Annular link Floer homology and $gl(1|1)$

(with A. Manion and M. Wong)

$$\begin{tikzcd} Kh \arrow[d] & \text{graded } \chi. & HFK \arrow[d] \\ \text{Jones poly} & & \text{Alexander poly} \end{tikzcd}$$

① A TQFT approach to link polynomials.

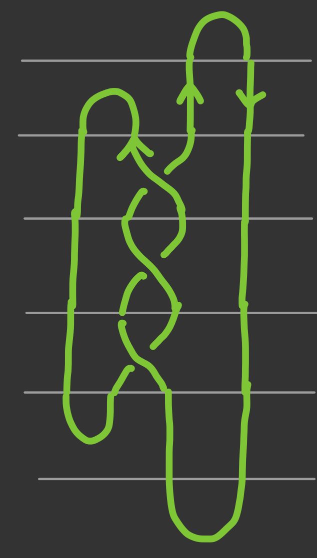
. Reshetikhin, Turaev, Viro

Δ_K: Murakami, Rozansky-Saleur, Reshetikhin

<u>Idea:</u>

- fix a quantum group $U_q(g)$ and a representation V of $U_q(g)$.
- Decompose link into elem. pieces
- Associate • $\otimes$ of V, V^* to cuts. $\uparrow = V$ $\downarrow = V^*$
  • maps of vs. to tangles
- Compose maps, get

$$\begin{array}{ccc} \Psi_L : \mathbb{C}[q,q^{-1}] & \longrightarrow & \mathbb{C}[q,q^{-1}] \\ \cup & & \cup \\ 1 & \longmapsto & \Psi_L(1) \end{array}$$ -link invt.



$\rightsquigarrow$

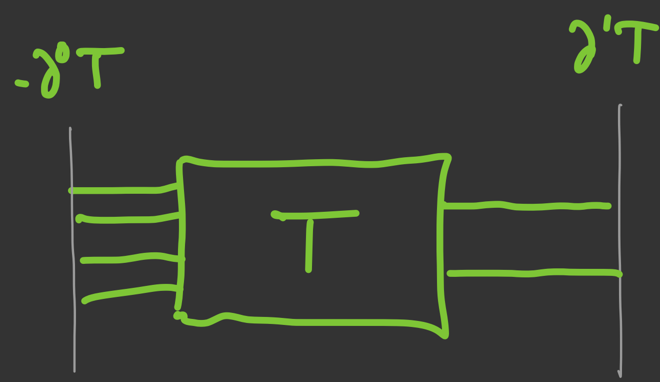
$$\begin{tikzcd}
\mathbb{C}[q,q^{-1}] \\
V \otimes V^* \arrow[u] \\
V^* \otimes V \otimes V \otimes V^* \arrow[u] \\
\vdots \arrow[u] \\
V \otimes V^* \arrow[u] \\
\mathbb{C}[q,q^{-1}] \arrow[u] \arrow[uuuuu, bend right=60, "\Psi_L"']
\end{tikzcd}$$

<u>Ex:</u> Jones poly $g = sl_2$ 2-diml vector rep. U

Alexander poly $g = gl_{1|1}$ V

<u>Goal:</u> "Categorify" this construction

Tangle Floer homology (P.-Vértesi '14)



bordered Floer theory ...

