Recovering a Riemannian metric from least-area data

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Outline

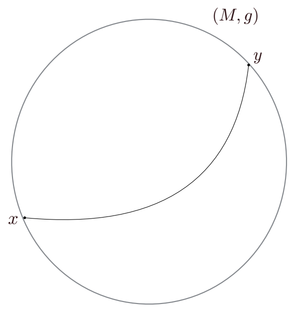
1. Boundary rigidity
2. A lower codimensional rigidity problem
3. Sketch of the proof
4. Future projects

Boundary rigidity

A classical geometric question

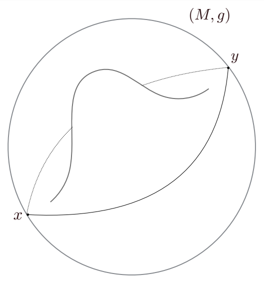
The boundary rigidity problem

- (M, g) a Riemannian manifold with boundary ∂M.
- Know the geodesic distance between any two boundary points $x, y \in \partial M$.
- Does this information determine the Riemannian metric g?



Obstructions

- Boundary-fixing diffeomorphisms.
- Regions of large positive curvature.



- Manifolds without such regions are called **simple**.
- Conjecture (Michel 1981): All simple manifolds are boundary rigid.

Selected results on boundary rigidity

- Special cases were shown by Michel, Gromov, and Croke.
- Lassas, Sharafutdinov, Uhlmann (2003): g is C^k-close to Euclidean.
- Stefanov and Uhlmann (2005): g, $\tilde{g}$ are simple and $\tilde{g}$ is C^k-close to g.
- Pestov and Uhlmann (2005): **Simple 2-manifolds are boundary rigid.**
- Burago and Ivanov (2010 and 2013): g is simple and either C^2-close to Euclidean or C^3-close to a hyperbolic metric.
- Graham, Guillarmou, Stefanov, Uhlmann (2019): Asymptotically hyperbolic setting.
- Stefanov, Uhlmann, and Vasy (2021): Manifolds with a convex foliation condition + lens data.

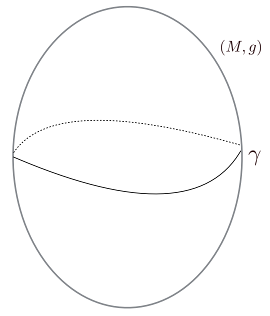
A lower codimensional rigidity problem

Determining the metric from area data

- Let us consider a codimension $n-2$ version of boundary rigidity.
- Consider least-areas of minimal surfaces instead of distances of geodesics.

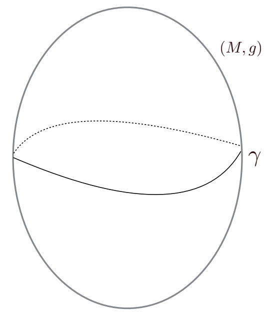
Question

- (M, g) a Riemannian manifold with boundary ∂M.
- For any simple closed curve $\gamma \subset \partial M$, we know the area of the least-area surface(s) circumscribed by γ.
- Does this information determine the Riemannian metric?



Answer

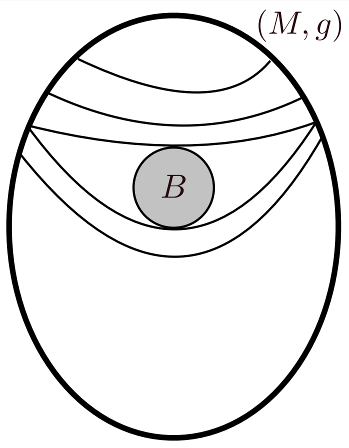
- **Yes!** (under certain geometric conditions.)
- In some cases, we only require the area data for a much smaller subclass of curves.



Further work

- N. Bao, CJ. Cao, S. Fischetti, C. Keeler (2019). Higher dimensions.
- N. Bao, CJ. Cao, S. Fischetti, J. Pollack, Y. Zhong (2020). Higher genus minimal surfaces + higher dimensions.

Obstruction: minimal spheres



- Minimal surfaces never pass through the sphere.
- Can detect this: area functional is not C^1.

Our main results on recovering a Riemannian metric from area data:

We can determine a Riemannian metric from knowledge of least-areas for **three** classes of manifolds, as well as a local result.

Briefly:

- The first two classes of manifolds arise from the tradeoff:
  less area data available $\longrightarrow$ more restrictions on the geometry.
- The third class of manifolds arise from the tradeoff:
  more data available $\longrightarrow$ fewer restrictions on the metric.

I will discuss a result for classes 1 and 2 today.

