On Eisenstein cocycles on GL(n) over imaginary quadratic fields

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Eisenstein cocycles: totally real fields

Let $\eta(\tau)$ be the Dedekind η-function. Consider the period of Eisenstein series:

$$\Phi(A) := \log \eta(\tau) - \log \eta(A\tau) = \frac{1}{4\pi i}\int_{\tau}^{A\tau} \sum_{m,n}{}' (mz+n)^{-2} dz$$

for $A \in SL(2,\mathbb{Z})$. The series converges conditionally, so a limiting process must be specified. We can view Φ as a cocycle on $SL(2,\mathbb{Z})$ satisfying

$$\Phi(AB) = \Phi(A) + A\Phi(B).$$

Sczech instead considers integration term by term:

$$\Phi(A) = \lim_{t\to\infty} \sum_{|Q(m,n)|<t} \frac{A\tau - \tau}{(mA\tau + n)(m\tau + n)}$$

where Q is a fixed binary form. The advantage of this definition is that it generalizes well to GL(n).

Eisenstein cocycles: totally real fields

Let $G = GL(n, \mathbb{Q})$. Given $A_1, \ldots, A_n \in G$ and a nonzero $x \in \mathbb{R}^n$, let σ_i be the first column of A_i such that $\langle x, \sigma_i \rangle \neq 0$. Then define

$$\psi(A_1, \ldots, A_n)(x) = \frac{\det(\sigma_1, \ldots, \sigma_n)}{\langle x, \sigma_1 \rangle \ldots \langle x, \sigma_n \rangle}.$$

Next, given a homogeneous polynomial P in n-variables, define

$$\psi(A_1, \ldots, A_n)(P, x) = P(-\partial_{x_1}, \ldots, -\partial_{x_n})\psi(A_1, \ldots, A_n)(x).$$

Finally, given a family of linear forms $Q_1, \ldots, Q_m$ on $\mathbb{R}^n$, set $Q = \prod Q_i$ and define *Eisenstein cocycle*

$$\Psi(A_1, \ldots, A_n)(P, Q, u, v) = \lim_{t \to \infty} \sum_{\substack{x \in \mathbb{Z}^n + u \\ |Q(x)| < t}} e(\langle u - x, v \rangle)\psi(A_1, \ldots, A_n)(P, x)$$

for $u \in \mathbb{Q}^n$ and $v \in \mathbb{Q}^n/\mathbb{Z}^n$. It is an $(n-1)$-cocycle on G with coefficients in the G-module

$$M = \text{Functions}(\{(P, Q, u, v)\}, \mathbb{C})$$

Eisenstein cocycles: totally real fields

Let F be a totally real field and $\mathfrak{f}$ an integral ideal of F. Define the partial zeta function, ranging over integral ideals $\mathfrak{b}$ equivalent to $\mathfrak{a}$ in the narrow class group mod $\mathfrak{f}$, is

$$\zeta_{\mathfrak{f}}(\mathfrak{a}, s) := \sum_{\mathfrak{b} \sim \mathfrak{a}} \frac{1}{N_{F/\mathbb{Q}}(\mathfrak{b})}$$

Theorem (Sczech, 1993)

There is an $(n-1)$-cycle $\mathcal{E}$ in $\mathbb{Z}[G^n] \otimes_G M^\vee$ such that

$$\zeta_{\mathfrak{f}}(\mathfrak{a}, s) = \langle [\Psi_s], [\mathcal{E}] \rangle \in \mathbb{Q}$$

where the pairing is given by the cup product

$$\langle \cdot, \cdot \rangle : H^{n-1}(G, M) \times H_{n-1}(G, M^\vee) \to \mathbb{Q}.$$

This can be viewed as a cohomological proof of the Klingen-Siegel rationality theorem. The Eisenstein cocycle represents an Eisenstein cohomology class in $H^{n-1}(G, M)$.

Eisenstein cocycles: totally real fields

Charollois and Dasgupta constructed a *smoothed* Eisenstein cocycle $\Psi_{s,\ell}$ that parametrizes values of a *smoothed* partial zeta function

$$\zeta_{\mathfrak{f},\mathfrak{c}}(\mathfrak{a}, s) := \zeta_{\mathfrak{f}}(\mathfrak{a}\mathfrak{c}, s) - N\mathfrak{c}^{1-s}\zeta_{\mathfrak{f}}(\mathfrak{a}, s).$$

where $\mathfrak{c}$ is an integral ideal of F prime to $\mathfrak{f}$ with norm ℓ.

Theorem (Charollois-Dasgupta, 2014)

For $k = 0, -1, -2, \ldots,$ we have $\zeta_{\mathfrak{f},\mathfrak{c}}(\mathfrak{a}, k) = \langle [\Psi_{k,\ell}], [\mathcal{E}_\ell] \rangle \in \mathbb{Z}\left[\frac{1}{\ell}\right].$

Assume moreover that $\mathfrak{c}$ is prime to $p\mathfrak{f}$. Define $\zeta^*_{\mathfrak{f},\mathfrak{c}}(\mathfrak{a}, s)$ as in $\zeta_{\mathfrak{f},\mathfrak{c}}(\mathfrak{a}, s)$, but with sums taken over prime ideals relatively prime to p.

Theorem (Charollois-Dasgupta, 2014)

There exists a unique $\mathbb{Z}_p$-analytic function of $s \in Hom_{cont}(\mathbb{Z}_p^\times, \mathbb{C}_p^\times)$ such that $\zeta_{\mathfrak{f},\mathfrak{c},p}(\mathfrak{a}, k) = \zeta^*_{\mathfrak{f},\mathfrak{c}}(\mathfrak{a}, k)$.