$A(-\partial^0 T)$ — DGA

$CFT(T)$ — dg bimodule over $A(-\partial^0 T)$, $A(\partial^1 T)$

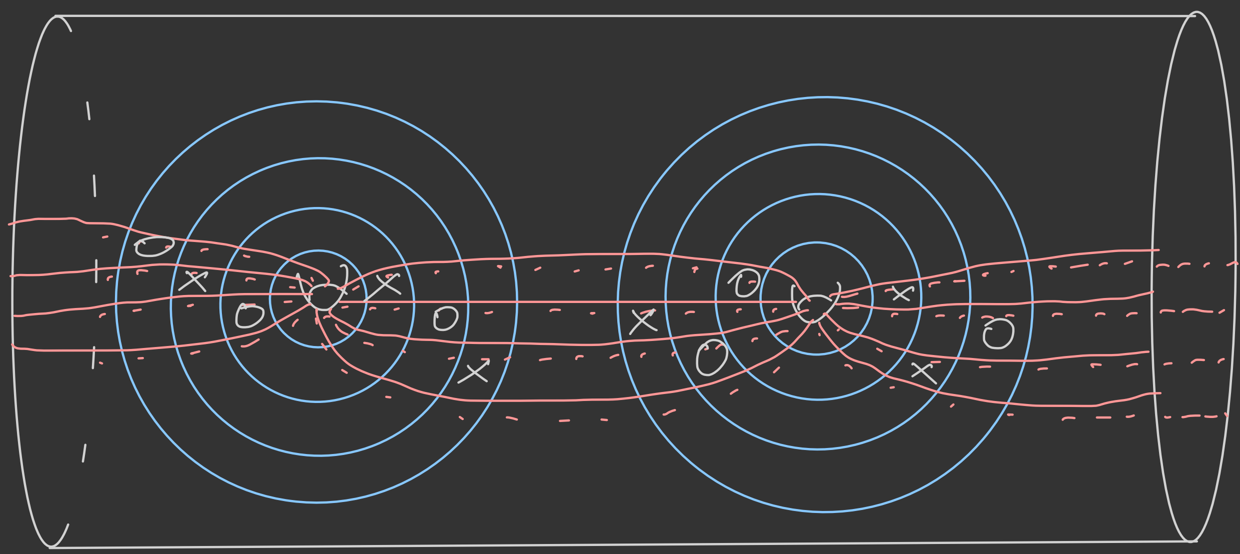
$A(\partial^1 T)$ — DGA

Thm: (P.-Vértesi)

1. $CFT(T)$ is an invariant of T.
2. 

   $$CFT(T_1) \underset{A(\partial^1 T_1)}{\tilde{\otimes}} CFT(T_2) \simeq CFT(T_1 \circ T_2)$$
3. $CFT(L) \simeq \widehat{HFL}(L) \otimes (\mathbb{F}_2 \oplus \mathbb{F}_2[1])$.

$\boxed{\text{CFT}(T)}$ Given a tangle T, represent by a multipointed bordered Heegaard diagram

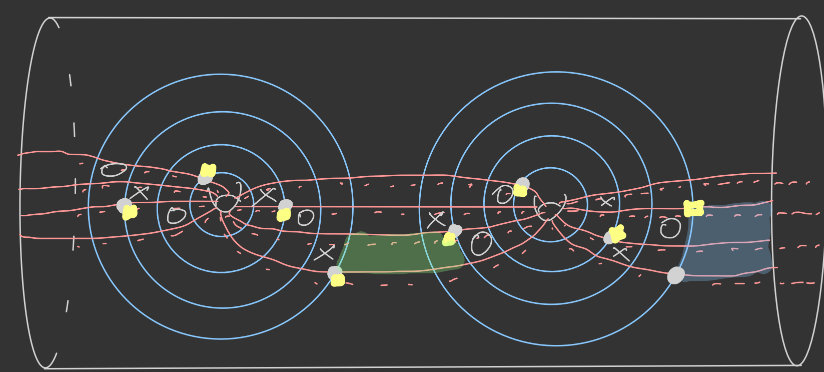


- Count hol. curves in $\Sigma \times I \times \mathbb{R}$ with certain asymptotics.

  '06 Lipshitz - Cyl. reform. of HF

  '08 Lipshitz-Ozsváth-Thurston Bordered HF

- Record $\partial\Sigma$ data of curves as algebra action.



$dx = \dots$

$x \cdot \quad = y$

$\boxed{\text{CFT \& } U_q(\mathfrak{gl}_{1|1})}$

$V_{p_i} := \begin{cases} V & \text{if } p_i = + \\ V^* & \text{if } p_i = - \end{cases}$

Thm: (Ellis-P.-Vertesi)

$V_{\underline{p}} := V_{p_1} \otimes V_{p_2} \otimes \cdots \otimes V_{p_n}.$

$\boxed{\text{CFT categorifies } RT_{\mathfrak{gl}_{1|1}}}$

$\boxed{\text{basis for } V_{\underline{p}} \otimes L(\lambda_{\underline{p}})}$

$\updownarrow$

More concretely:

① $K_0(A(P))$ - free $\mathbb{Z}[q,q^{-1}]$-module w/ basis $\boxed{\{ [A(P)e_s] \mid s \in 2^P \}}$

so $K_0(A(P)) \otimes_{\mathbb{Z}[q^{\pm 1}]} \mathbb{C}(q) \cong V_P \otimes L(\lambda_P)$