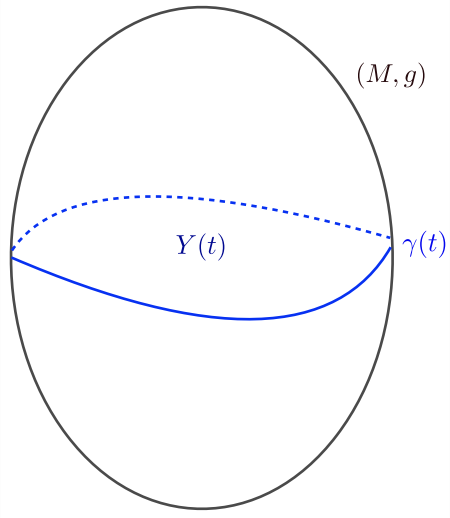
Our global result

Theorem (Alexakis, B., Nachman, 2020)

- (M, g) a manifold of Class 1 or Class 2.
- $g|_{\partial M}$ given.
- Suppose for the given family of simple closed curves $\gamma(t) \subset \partial M$ and any nearby perturbations $\gamma(s,t) \subset \partial M$, we know the area of the properly embedded surface $Y(s,t) \subset M$ which solves the least-area problem for $\gamma(s,t)$.

Then, the metric g is uniquely determined up to diffeomorphisms which fix ∂M.

Set up



Determining a Riemannian metric from least-area surfaces:

The **first** and **second** class of manifolds:

Let (M, g) be a Riemannian manifold with boundary ∂M satisfying

- (M, g) is C^4-smooth.
- $\dim(M) = 3$.
- (M, g) has strictly mean convex boundary ∂M.
- there is a foliation of ∂M by simple closed curves $\{\gamma(t)\}_{t\in(-1,1)}$ which satisfy some technical curvature bounds.
- the foliation $\{\gamma(t)\}_{t\in(-1,1)}$ induces a foliation of M by area minimizing discs $\{Y(t)\}_{t\in(-1,1)}$.

Class 1: For (M, g) as described, we additionally have g is C^3-close to Euclidean.

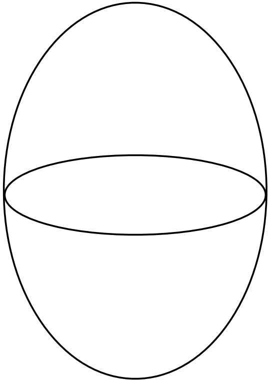


Figure 1: g "looks flat" even when zoomed to level of curvature.

Class 2: For (M, g) as described, (M, g) is also **straight-thin**: the minimal surfaces $Y(t)$ have area bounded above by a (small) number and (M, g) is not too "curvy".

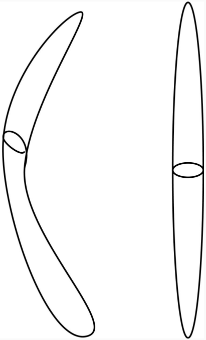


Figure 2: Cross-sectional area is small.

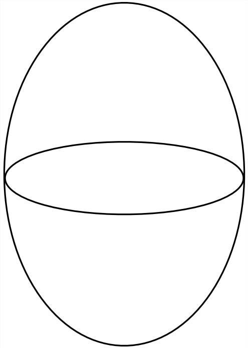


Figure 3: Wider cross-section compensated by "straightness".

Sketch of the proof

Overview of global result proof:

Want to show: least-area data for the foliation $\{Y(t) : t \in (-1,1)\} = M$ and its nearby perturbations $\implies g$ is uniquely determined.

- Solve for the metric by moving along the foliation $Y(t)$.
- Use conformal structure of each $Y(t)$ to write the metric as

$$g = \begin{pmatrix} e^{2\phi} & 0 & g_{31} \\ 0 & e^{2\phi} & g_{32} \\ g_{13} & g_{23} & g_{33} \end{pmatrix}.$$

- Note: by extending (M, g) to an asymptotically flat manifold, ϕ is unique on each $Y(t)$.

Main proof ideas:

- **Key:** Use variations of the foliation to relate geometric data to PDE data.
- By considering a normal variation of $Y(0)$ to $Y(t)$, we find that $\frac{\partial^2}{\partial t^2} A(Y(t))\Big|_{t=0}$ determines the **Dirichlet-to-Neumann map**

$$\Lambda_{g_{Y(0)}} : \psi_0 \mapsto \frac{\partial \psi}{\partial \nu}$$

for

$$\begin{aligned} \Delta_{g_{Y(0)}} \psi + \left(\mathrm{Ric}_g(\vec{n}, \vec{n}) + ||A||_g^2\right) \psi &= 0 \qquad \text{on } Y(0) \\ \psi &= \psi_0 \qquad \text{on } \partial Y(0). \end{aligned} \tag{1}$$

where $\psi : Y(0) \to \mathbb{R}$, $\vec{n}$ is a unit normal vector field on $Y(0)$, and ν is the outward unit normal vector field on $\partial Y(0)$.

