a different construction, which can be rephrased in terms of RBG links). This produced infinitely many homotopy 4-spheres as in (1), but no new examples of type (2).

New examples of homotopy 4-spheres

The following family is obtained by annulus twisting the slice knot $J_0 = 8_8$ to produce slice knots J_n with the same 0-surgery. (Left: $n > 0$. Right: $n < 0$.)



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It is still unclear whether the homotopy 4-spheres obtained from J_m and J_n for $m \neq n$ nonzero are standard.

X_1 and X_{-1} were 2 of the 10 examples from our RBG family. The other 8 homotopy 4-spheres remain as potential counterexamples to SPC4.

Non-slice knots

We found 21 examples where K_B or K_G has $s = -2$ (hence is not slice), and we could not immediately determine if the companion is slice.

Thus, if any of the following 21 knots had been slice, then SPC4 would have been false.

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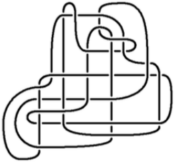
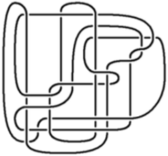
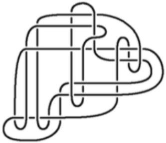


K_1

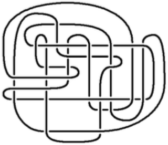


K_2



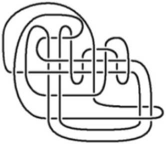
K_3

Non-slice knots

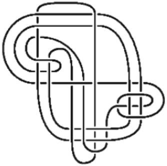


K_4

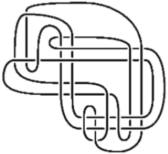
K_5



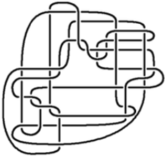
K_6



K_7



K_8



K_9

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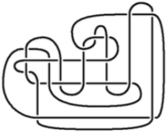
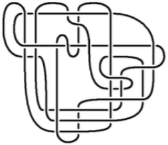


K_{10}

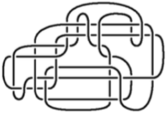


K_{11}

K_{12}

K_{13}

K_{14}



K_{15}

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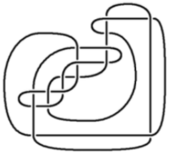
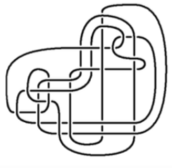
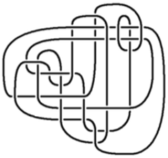
K_{16}



K_{17}



K_{18}



K_{19}

K_{20}

K_{21}

Looking for slice knots

The 21 knots passed many of the known obstructions to sliceness: their Alexander polynomial satisfies the Fox-Milnor condition; s and its variants $s^{\mathbb{F}_2}$, $s^{\mathbb{F}_3}$, s^{Sq^1} all vanish; the knot Floer homology invariants $\epsilon = \tau = \nu = 0$. For at least 12 of the 21, the 0-surgery homeomorphism does not extend to traces.

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The strategy still stands, and we are currently investigating other families.

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Recall (**MMSW, 2018**): If K is H-slice in $\#^n\mathbb{CP}^2$, then $s(K) \geq 0$.

Suppose we found a pair of knots K, K' with $S^3_0(K) \cong S^3_0(K')$, K H-slice in $\#^n\mathbb{CP}^2$, and $s(K') < 0$. Then K' is not H-slice in $\#^n\mathbb{CP}^2$, but is H-slice in a homotopy $\#^n\mathbb{CP}^2$. Therefore, we would obtain an exotic smooth structure on $\#^n\mathbb{CP}^2$.

A strategy for finding exotic $\#^n\mathbb{CP}^2$

A slightly less ambitious goal than disproving SPC4 is to find exotic structures on $\#^n\mathbb{CP}^2$ for some n.

Freedman + Donaldson: Every closed, smooth, simply connected 4-manifold with positive definite intersection form is homeomorphic to $\#^n\mathbb{CP}^2$ for some n.

Recall (**MMSW, 2018**): If K is H-slice in $\#^n\mathbb{CP}^2$, then $s(K) \geq 0$.

Suppose we found a pair of knots K, K' with $S^3_0(K) \cong S^3_0(K')$, K H-slice in $\#^n\mathbb{CP}^2$, and $s(K') < 0$. Then K' is not H-slice in $\#^n\mathbb{CP}^2$, but is H-slice in a homotopy $\#^n\mathbb{CP}^2$. Therefore, we would obtain an exotic smooth structure on $\#^n\mathbb{CP}^2$.

It is easier to find knots that are H-slice in $\#^n\mathbb{CP}^2$ than slice: If there is a sequence of n crossing changes from positive to negative that turn K into the unknot, then K is H-slice in $\#^n\mathbb{CP}^2$.