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②
$$\begin{array}{ccc} K_0(-\partial^0 T) & \xrightarrow{[CFT(T) \tilde{\otimes} -]} & K_0(\partial^1 T) \\ \text{\rotatebox{90}{$\cong$}} & & \text{\rotatebox{90}{$\cong$}} \\ V_{-\partial^0 T} \otimes L(\lambda_P) & \xrightarrow{RT_{U_q(\mathfrak{gl}_{1|1})}(\vec{T}^{\lambda_P})} & V_{\partial^1 T} \otimes L(\lambda_P) \end{array}$$

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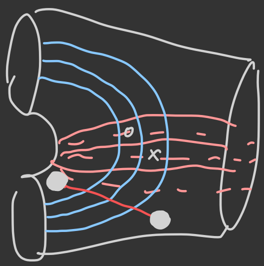
③ We construct functors on $A(P)$-Mod corresponding to the action of $E, F \in U_q(\mathfrak{gl}_{1|1})$ on $V_P \otimes L(\lambda_P)$.

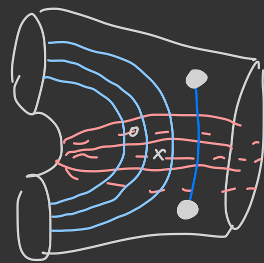
$\mathcal{E}(P),\ \mathcal{F}(P)$: bimodules

$\circledast$

They have H. diagrammatic descriptions:



E



F

$$\begin{tikzcd}
\mathcal{E}(P), \mathcal{F}(P) \arrow[d, bend right] & & \mathcal{E}(P'), \mathcal{F}(P') \arrow[d, bend right] \\
A(P)\text{-Mod} \arrow[rr, "CFT(T)"] \arrow[dd, "K_0"] & & A(P')\text{-Mod} \arrow[dd, "K_0"] \\
& & \\
V_P \otimes L(\underline{\lambda_P}) \arrow[rr, "RT_{U_q(\mathfrak{gl}_{1|1})}(\overrightarrow{\equiv T \equiv})^{\vee}"] & & V_{P'} \otimes L(\underline{\lambda_P}) \\
U_q(\mathfrak{gl}_{1|1}) \arrow[u, bend right] & & U_q(\mathfrak{gl}_{1|1}) \arrow[u, bend right]
\end{tikzcd}$$

$$\mathcal{E}\,\tilde{\otimes}\, CFT(T) \simeq CFT(T)\,\tilde{\otimes}\,\mathcal{E}$$

$$\mathcal{F}\,\tilde{\otimes}\, CFT(T) \simeq CFT(T)\,\tilde{\otimes}\,\mathcal{F}$$

$$\mathcal{E}\,\tilde{\otimes}\,\mathcal{E} \simeq 0 \qquad \mathcal{F}\,\tilde{\otimes}\,\mathcal{F} \simeq 0$$

$$\mathcal{E}\,\tilde{\otimes}\,\mathcal{F} \xrightarrow{f} A \longrightarrow \mathcal{F}\,\tilde{\otimes}\,\mathcal{E} \longrightarrow \mathcal{E}\,\tilde{\otimes}\,\mathcal{F}[1]$$

$$E\,RT(\vec{T}) = RT(\vec{T})\,E$$

$$F\,RT(\vec{T}) = RT(\vec{T})\,F$$

$$E^2 = 0 \qquad F^2 = 0$$

$$EF + FE = 1$$

Annularization

2-category $\mathcal{C}$:

$Ob(\mathcal{C}) = \{ A(P) \mid P \in \{\pm 1\}^n, n \geq 0 \}$

$1\text{-}Mor_{\mathcal{C}}$: bimodules

$2\text{-}Mor_{\mathcal{C}}$: bimodule homomorphisms

CH_*: $1\text{-}Mor(A(P), A(P))$ $M \longmapsto CH_* M$ - chain cx.

