Generic ABV-packets for p-adic groups

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Happy Birthday, Bill!

Abstract

In this talk we propose an adaptation of Shahidi's enhanced genericity conjecture to ABV-packets for p-adic groups: for every Langlands parameter, the associated ABV-packet contains a generic representation if and only if the orbit of the parameter in the moduli space is open. We relate this genericity conjecture for ABV-packets to other standard conjectures. Along the way we sketch a proof of the tempered parameter case of Vogan's conjecture on Arthur packets for p-adic groups and discuss the genericity conjecture of Gross-Prasad and Rallis.

Joint work with Andrew Fiori, Ahmed Moussaoui and Qing Zhang.

Shahidi's enhanced genericity conjecture

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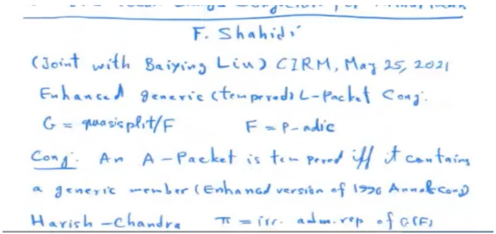
Conjecture (Shahidi)

An A-packet $\Pi_\psi(G)$ contains a generic representation if and only if it contains only tempered representations.



In this case, the A-packet $\Pi_\psi(G)$ is an L-packet.

Here, G must be a group for which A-packets are known.

Genericity conjecture for ABV-packets

Conjecture ([CFMZ])

An ABV-packet $\Pi_\phi^{\mathrm{ABV}}(G)$ *contains a generic representation if and only if* $L(s,\phi,\mathrm{Ad})$ *is regular at* $s=1$.

The objective of this talk is to

- recall ABV-packets for p-adic groups,
- explain how the conjecture above relates to Shahidi's enhanced genericity conjecture, and
- show how the geometric perspective gives a proof of one part of the genericity conjecture of Gross-Prasad and Rallis.

Open parameters

Definition

The *infinitesimal parameter* of a Langlands parameter $\phi : W'_F \to {}^LG$ is the homomorphism $\lambda_\phi : W_F \to {}^LG$ defined by $\lambda_\phi(w) := \phi(w, \mathrm{diag}(|w|^{1/2}, |w|^{-1/2}))$.

Proposition ([CFMZ])

$L(s, \phi, \mathrm{Ad})$ is regular at $s = 1$ if and only if the $\widehat{G}$-orbit of ϕ is Zariski-open in the moduli space of Langlands parameters with shared infinitesimal parameter.

In this case we say ϕ is *open*.

Proposition ([CFMZ])

ϕ is tempered if and only if ϕ is open and of Arthur type.

Langlands parameters with shared infinitesimal parameter

For fixed infinitesimal parameter $\lambda; W_F \to {}^LG$, consider the set

$$\left\{\phi : W_F' \to {}^LG \mid \phi\left(w, \begin{pmatrix} |w|^{1/2} & 0 \\ 0 & |w|^{-1/2} \end{pmatrix}\right) = \lambda(w)\right\}.$$

Every such ϕ is determined by λ and $x \in \operatorname{Lie} \widehat{G}$ such that

$$\exp(x) = \phi\left(1, \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\right).$$

Langlands parameters with shared infinitesimal parameter

The moduli space of Langlands parameters with shared infinitesimal parameter λ is

$$V_\lambda \quad := \quad \{x \in \operatorname{Lie} \widehat{G} \mid \operatorname{Ad}(\lambda(w))(x) = |w|\, x,\ \forall w \in W_F\}.$$

which carries an action by

$$H_\lambda \quad := \quad \{g \in \widehat{G} \mid \operatorname{Inn}(\lambda(w))(g) = g,\ \forall w \in W_F\}.$$

inherited from the adjoint action of $\widehat{G}$ on $\operatorname{Lie} \widehat{G}$.