Main proof ideas:

- In our conformal coordinates, we determine the **Dirichlet-to-Neumann map**

$$\Lambda_{g_{\mathbb{E}}} : \psi_0 \mapsto \frac{\partial \psi}{\partial \nu}$$

for

$$\begin{aligned} \Delta_{g_{\mathbb{E}}} \psi + e^{2\phi} \left(\mathrm{Ric}_g(\vec{n}, \vec{n}) + \|A\|_g^2 \right) \psi &= 0 \quad \text{on } D \subset \mathbb{R}^2 \\ \psi &= \psi_0 \quad \text{on } \partial D. \end{aligned} \tag{2}$$

- **Nachman (1996):**

$\Lambda_{g_{\mathbb{E}}}$ determines $e^{2\phi} \left(\mathrm{Ric}_g(\vec{n}, \vec{n}) + \|A\|_g^2 \right)$.

- Thus we know **any** solution ψ to (1).

Main proof ideas:

- For the foliation $\{Y(t)\}_{t\in(-1,1)}$, the lapse function $\psi := ||N||_g$ is a solution to (1).

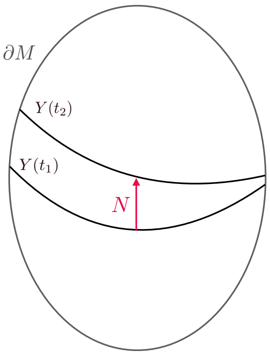
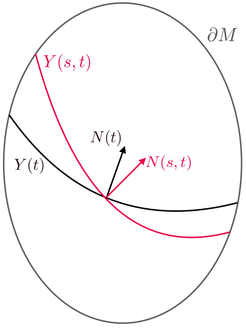


Figure 4: The lapse function is $||N||_g$.

Main proof ideas:

- Variations $Y(s,t)$ of $Y(t)$ lead to knowledge of **new** lapse functions $\psi(s,t) := ||N(s,t)||_g$.



Main proof ideas:

Next steps:

- Linearizing $||N(s,t)||_g$ about $s = 0$ gives nonlinear, non-local equations for the components of g^{-1}.
- Get an evolution equation for ϕ from the minimality of each $Y(t)$.
- We show uniqueness for this system by considering two metrics g_1 and g_2 for which we have the same area data.

Main proof ideas:

- Obtain $\delta g^{33} := g_1^{33} - g_2^{33} = 0$ in the coordinates (x^α).
- Taking differences of the equations we derived:

$$\begin{aligned}
0 &= \delta g^{31}(p) + \partial_k ||\nabla x^3||_{g_1}(p) \delta \dot{x}_1^k(p) \\
0 &= \delta g^{32}(p) + \partial_k ||\nabla x^3||_{g_1}(p) \delta \dot{x}_2^k(p) \\
0 &= g_1^{k3} \partial_k(\delta\phi) + g_1^{33} \partial_3(\delta\phi) \\
&\qquad + \left(\partial_k \phi_2 - \frac{1}{2}\partial_k \log(g_1^{33})\right) \delta g^{k3} + \frac{1}{2}\partial_k(\delta g^{3k}).
\end{aligned}$$

  in the differences δg^{31}, δg^{32}, and $\delta\phi$.
- Here $\delta \dot{x}_i^k$ is a pseudodifferential operator (ΨDO) acting on δg^{31}, δg^{32}, $\delta\phi$ and $\partial_3 \delta\phi$.

Main ideas of the proof

- We show δg^{31}, δg^{32} are ΨDOs acting on $\delta\phi$ and $\partial_3\delta\phi$.
  - The conditions of close to Euclidean or straight-thin are used to invert the system.
- Then, the equation for $\delta\phi$ becomes a hyperbolic Cauchy problem:

$$\begin{aligned} \partial_3\delta\phi + P(\delta\phi) &= 0 \quad \text{on } M \\ \delta\phi &= 0 \quad \text{on } \partial M. \end{aligned}$$

  where P is an order 1 ΨDO in the tangential directions.
- The uniqueness of this Cauchy problem gives us uniqueness of the metric components.

Future projects

Future projects

- Asymptotically hyperbolic + renormalized area information.
- Larger classes of 3-manifolds.
- Higher co-dimensional surfaces.
- When does a manifold admit a foliation by minimal surfaces?

Thanks!