$2\text{-}Mor({}_A M_A, {}_A N_A)$ $f \longmapsto CH_*(f)$

$\overline{T}$

Thm (P-Vértesi '15) $\boxed{HH_*(CFT(T)) \simeq AHFL(\overline{T}) = HFL(\overline{T} \cup O)}$

Goal 1: $\boxed{\text{2-morphisms in } \mathcal{C} \text{ related to } \mathcal{E}, \mathcal{F} \text{ satisfy certain conditions}}$ $\rightsquigarrow$ $\boxed{\text{define a } U_q(gl_{1|1}) \text{ action on } hTr(\mathcal{C}).}$

Goal 2: $\boxed{U_q(gl_{1|1}) \curvearrowright hTr(\mathcal{C})}$ "familiar recipe" $\rightsquigarrow$ $\boxed{U_q(gl_{1|1}) \curvearrowright Ch}$

Goal 3: $\boxed{\text{"Compute" the } U_q(gl_{1|1})\text{-action on AHFL.}}$

↑ interpret in a Floer-theoretic way (count of polygons)

① Horizontal trace

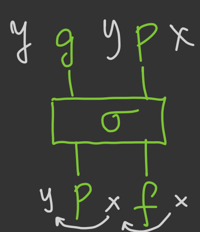
Def (Beliakova-Habiro-Lauda-Zivkovic '14)

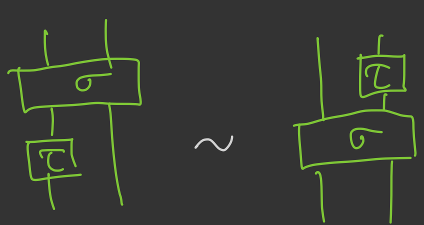
The $\boxed{\text{horizontal trace}}$ of a bicategory $\mathcal{C}$ is the category $\boxed{hTr(\mathcal{C})}$ with

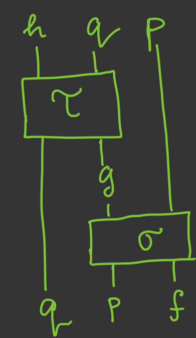
$\boxed{Ob(hTr(\mathcal{C}))}$: 1-endomorphisms $f: x \to x$, $x \in Ob(\mathcal{C})$

$\boxed{Mor_{hTr(\mathcal{C})}(f: x \to x,\ g: y \to y)}$: classes $[p, \sigma]$ s.t. $p: x \to y$ · 1-morphism in $\mathcal{C}$

$\sigma: p \circ f \Rightarrow g \circ p$ · 2-morphism in $\mathcal{C}$

Graphically: $[p, \alpha] =$ 



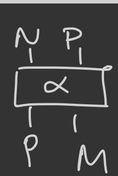
$\boxed{\text{Composition:}}$ $[q, \tau][p, \sigma] =$ 

$\boxed{\text{Identity morphism:}}$ $1_f = [1_x, 1_f]$.

$\boxed{\text{Properties of the 2-morphisms related to } \mathcal{E}, \mathcal{F}.}$ (Write $M = CFT(T)$)

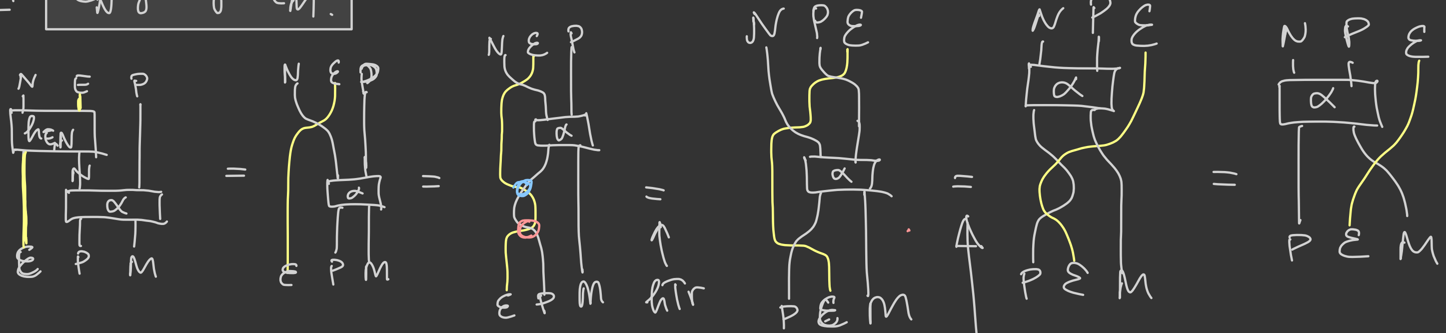
$$\mathcal{E} \tilde{\otimes} M \underset{h_{\mathcal{E},M}}{\overset{h_{M,\mathcal{E}}}{\rightleftarrows}} M \tilde{\otimes} \mathcal{E} \qquad e_M := [\mathcal{E}, h_{\mathcal{E}M}] \in \text{1-Mor}_{hTr(C)}(M,M).$$