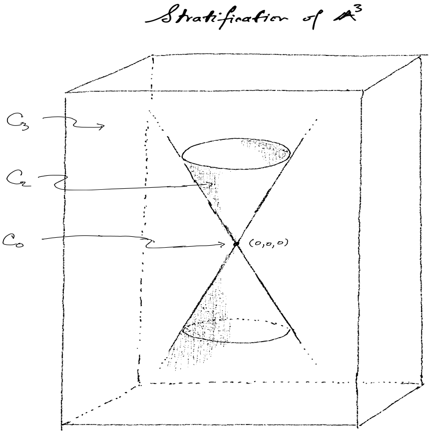
Remark: H_λ is a reductive algebraic group, not necessarily connected.

Prehomogeneous vector space

V_λ is a prehomogeneous vector space for the action of H_λ.

In particular, V_λ is a finite-dimensional vector space, stratified into H_λ-orbits, with a unique open orbit.

Eg. $V_\lambda = \mathrm{Sym}^2(\mathbb{C}^2)$ (vector space of homogenous quadratics in two variables) with $H_\lambda = \mathrm{GL}_2(\mathbb{C})$ action.



Component groups are fundamental groups

The component group $A_\phi = Z_{\widehat{G}}(\phi)/Z_{\widehat{G}}(\phi)^\circ$ is an equivariant fundamental group:

$$A_\phi = \pi_0(Z_{\widehat{G}}(\phi)) = \pi_1^{H_\lambda}(C_\phi, x_\phi).$$

So,

$$\mathrm{Rep}(A_\phi) = \mathrm{Rep}(\pi_1^{H_\lambda}(C_\phi, x_\phi)) \equiv \mathrm{Loc}_{H_\lambda}(C_\phi).$$

Admissible representations and equivariant perverse sheaves

$$
\begin{array}{ccccc}
\Pi_\lambda(G) & & \pi & & \text{irred admissible representation} \\
\Big\downarrow \mathcal{P} & & \Big\downarrow \text{LLC} & & \\
 & & (\phi, \rho) & & \text{enhanced Langlands parameter} \\
 & & \Big\downarrow & & \\
 & & \mathcal{L}_C & & \text{geometric parameter} \\
 & & \Big\downarrow & & \\
\mathrm{Per}_{H_\lambda}(V_\lambda)^{\text{simple}}_{/\text{iso}} & & \mathcal{IC}(\mathcal{L}_C) =: \mathcal{P}(\pi) & & \text{irred equiv perverse sheaf}
\end{array}
$$

Moduli space of Langlands parameters

The quotient

$$X_\lambda := (\widehat{G} \times V_\lambda)/H_\lambda,$$

exists in varieties, for the action $h \cdot (g, x) := (gh^{-1}, Ad(h)x)$, and carries a $\widehat{G}$-action

$$g' \cdot [g, x] := [g'g, x].$$

Then X_λ is the moduli space of Langlands parameters ϕ for which λ_ϕ is $\widehat{G}$-conjugate to λ.

Proposition ([CFMZ])

$L(s, \phi, \mathrm{Ad})$ is regular at $s = 1$ if and only if the $\widehat{G}$-orbit of ϕ is open in X_{λ_ϕ}, or equivalently, if the H_{λ_ϕ}-orbit of ϕ is open in V_{λ_ϕ}.

Lie algebra perspective on the cotangent space

The cotangent space for V_λ:

$$T^*(V_\lambda) \quad = \quad V_\lambda \times V_\lambda^*$$

where

$$V_\lambda^* := \{x \in \operatorname{Lie} \widehat{G} \mid \operatorname{Ad}(\lambda(w))(x) = |w|^{-1} x, \ \forall w \in W_F\},$$

which also carries an action by H_λ inherited from the adjoint action of $\widehat{G}$ on Lie $\widehat{G}$. The action of V_λ^* on V_λ is given by the Killing form on Lie $\widehat{G}$, which in turn defines

$$f : T^*(V_\lambda) \to \mathbb{A}^1.$$

Regular part of the conormal bundle

The Lie algebra perspective on the cotangent space $T^*(V_\lambda)$ also leads to a pleasant description of conormal bundle

$$\Lambda_\lambda := \{(x,y) \in T^*(V_\lambda) \mid [x,y] = 0\}.$$

For any H_λ-orbit $C \subset V_\lambda$, set

$$\Lambda_C := \{(x,y) \in \Lambda_\lambda \mid x \in C\}$$

and

$$\Lambda_C^{\text{reg}} := \Lambda_C \setminus \bigcup_{C<C'} \overline{\Lambda_{C'}}.$$