$\dot{x}^k$ equation

$$
\begin{aligned}
\Delta_{g_0}\dot{x}^k &= -2\psi_p A^{ij}\Gamma^k_{ij}(g_0) - 2g_0^{ij}\nabla_j(\psi_p A^k_i) \\
&= -g_0^{ij}\nabla_j(\psi_p)\frac{e^{-2\phi}(g_{\mathbb{E}})^{jk}}{2||\nabla x^3||_g}(g_{\alpha j}\partial_i g^{3\alpha} + g_{i\alpha}\partial_j g^{3\alpha} + g^{3\alpha}\partial_\alpha g_{ij}) \\
&\quad + 2g_0^{ij}\psi_p\frac{e^{-2\phi}(g_{\mathbb{E}})^{jk}}{2||\nabla x^3||_g}(g_{\alpha j}\partial_i g^{3\alpha} + g_{i\alpha}\partial_j g^{3\alpha} + g^{3\alpha}\partial_\alpha g_{ij}) \\
&\quad + g_0^{ij}\psi_p\frac{1}{||\nabla x^3||_g}\partial_j||\nabla x^3||_g\frac{e^{-2\phi}(g_{\mathbb{E}})^{jk}}{2||\nabla x^3||_g}(g_{\alpha j}\partial_i g^{3\alpha} + g_{i\alpha}\partial_j g^{3\alpha} + g^{3\alpha}\partial_\alpha g_{ij}) \\
&\quad - g_0^{ij}\psi_p\frac{e^{-2\phi}(g_{\mathbb{E}})^{jk}}{||\nabla x^3||_g}\Big\{\partial_j g_{\alpha m}\partial_i g^{3\alpha} + g_{\alpha m}\partial_j\partial_i g^{3\alpha} + \partial_j g_{\alpha i}\partial_m g^{3\alpha} + g_{\alpha i}\partial_j\partial_m g^{3\alpha} \\
&\quad + 2e^{2\phi}(g_{\mathbb{E}})_{im}\partial_j g^{3\alpha}\partial_\alpha\phi + 2e^{2\phi}(g_{\mathbb{E}})_{im}g^{3\alpha}\partial_j\partial_\alpha\phi - 4e^{2\phi}(g_{\mathbb{E}})_{im}g^{3\alpha}\partial_\alpha\phi\partial_j\phi\Big\} \\
&\quad - 8\psi_p e^{-4\phi}\Big\{g_{\mathbb{E}}^{km}g_{\mathbb{E}}^{jl}g_{3l}\partial_m g^{33}\partial_j\phi + g_{\mathbb{E}}^{im}g_{\mathbb{E}}^{kl}g_{3l}\partial_m g^{33}\partial_i\phi - g_{\mathbb{E}}^{kj}g_{\mathbb{E}}^{ml}g_{3l}\partial_m g^{33}e^{2\phi}\partial_j\phi \\
&\quad + g_{\mathbb{E}}^{km}e^{2\phi}\partial_m g^{3j}\partial_j\phi + g_{\mathbb{E}}^{im}e^{2\phi}\partial_m g^{3k}\partial_i\phi - g_{\mathbb{E}}^{kj}\partial_m g^{3m}e^{4\phi}\partial_j\phi\Big\} \\
&=: \mathcal{F}^k(g^{13}, g^{23}, \phi, \psi_{p,i}, d\psi_{p,i}, p).
\end{aligned}
$$

$\delta\dot{x}^k$ equation

Here $\delta\dot{x}_i^k$ is a pseudodifferential operator (ΨDO) acting on δg^{31}, δg^{32}, $\delta\phi$ and $\partial_3\delta\phi$:

$$\begin{aligned}\Delta_{g_{\mathbb{E}}}\delta\dot{x}^k = {} & \psi_{p,i}\bar{A}_m^{jkl}\partial_l\partial_j\delta g^{3m}(w) + \psi_{p,i}\bar{B}^{jk\alpha}\partial_j\partial_\alpha\delta\phi(w) \\ & + (\psi_{p,i}\bar{C}_1^{k\alpha} + \partial_j\psi_{p,i}\bar{C}_2^{jk\alpha})(w)\partial_\alpha\delta\phi(w) \\ & + (\psi_{p,i}\bar{C}_3 + \partial_j\psi_{p,i}\bar{C}_4^j)\delta\phi \\ & + (\psi_{p,i}\bar{D}_{1m}^{jk} + \partial_l\psi_{p,i}\bar{D}_{2m}^{jkl})(w)\partial_j\delta g^{3m}(w) \\ & + (\psi_{p,i}\bar{F}_{1m}^{k} + \partial_l\psi_{p,i}\bar{F}_{2m}^{kl})(w)\delta g^{3m}(w),\end{aligned}$$

for smooth functions $\bar{A}_m^{jkl}, \ldots, \bar{F}_{2m}^{kl}$ in the unknown metric coefficients g_1^{13}, g_1^{23} and g_2^{13}, g_2^{23} and their first and second derivatives at q.