Explicit asymptotic expansions of supercuspidal characters II

A good day for asymptotic expansions; or, 2 Asymptotic 2 Expansion

Loren Spice

2021-11-16

- k is a field,
- $\mathbf{G}$ is a smooth, reductive, linear algebraic group over k, and
- $G = \mathbf{G}(k)$.

In this talk, k is a p-adic field (finite extension of $\mathbb{Q}_p$ or $\mathbb{F}_p(\!(t)\!)$)

- k is a field,
- $\mathbf{G}$ is a smooth, reductive, linear algebraic group over k, and
- $G = \mathbf{G}(k)$.

In this talk, k is a p-adic field (finite extension of $\mathbb{Q}_p$ or $\mathbb{F}_p(\!(t)\!)$)

The machinery that is developed in S 2021 to handle the harmonic analysis described here does not assume that $\mathbf{G}$ is connected * (so "$\mathbf{G}$ reductive" really means "$\mathbf{G}^\circ$ reductive"), but so far we need connectedness for *applications*

* Blame Jeff.

Hales: p-adic harmonic analysis is not elementary

Hales: p-adic harmonic analysis is not elementary

Specifically,

- characters are not elementary and
- Fourier transforms of nilpotent orbital integrals are not elementary

Hales: p-adic harmonic analysis is not elementary

Specifically,

- characters are not elementary and
- Fourier transforms of nilpotent orbital integrals are not elementary

(among other things)

Hales: p-adic harmonic analysis is not elementary

Specifically,

- characters are not elementary and
- Fourier transforms of nilpotent orbital integrals are not elementary

(among other things)

Maybe characters are *no more* non-elementary than Fourier transforms of orbital integrals

Harish-Chandra–Howe local character expansion:

$$\Phi_\pi(\gamma \cdot \exp(Y)) = \sum_{\mathcal{O} \in \mathcal{O}^H(0)} c_{\mathcal{O}}(\pi, \gamma) \hat{\mu}^H_{\mathcal{O}}(Y),$$

where H is the connected centraliser of γ, for Y near 0

Harish-Chandra–Howe local character expansion:

$$\Phi_\pi(\gamma \cdot \exp(Y)) = \sum_{\mathcal{O} \in \mathcal{O}^H(0)} c_{\mathcal{O}}(\pi, \gamma) \hat{\mu}^H_{\mathcal{O}}(Y),$$

where H is the connected centraliser of γ, for Y near 0

"Near" made precise by

- DeBacker for $\gamma = 1$: $G_{>r}$, where r is the depth of π; and
- Adler–Korman for general γ: depends also on the depths of non-trivial root values of γ.

Harish-Chandra–Howe local character expansion:

$$\Phi_\pi(\gamma \cdot \exp(Y)) = \sum_{\mathcal{O} \in \mathcal{O}^H(0)} c_{\mathcal{O}}(\pi, \gamma) \hat{\mu}^H_{\mathcal{O}}(Y),$$

where H is the connected centraliser of γ, for Y near 0

"Near" made precise by

- DeBacker for $\gamma = 1$: $G_{>r}$, where r is the depth of π; and
- Adler–Korman for general γ: depends also on the depths of non-trivial root values of γ.

How elementary are the coefficients $c_{\mathcal{O}}(\pi, \gamma)$? Long-term project of Gordon–Hales–S

Murnaghan–Kirillov asymptotic expansion:

$$\Phi_\pi(\exp(Y)) = \sum_{\mathcal{O}\in\mathcal{O}^G(\Gamma_\pi)} c_{\mathcal{O}}(\pi)\widehat{O}^G_{\mathcal{O}}(Y),$$

where Γ_π is a semisimple element of $\mathfrak{g}^*$ associated to π, for Y near 0

* Slightly larger than the domain of validity of the LCE.