These are components in

$$\Lambda_\lambda^{\text{reg}} = \bigcup_C \Lambda_C^{\text{reg}}.$$

Connection with Arthur parameters

Let ψ be an Arthur parameter with infinitesimal parameter λ and let $x_\psi \in V_\lambda$ be the Langlands parameter ϕ_ψ as an element of the moduli space V_λ. Then

$$\exp(x_\psi) = \psi\left(1, \left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right), 1\right).$$

Define $y_\psi \in V_\lambda^*$ by

$$\exp(y_\psi) = \psi\left(1, 1, \left(\begin{smallmatrix}1 & 0\\ 1 & 1\end{smallmatrix}\right)\right).$$

Then

$$(x_\psi, y_\psi) \in \Lambda^{\mathrm{reg}}_{C_\psi}.$$

Component groups are fundamental groups again

A second miracle about component groups and equivariant fundamental groups:

$$H_\lambda \cdot (x_\psi, y_\psi) = \Lambda^{\mathrm{reg}}_{C_\psi}$$

and

$$A_\psi = \pi_0(Z_{\widehat{G}}(\psi)) = \pi_1^{H_\lambda}(\Lambda^{\mathrm{reg}}_{C_\psi}, (x_\psi, y_\psi)).$$

Consequently

$$\mathrm{Rep}(A_\psi) = \mathrm{Rep}(\pi_1^{H_\lambda}(\Lambda^{\mathrm{reg}}_{C_\psi}, (x_\psi, y_\psi))) \equiv \mathrm{Loc}_{H_\lambda}(\Lambda^{\mathrm{reg}}_{C_\psi})$$

Microlocal vanishing cycles

For any H_λ-orbit $C \subset V_\lambda$, consider the functor

$$\mathsf{Evs}_C : \mathrm{Per}_{H_\lambda}(V_\lambda) \to \mathrm{Loc}_{H_\lambda}(\Lambda_C^{\mathrm{reg}})$$

defined [CFM+22] by

$$\mathsf{Evs}_C\, \mathcal{F} := \left(\mathrm{R}\Phi_f[-1](\mathcal{F} \boxtimes \mathbb{1}^!_{C^*})\right)|_{\Lambda_C^{\mathrm{reg}}}[-\operatorname{codim} C^*],$$

where $C^* \subset V_\lambda^*$ is the Pyasetskii dual orbit.

This functor is designed so that

$$(\mathsf{Evs}_C\, \mathcal{F})_{(x,y)} = (\mathrm{R}\Phi_y[-1]\mathcal{F})_x\,[-\operatorname{codim} C^*], \qquad (x,y) \in \Lambda_C^{\mathrm{reg}}.$$

ABV-packets and coefficients

For any Langlands parameter ϕ with infinitesimal parameter λ,

$$\Pi_\phi^{\mathrm{ABV}}(G) := \left\{\pi \in \Pi_\lambda(G) \mid \operatorname{Evs}_{C_\phi} \mathcal{P}(\pi) \neq 0\right\}.$$

ABV-packet coefficients are defined by

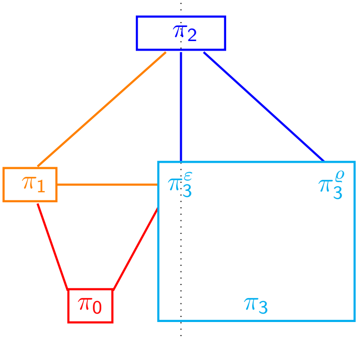
$$\begin{array}{ccccccc} \Pi_\phi^{\mathrm{ABV}}(G) & \xrightarrow{\mathrm{LLC}} & \operatorname{Per}_{H_\lambda}(V_\lambda) & \xrightarrow{\operatorname{NEvs}_{C_\phi}} & \operatorname{Loc}_{H_\lambda}(\Lambda^{\mathrm{gen}}_{C_\phi}) & \longrightarrow & \operatorname{Rep}(A_\phi^{\mathrm{ABV}}) \\ \pi & \mapsto & \mathcal{P}(\pi) & \mapsto & \operatorname{NEvs}_{C_\phi} \mathcal{P}(\pi) & \mapsto & \langle\ ,\pi\rangle_\phi^{\mathrm{ABV}} \end{array}$$

Remark: The normalization NEvs of Evs and the definition of Λ^{gen} is subtle and explained in [CFM+22].