$$\mathcal{F} \tilde{\otimes} M \underset{h_{\mathcal{F},M}}{\overset{h_{M,\mathcal{F}}}{\rightleftarrows}} M \tilde{\otimes} \mathcal{F} \qquad f_M := [\mathcal{F}, h_{\mathcal{F}M}] \in$$

Let $\gamma = [P, \alpha]$ s.t.  $\cdot$ 2-Mor(PM, NP) for a tangle cobordism.

Claim: $\boxed{e_N \circ \gamma = \gamma \circ e_M.}$

Pf:



Denote

$h_{\mathcal{E}M}$ by 

$h_{M\mathcal{E}}$ by 

We use Heegaard-Floer arguments to prove that the 2-morpisms $h_{\mathcal{E}M}$ in C commute w/ 2-morphisms for tangle cobordisms.

$h_{\mathcal{E},NP} \circ \alpha = \alpha \circ h_{\mathcal{E},PM}$

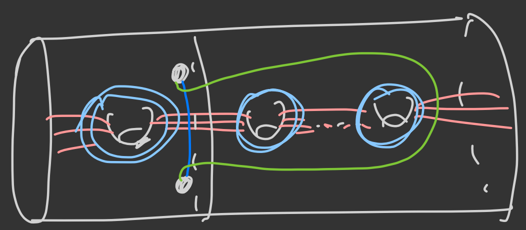
(*)

Similarly, we get $f_N \circ \gamma = \gamma \circ f_N$

$e_M \circ e_M = 0 \quad f_M \circ f_M = 0$

$e_M \circ f_M + f_M \circ e_M = id_M$

Why is ⊛ true?



$\mathcal{F} \,\tilde{\otimes}\, CFT(T)$

$$\begin{tikzcd}
 & \Sigma(\alpha, {\color{blue}\beta}) \arrow[dl, "\alpha\beta\gamma"', "h_{\varepsilon, T}"] \arrow[dr, "\text{cob.}", "\alpha\beta\varepsilon"'] & \\
\Sigma(\alpha, {\color{green}\gamma}) \arrow[dr, "\text{cob.}", "\alpha\gamma\delta"'] & & \Sigma(\alpha, \varepsilon) \arrow[dl, "\alpha\varepsilon\delta"', "h_{\varepsilon, T'}"] \\
 & \Sigma(\alpha, \delta) &
\end{tikzcd}$$

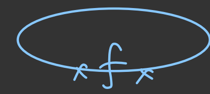
$CFT(T) \,\tilde{\otimes}\, \mathcal{F}$

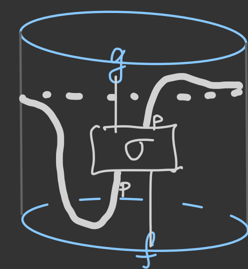
$\mathcal{F} \,\tilde{\otimes}\, CFT(T')$

$CFT(T') \,\tilde{\otimes}\, \mathcal{F}$

Floer-theoretic arguments. ('01, ...)

② From $\mathrm{hTr}(\mathcal{C})$ to Ch

$[f:x\to x]\in \mathrm{Ob}(\mathrm{hTr}(\mathcal{C})) \rightsquigarrow$ 

$[p,\sigma]\in \mathrm{Mor}_{\mathrm{hTr}(\mathcal{C})}(f,g) \rightsquigarrow$ 

What do $\cup$, $\cap$ "mean"?

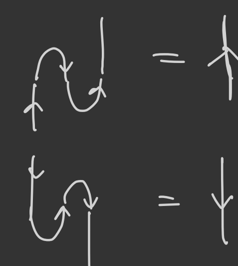
If p has a left dual ${}^*p$, then we can think of $\cup$ as coev, $\cap$ as ev.