Murnaghan–Kirillov asymptotic expansion:

$$\Phi_\pi(\exp(Y)) = \sum_{\mathcal{O} \in \mathcal{O}^G(\Gamma_\pi)} c_{\mathcal{O}}(\pi) \widehat{O}^G_{\mathcal{O}}(Y),$$

where Γ_π is a semisimple element of $\mathfrak{g}^*$ associated to π, for Y near 0

"Near" made precise by Kim–Murnaghan: $G_{\geq r}$, where r is the depth of π (2003) *;

* Slightly larger than the domain of validity of the LCE.

Murnaghan–Kirillov asymptotic expansion:

$$\Phi_\pi(\exp(Y)) = \sum_{\mathcal{O} \in \mathcal{O}^G(\Gamma_\pi)} c_{\mathcal{O}}(\pi) \widehat{O}^G_{\mathcal{O}}(Y),$$

where Γ_π is a semisimple element of $\mathfrak{g}^*$ associated to π, for Y near 0

"Near" made precise by Kim–Murnaghan: $G_{\geq r}$, where r is the depth of π (2003) *; or, with a different choice of Γ_π, even $G_{>r/2}$ (2006)

Theorem (S 2018)

$$\Phi_\pi(\gamma \cdot \exp(Y)) = \sum_{\mathcal{O} \in \mathcal{O}^H(\Gamma_\pi)} c_{\mathcal{O}}(\pi, \gamma) \widehat{O}^G_{\mathcal{O}}(Y)$$

for Y sufficiently close to 0, depending explicitly on the depth of π and the singular depth of γ.

* Slightly larger than the domain of validity of the LCE.

Murnaghan–Kirillov asymptotic expansion:

$$\Phi_\pi(\exp(Y)) = \sum_{\mathcal{O} \in \mathcal{O}^G(\Gamma_\pi)} c_{\mathcal{O}}(\pi) \widehat{O}^G_{\mathcal{O}}(Y),$$

where Γ_π is a semisimple element of $\mathfrak{g}^*$ associated to π, for Y near 0

"Near" made precise by Kim–Murnaghan: $G_{\geq r}$, where r is the depth of π (2003) *; or, with a different choice of Γ_π, even $G_{>r/2}$ (2006)

Theorem (S 2018)

$$\Phi_\pi(\gamma \cdot \exp(Y)) = \sum_{\mathcal{O} \in \mathcal{O}^H(\Gamma_\pi)} c_{\mathcal{O}}(\pi, \gamma) \widehat{O}^G_{\mathcal{O}}(Y)$$

for Y sufficiently close to 0, depending explicitly on the depth of π and the singular depth of γ.

For example, $\Gamma_\pi = 0$ generalises the Adler–Korman results on the local character expansion

* Slightly larger than the domain of validity of the LCE.

Theorem (S 2018)

$$\Phi_\pi(\gamma \cdot \exp(Y)) = \sum_{\mathcal{O} \in \mathcal{O}^H(\Gamma_\pi)} c_{\mathcal{O}}(\pi, \gamma) \widehat{O}^G_{\mathcal{O}}(Y)$$

for Y sufficiently close to 0, depending explicitly on the depth of π and the singular depth of γ.

This is a pure existence result; it gives no information about how to compute the $c_{\mathcal{O}}$'s

Theorem (S 2018)

$$\Phi_\pi(\gamma \cdot \exp(Y)) = \sum_{\mathcal{O} \in \mathcal{O}^H(\Gamma_\pi)} c_{\mathcal{O}}(\pi, \gamma) \widehat{O}^G_{\mathcal{O}}(Y)$$

for Y sufficiently close to 0, depending explicitly on the depth of π and the singular depth of γ.

This is a pure existence result; it gives no information about how to compute the $c_{\mathcal{O}}$'s

So we've turned the question from "compute $\Phi_\pi(\gamma \cdot \exp(Y))$" to "compute $c_{\mathcal{O}}(\pi, \gamma)$"

What does it mean to compute an object in harmonic analysis, such as $\Phi_\pi(\gamma \cdot \exp(Y))$ or $c_{\mathcal{O}}(\pi, \gamma)$?