ABV-packets and coefficients

- We compute numerous examples of ABV-packets in [CFM+22], mainly for $G = \mathrm{SO}_{2n+1}$.
- We compute ABV-packets for all unipotent representations of $G_2(F)$ in [CFZ21] and [CFZ].
- See [CFK] for non-singleton ABV-packets for general linear groups.

Some ABV-packets for $G_2(F)$



$$
\begin{aligned}
\Theta_{\psi_0} &= \Theta_{\pi_0} + 2\Theta_{\pi_1} + \Theta_{\pi_3^\varepsilon} \\
\Theta_{\psi_1} &= \Theta_{\pi_1} - \Theta_{\pi_2} + \Theta_{\pi_3^\varepsilon} \\
\Theta_{\psi_2} &= \Theta_{\pi_2} - \Theta_{\pi_3^\varrho} - \Theta_{\pi_3^\varepsilon} \\
\Theta_{\psi_3} &= \Theta_{\pi_3} + 2\Theta_{\pi_3^\varrho} + \Theta_{\pi_3^\varepsilon}
\end{aligned}
$$

Vogan's conjecture on Arthur packets

Conjecture (Vogan,[CFM+22])

For any Arthur parameter $\psi : W''_F \to {}^LG$,

$$\Pi_\psi(G) = \Pi^{\mathrm{ABV}}_{\phi_\psi}(G),$$

where $\phi_\psi : W'_F \to {}^LG$ is the Langlands parameter defined by $\phi_\psi(w,x) = \psi(w,x,dw)$.

- This conjecture is verified in numerous examples in [CFM+22], mainly for odd special orthogonal groups SO_{2n+1}.
- The proof for general linear groups is under construction, joint with Mishty Ray.

Arthur-type case of the genericity conjecture for ABV-packets

Thus, the Arthur-type case of the genericity conjecture for ABV-packets, together with Vogan's conjecture on Arthur parameters, is precisely Shahidi's enhanced genericity conjecture.

Tempered case of Vogan's conjecture

Proposition ([CFMZ])

For any connected reductive G, if ϕ is open, then

$$\Pi^{\mathrm{ABV}}_{\phi}(G) = \Pi_{\phi}(G).$$

Consequently, if ψ is tempered, then Vogan's conjecture is true:

$$\Pi_{\psi}(G) = \Pi^{\mathrm{ABV}}_{\phi_\psi}(G).$$

Crux.

If the orbit C_ϕ of ϕ in the moduli space V_{λ_ϕ} is open then its Pyasetskii dual C^*_ϕ is closed and in fact $C^*_\phi = \{0\}$. In this case we can compute the vanishing cycles:

$$\mathsf{Evs}_{C_\phi}\, \mathcal{IC}(\mathcal{L}_C) = \left(\mathsf{R}\Phi_f(\mathcal{IC}(\mathcal{L}_C) \boxtimes \mathbb{1}^{!}_{0})\right)|_{\Lambda^{\mathrm{reg}}_{C_\phi}}[-\dim V_{\lambda_\phi}] = \begin{cases} 0, & C \neq C_\phi; \\ \mathcal{L}_{\Lambda^{\mathrm{reg}}_{C_\phi}}, & C = C_\phi. \end{cases}$$

Genericity conjecture of Gross-Prasad and Rallis

Conjecture (Gross-Prasad and Rallis)

$L(s, \phi, \mathrm{Ad})$ is regular at $s = 1$ if and only if $\Pi_\phi(G)$ contains a generic representation.

The genericity conjecture for ABV-packets now implies one direction of the conjecture of Gross-Prasad and Rallis: if $L(s, \phi, \mathrm{Ad})$ is regular at $s = 1$ then $\Pi_\phi(G)$ contains a generic representation.

More information

Fields Institute course: January 2022

http://www.fields.utoronto.ca/activities/21-22/local-arthur-packets

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