Upsilon-like Invariants from Khovanov Homology

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based joint work with **Linh Truong** (UMich)
and work-in-progress with **Ross Akhmechet** (UVA)

We'll focus on knots today.

smooth, concordance

$K \subset S^3$ $\quad (F, \partial F = K) \subset (B^4, \partial B^4 = S^3)$

$K \subset Y^3$ $\quad (F, \partial F = K) \subset (W^4, \partial W^4 = Y^3)$

geometric

$K \subset (S^3, \xi_{std})$ $\quad (F, \partial F = K) \subset (W^4, \omega)$

Legendrian $(TK \subset \xi)$ $\quad$ Lagrangian F

Transverse $(TK \perp \xi)$ $\quad$ Symplectic F

related to braids via Transverse Markov Theorem

Pattern for today:

$(\mathcal{C}, d_{tot})$ equipped with filtration grading gr

- $\mathcal{C}$ = chains

  Kc = Khovanov chains

  $= \mathbb{F}\langle$ Kauffman states $\rangle$

- d_{tot} = total differential

  $$\left.\begin{array}{l} d_{Kh} = \text{Kh differential} \\ \Phi_{Lee} = \text{Lee's perturbation} \\ d_{tot} = d_{Kh} + \Phi_{Lee} \end{array}\right\} \text{Kh-Lee}$$

- $\mathcal{C}$ is generated by a gr-homogeneous distinguished basis

  Kg = Khovanov generators

  $=$ {Kauffman states}

  gr_q = quantum grading

$(\mathcal{C}, d_{tot})$ equipped with filtration grading gr

Extract numerical invariants by computing:

captures same information

- filtration grading of distinguished homology class

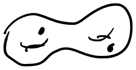
  Rasmussen-Lee: $s(K) = gr_q([\mathfrak{s}_o]) + 1$

- grading of generator of nontorsion tower

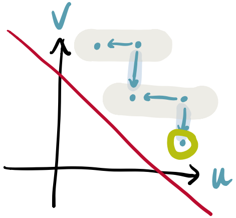
  $(\mathcal{C} = Kc \otimes_{\mathbb{F}} \mathbb{F}[T], \; d = d_{Kh} + T \cdot \Phi_{Lee})$ is graded now

- extremal filtration level where an induced map is nonzero

[Ozsváth-Szabó] (CFK^-, d), gr = Alexander



τ is a knot concordance invariant coming from knot Floer homology [Ozsváth-Szabó, Rasmussen]



Similar in spirit to the s-invariant from Kh-Lee

[Ozsváth-Stipsicz-Szabó] (CFK^-, d), $gr_t =$ Maslov $- t \cdot$ Alexander

$\Upsilon_K(t) : [0,2] \longrightarrow \mathbb{R}$ is a PL function for each knot K

- $\Upsilon_K'(0) = -\tau$

"Υ-like": 1-parameter family of invariants obtained by by letting gr be a linear combination of gradings

or more

Some Notable Properties of Υ

- $\Upsilon_K(\cdot) : [0,2] \longrightarrow \mathbb{R}$ piecewise linear

- $\Upsilon_\cdot(t)$: Knot concordance group $\longrightarrow$ PL functions $[0,2] \longrightarrow \mathbb{R}$ is a homomorphism

- 4-ball genus bounds :

$$\left| \Upsilon_K(t) \right| \leq t \cdot g_4(K) \qquad \forall t.$$

- nonorientable 4-ball genus bound at $t=1$

$$\left| \Upsilon_K(1) - \frac{\sigma(K)}{2} \right| \leq \gamma_4(K)$$

Question

Q. Is there an analogue to Υ coming from Khovanov Homology?

geometric	quantum
$CFK^-(K)$	$Kh(K)$
τ gr = Alex	s gr = quantum
Υ gr = Maslov − t · Alex	?

An algebraic analogue to Υ from Kh may contain different topological information.

Recall

Extracting Numerical Invariants

Pattern for today:

$(\mathcal{C}, d_{tot})$ equipped with filtration grading gr

Replace gr with gr_t

- $\mathcal{C}$ = chains

  Kc = Khovanov chains
  = $\mathbb{F}\langle$ Kauffman states $\rangle$

- d_{tot} = total differential

  d_{Kh} = Kh differential

  Φ_{Lee} = Lee's perturbation

  $d_{tot} = d_{Kh} + \Phi_{Lee}$

- $\mathcal{C}$ is generated by a gr-homogeneous distinguished basis

  Kg = Khovanov generators
  = $\{$ Kauffman states $\}$

  gr_q = quantum grading

Melissa Zhang (UGA) Υ-like invariants from *Kh* 16 May 2021

SARKAR - SEED - SZABÓ s^u (Figures below are from their paper)

their construction is more general

- $\mathcal{C} = \mathcal{C}_{f_{tot}} = Kc$

$d_{Kh} = d_1 + h_1$

- $d_{f_{tot}} = d_{Sz} + h_{BN} = \sum_{i=1}^{\infty} d_i + \sum_{j=1}^{\infty} h_i$

(gr_h, gr_q) degrees: $(i, 2i-2)$ $(i, 2i)$

- gr_t mixes gr_h and gr_q

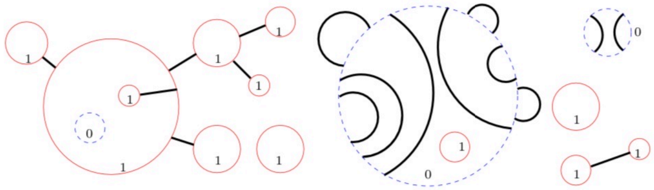


FIGURE 3.1. A labeled configuration that contributes to $\mathfrak{h}$.