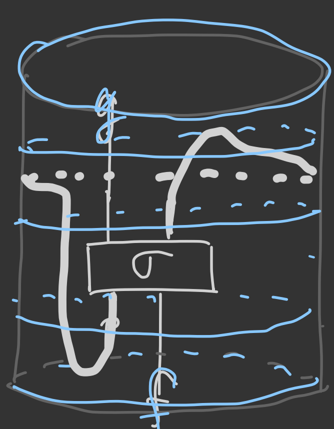
${}^*f\in \mathcal{C}(y,x)$ is a $\boxed{\text{left-dual}}$ of $f\in\mathcal{C}(x,y)$ if we have

2-morphisms $\quad f\circ {}^*f \xRightarrow{\ \varepsilon\ } \mathrm{id}_y \quad$ s.t. $\quad f \xRightarrow{1_f\circ\eta} f\circ{}^*f\circ f \xRightarrow{\varepsilon\circ 1_f} f = 1_f$

$\quad \mathrm{id}_x \xRightarrow{\ \eta\ } {}^*f\circ f \quad\quad {}^*f \xRightarrow{\eta\circ 1_{*f}} {}^*f\circ f\circ{}^*f \xRightarrow{1_{*f}\circ\varepsilon} {}^*f = 1_{*f}$

Graphically

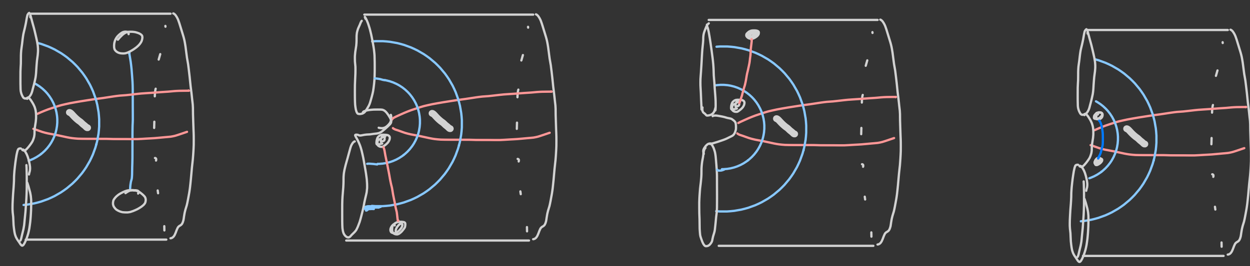


If so 

$$
\begin{array}{l}
(g)\\
\uparrow (\mathrm{id}_g\otimes\varepsilon)\\
(gpp^*)\\
\uparrow \theta\\
(p^*gp)\\
\uparrow (\mathrm{id}_{p^*}\otimes\sigma)\\
(p^*pf)\\
\uparrow (\eta\otimes\mathrm{id}_f)\\
\mathrm{CH}_*(f)
\end{array}
$$

Get a well-defd map T to Ch

Turns out $\mathcal{E}(P), \mathcal{F}(P)$ have left adjoints!



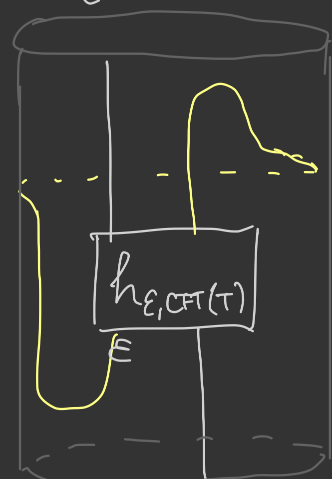
$\mathcal{F} = {}^*\mathcal{E}$ $\qquad \mathcal{E}$ $\qquad {}^*\mathcal{F}$ $\qquad \mathcal{E}^*$

Pf: Floer theoretic.