Eisenstein cocycles: totally complex fields

Let F be a degree n extension of K an imaginary quadratic field. We want to prove analogue of these theorems.
Our Eisenstein cocycle is

$$\Psi_s(A_1, \dots, A_n)(P, M, u) = \sum_{x \in \Lambda + u} \psi(A_1, \dots, A_n)(P, x)\Omega_s^k(x, M)$$

where $u \in F^n/\Lambda$, Λ is a product of n lattices in $\mathbb{C}$ with the same ring of multipliers $\mathcal{O}_F$, M a matrix whose columns are conjugate over K. Also

$$\Omega_s^k(x, M) = \prod_{i=1}^{n} \frac{\overline{xM_i}^k}{|xM_i|^{2s}}$$

is a convergence factor inspired by work of Colmez, converging for $s \gg 0$.
It plays the role of the binary form $Q(x)$.
The cocycle takes values in the G-module

$$S = \text{Functions}(\{(P, M, u)\}, \mathbb{C})$$

Eisenstein cocycles: totally complex fields

Let $I(\mathfrak{f})$ be the group of fractional ideals prime to $\mathfrak{f} \subset F$. Fix a character

- a residue class character $\varphi : (\mathcal{O}_F/\mathfrak{f})^\times \to \mathbb{C}^\times$
- $\lambda(a) = \overline{N_{F/K}(a)}^k N_{F/F}(a)^{-l}$ for integers $k \geq 0, l > 0$ s.t. $\lambda(\epsilon) = 1$
- $\mathfrak{b}$ an integral ideal prime to $\mathfrak{f}$ and r in $\mathfrak{b}^{-1}$

Theorem (Flórez-Karabulut-W., 2019)

For $Re(s) > 1 + k/2$, there is a cycle $\mathcal{E}$ such that

$$L(s, \varphi \cdot \lambda) = \sum_{\mathfrak{b}} \frac{\chi(\mathfrak{b})}{N_{F/\mathbb{Q}}(\mathfrak{b})^s} \sum_{(r) \in I(\mathfrak{f}), r \in \mathfrak{b}^{-1}} \varphi(r) \Psi_s(\mathcal{E})(P^{l-1}, u, M)$$

At integer values, this should give a cohomological interpretation of an algebraicity result of Colmez (1989). The result parametrizes the L-function but does not give a new proof of algebraicity.

Eisenstein cocycles: totally complex fields

For two row vectors $x = (x_1, \ldots, x_n)$ and $v = (v_1, \ldots, v_n)$ in K^n define the scalar product $\langle x|v\rangle = 2\mathrm{Re}\left(\sum_{i=1}^n x_i \overline{v_i}\right) = \mathrm{Tr}_{K/\mathbb{Q}}\left(\sum_{i=1}^n x_i \overline{v_i}\right)$ and the pairing $e(x|v) = e^{2\pi i \langle x|v\rangle}$. Fix a product $\Lambda = \Lambda_1 \times \cdots \times \Lambda_n$ of fractional ideals of K and let $\Gamma_\Lambda = \mathrm{Aut}(\Lambda) \subset GL_n(K)$.

Define the twisted Eisenstein cocycle

$$\Psi^s_{k,\Lambda}(A)(P,u,v,M) = \sum_{x \in \Lambda + u} e(x|v)\psi(\mathfrak{A})(P,x)\Omega^k_s(x,M),$$

representing a cohomology class in $H^{n-1}(\Gamma_\Lambda, S)$.

Let $\Lambda_\ell := \Lambda_1 \ell \times \cdots \times \Lambda_n$ and $\ell = N_{F/\mathbb{Q}}(\mathfrak{c})$. Define the smoothed cocycle

$$\Psi^{s,\ell}_{k,\Lambda}(A)(P,v,M) := \Psi^s_{k,\Lambda_\ell}(A)(P,0,v,M) - \Psi^s_{k,\Lambda}(A)(P,0,v,M).$$

Also define the smoothed partial L-function

$$\mathcal{L}_{\mathfrak{f},\mathfrak{c}}(\mathfrak{a},1,s) = \mathcal{L}_{\mathfrak{f}}(\mathfrak{a}\mathfrak{c},1,s) - N_{F/Q}(\mathfrak{c})\mathcal{L}_{\mathfrak{f}}(\mathfrak{a},1,s).$$

where

$$L(\chi,s) = \sum_{\mathfrak{a} \in G_{\mathfrak{f}}} \chi(\mathfrak{a}) N_{F/Q}(\mathfrak{a})^{-s} \mathcal{L}_{\mathfrak{f}}(\mathfrak{a},1,s),$$

where $G_{\mathfrak{f}}$ is the ray class group of F modulo $\mathfrak{f}$.

Eisenstein cocycles: totally complex fields

Theorem (Flórez-Karabulut-W., 2021)

Let $\mathfrak{p}$ be a prime ideal that splits in K, satisfying some natural conditions. Then up to explicit constants, we have that

$$\mathcal{L}_{\mathfrak{f},\mathfrak{c}}(\mathfrak{a},1,0) = \Psi^{0,\ell}_{k,\Lambda}(A)(P,v,M) \in \mathcal{O}_{K(\mathfrak{f}_0)(E[\mathfrak{m}])}\left[\frac{1}{N\mathfrak{p}}\right].$$

This generalizes a result of Colmez-Schneps (1992).

1. From here, the construction of a p-adic L-function is not far away. (In progress)
2. N. Bergeron, P. Charollois, and L. Garcia proved a similar (rationality) result (2021) using topological methods.
3. G. Kings and J. Sprang proved a far much stronger result using algebraic methods (2019). Namely, they prove integrality and the associated p-adic L-function for Hecke characters of arbitrary extensions of CM fields, but without explicit formulas.

Proof sketch

Thank you!