h_{BN}

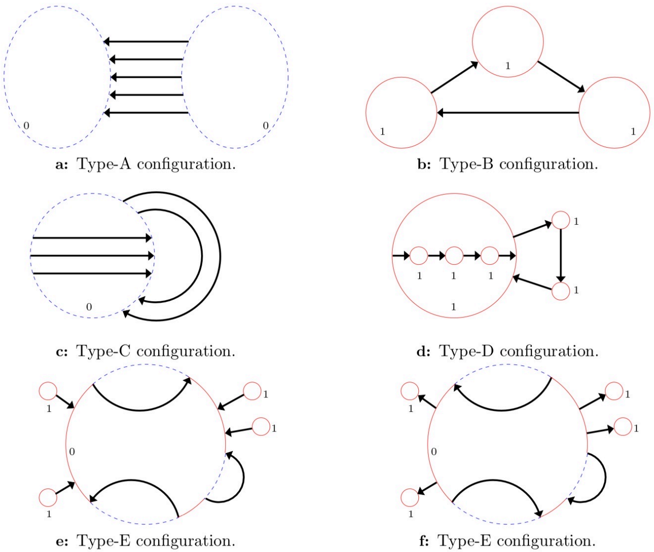


FIGURE 2.6. The oriented labeled configurations that contribute to the function $\mathfrak{d}$.

d_{Sz}

Quantum invariants are well-suited for annular refinements.

$K \subset A \times I$

A = annulus = $\mathbb{R}^2 \setminus \{0\}$

$K = \hat{\beta}$

braid conjugacy classes

$= \dfrac{\text{braid closures}}{\text{annular isotopy}}$

fibered knots in $\Sigma(K)$, 3D contact topology (eg. FDTC)

$K = [\hat{\beta}]$

transverse knots in (S^3, ξ_{rot})

$= \dfrac{\text{braid closures}}{\text{annular isotopy, positive stabilization (positive Markov 2)}}$

$K = P$ (any annular knot)

patterns for satellite operators on the smooth knot concordance group

P: knot conc group $\to$ knot conc group

$[J] \longmapsto [P(J)]$

GRIGSBY - LICATA - WEHRLI d_t

$d_t : [0,2] \longrightarrow \mathbb{R}$ over $\mathbb{C}$

$\mathcal{C} = Kc(K)$

$d_{tot} = d_{Kh} + \Phi_{Lee}$

Khovanov-Lee complex

$= d_{0,0} + d_{0,-2} + \Phi_{4,0} + \Phi_{4,2}$

$gr_t = gr_q - t \cdot gr_a$

gr_a = winding number grading

Bigradings above are (gr_q, gr_a)

TRUONG-Z $s_{r,t}$

$s_{r,t} : [0,1] \times [0,1] \longrightarrow \mathbb{R}$ over $\mathbb{F}_2$

$\mathcal{C} = Kc(K)$

$d_{tot} = d_{Sz} + h_{BN}$

Sarkar-Seed-Szabó's $\mathcal{C}_{ftot}$

$= \sum_{i=1}^{\infty} d_{i,-2} + d_{i,0} + d_{i,2}$

$+ \sum_{i=1}^{\infty} h_{i,0} + h_{i,2} + h_{i,4} + \dots + h_{i,i+1}$ (even only)

$gr_{r,t} = r \cdot gr_h + (1-r) \cdot (gr_q - t \cdot gr_a)$

$d_t : [0,2] \longrightarrow \mathbb{R}$ and $s_{r,t} : [0,1] \times [0,1] \longrightarrow \mathbb{R}$

have similar properties and applications:

- annular concordance invariants
- s-Bennequin inequalities
- braid quasipositivity; right-veeringness for braid (as element of $MCG(\mathbb{D}^2, n \text{ pts})$)

[G. Martin] uses restrictions on the shape of d_t for 3-braid closures to compute Rasmussen's s for all 3-braid closures.

LEWARK–LOBB $\gimel$ (gimel!)

$\gimel_n(K) : [0,1] \longrightarrow \mathbb{R}$ for $n \geq 2$

$(\mathcal{C}, d_{tot}) =$ $\mathfrak{sl}_n$ complex of $\mathbb{C}[x]/(x^n - x^{n-1})$ -modules

gr_q = quantum filtration grading

gr_x = x-filtration (x-power)

$gr_t = t(gr_q + gr_x) - gr_x$ (they actually use nonvanishing of an inclusion map)

$\gimel$ is interesting for quasi-alternating knots, while Υ cannot be.

This supports the idea that quantum invariants may contain different information.

BALLINGER (2020) $\Phi_K(\alpha)$

$\mathcal{C} = Kc(K)$

$$d_{tot} = \underbrace{\left(d_{Kh} + \Phi_{Lee}\right)}_{d_{Lee}} + \underbrace{d_{-1}}_{E(-1)\text{ differential}}$$

has $\deg_h = -1$

gr_t mixes gr_h and gr_q.

d_{Lee} and $E(-1)$ have analogues for $\mathfrak{sl}_n$ homology — could be extended

- lower bounds on g_4
- γ_4 bound at $\alpha = 1$
- no symmetry

$\alpha \longleftrightarrow 2 - \alpha$

(perhaps more information?)

? $\longrightarrow$ ✓ !

Alternate Approach: Akhmechet - Z (WIP)

$\mathcal{C} = Kc(K) \otimes_{\mathbb{F}} \mathbb{F}[U,V]$, complex of $\mathbb{F}[U,V] / ((X-U)(X-V))$ - modules

[Khovanov - Robert]

d from the Frobenius system

$\mathbb{F}[X,h,t] / (X^2 - hX - t)$

gr_t mixes gr_U = U-power grading
gr_V = V-power grading

We are looking at applications to transverse knots as well as smooth concordance.

Thank You!