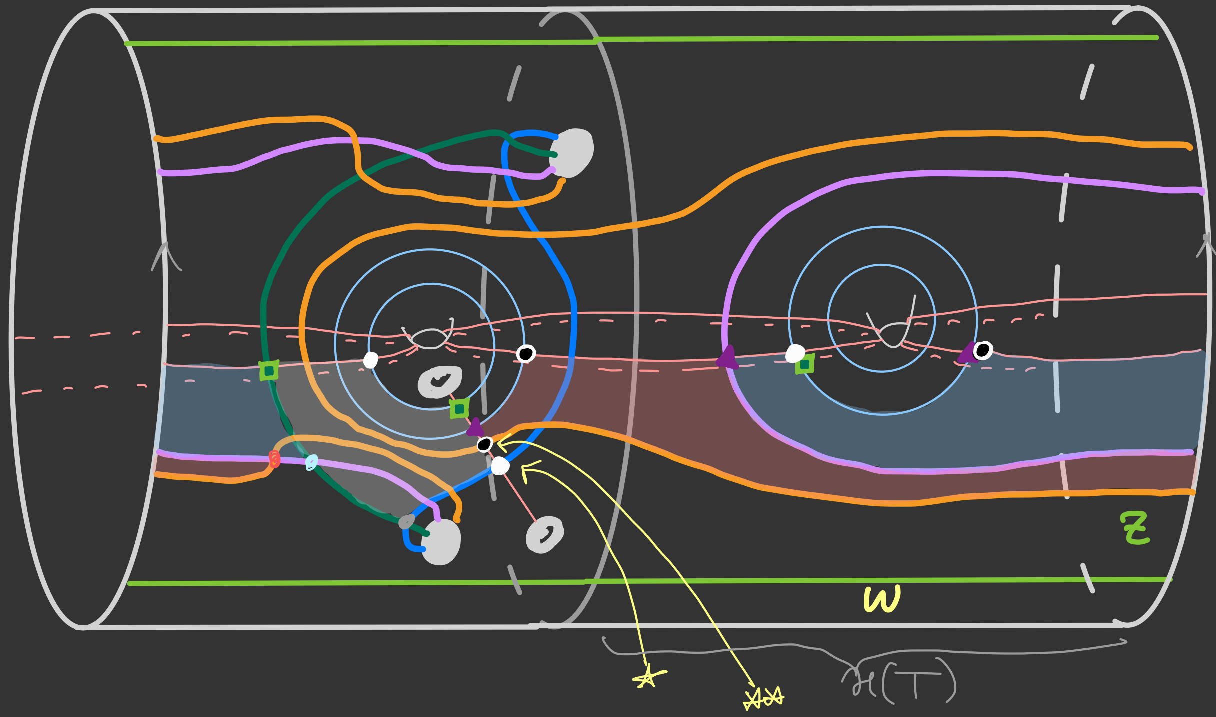
So let's compute

$T(\ell_{CFT(T)})$

$CFT(T)$



$CFT(T)$



Gen. ● $\xrightarrow[\text{grey polygons}]{\eta_{\mathcal{E}} \tilde{\otimes}\, id_{CFT(T)}}$ $\blacksquare$ $\xrightarrow[\text{blue polygons}]{id_{*\mathcal{E}} \tilde{\otimes}\, h_{\mathcal{E},CFT(T)},\ \text{then "slide"}}$ ▲ $\xrightarrow[\text{red polygons}]{id_{CFT(T)} \tilde{\otimes}\, \varepsilon_{\mathcal{E}}}$ ◉

H.d. $\mathcal{H}_{\alpha\beta}$ $\mathcal{H}_{\alpha\gamma}$ $\mathcal{H}_{\alpha\delta}$ $\mathcal{H}_{\alpha\varepsilon}$

Bi-Module: $\mathrm{Cone}(\mathcal{E} \tilde{\otimes} {}^*\mathcal{E} \xrightarrow{f} A) \tilde{\otimes} CFT(T)$ ${}^*\mathcal{E} \tilde{\otimes} \mathcal{E} \tilde{\otimes} CFT(T)$ $\mathcal{E} \tilde{\otimes} {}^*\mathcal{E} \tilde{\otimes} CFT(T)$ $\mathrm{Cone}(A \xrightarrow{g} {}^*\mathcal{E} \tilde{\otimes} \mathcal{E}) \tilde{\otimes} CFT(T)$

★ occ: $A \tilde{\otimes} CFT(T)$

$\| \#$

$\mathcal{E} \tilde{\otimes} CFT(T) \tilde{\otimes} {}^*\mathcal{E}$

★★ occ: $A \tilde{\otimes} CFT(T)$

Composition $\hookrightarrow$ $\overset{T(e)}{\text{count}}$ of polygons through z on $\mathcal{H}(\overline{T} \cup O)$.

Similarly $T(f)$ counts polygons through w on $\mathcal{H}(\overline{T} \cup O)$.

$\Bigg\}$ automatically have

$T(e)T(e) = 0 = T(f)T(f)$

$T(e)T(f) + T(f)T(e) = 1$.

Get action on AHFL.