What does it mean to compute an object in harmonic analysis, such as $\Phi_\pi(\gamma \cdot \exp(Y))$ or $c_{\mathcal{O}}(\pi, \gamma)$?

We need:

- a description or parameterisation of representations of G, and then the formula will be written in terms of the parameters for π; and

What does it mean to compute an object in harmonic analysis, such as $\Phi_\pi(\gamma \cdot \exp(Y))$ or $c_{\mathcal{O}}(\pi, \gamma)$?

We need:

- a description or parameterisation of representations of G, and then the formula will be written in terms of the parameters for π; and
- a description or parameterisation of elements of G, and then the formula will be written in terms of the parameters for γ

Parameterising the elements of G is the easier of the two tasks

Parameterising the elements of G is the easier of the two tasks

It's always dangerous to attach names to frequently re-invented concepts

Parameterising the elements of G is the easier of the two tasks

It's always dangerous to attach names to frequently re-invented concepts

But I know about *good* elements (sometimes called 'equi-valued') from Adler

An element γ of G is *good* if it is semisimple, and all of its *non-trivial* root values lie at the same distance from 1

Parameterising the elements of G is the easier of the two tasks

It's always dangerous to attach names to frequently re-invented concepts

But I know about *good* elements (sometimes called 'equi-valued') from Adler

An element γ of G is *good* if it is semisimple, and all of its *non-trivial* root values lie at the same distance from 1

(I.e., it is the "most singular element nearby")

Parameterising the elements of G is the easier of the two tasks

It's always dangerous to attach names to frequently re-invented concepts

But I know about *good* elements (sometimes called 'equi-valued') from Adler

An element γ of G is *good* if it is semisimple, and all of its *non-trivial* root values lie at the same distance from 1

(I.e., it is the "most singular element nearby")

While computing characters of their representations, Corwin studied *normal* elements of division algebras: those elements γ such that all monomials in the p-adic expansion of γ commute

This depends on a choice of uniformiser, and every element is normal with respect to *some* uniformiser

Parameterising the elements of G is the easier of the two tasks

It's always dangerous to attach names to frequently re-invented concepts

But I know about *good* elements (sometimes called 'equi-valued') from Adler

An element γ of G is *good* if it is semisimple, and all of its *non-trivial* root values lie at the same distance from 1

(I.e., it is the "most singular element nearby")

While computing characters of their representations, Corwin studied *normal* elements of division algebras: those elements γ such that all monomials in the p-adic expansion of γ commute

This depends on a choice of uniformiser, and every element is normal with respect to *some* uniformiser

Adler and I generalised this notion to general p-adic groups: in the connected case, replace a commuting sum of monomials by a commuting product of good elements

Given $\gamma = \prod_{i \geq 0} \gamma_i$, we put $\gamma_{<r} = \prod_{0 \leq i < r} \gamma_i$, $\gamma_{\geq r}$, $Y_{\geq r} = \log(\gamma_{\geq r})$, ...

Given $\gamma = \prod_{i \geq 0} \gamma_i$, we put $\gamma_{<r} = \prod_{0 \leq i < r} \gamma_i$, $\gamma_{\geq r}$, $Y_{\geq r} = \log(\gamma_{\geq r})$, ...

The decomposition is not unique, but, for $\gamma \in G^\circ$, the connected centraliser

$$\mathrm{Cent}_{\mathbf{G}}^{(<r)}(\gamma)^\circ := \mathrm{Cent}_{\mathbf{G}}(\gamma_i : 0 \leq i < r)^\circ$$

is well defined for every r

* See also the general discussion of quasisemisimple groups of automorphisms in Adler–Lansky–S (in progress).

Given $\gamma = \prod_{i \geq 0} \gamma_i$, we put $\gamma_{<r} = \prod_{0 \leq i < r} \gamma_i$, $\gamma_{\geq r}$, $Y_{\geq r} = \log(\gamma_{\geq r})$, ...

The decomposition is not unique, but, for $\gamma \in G^\circ$, the connected centraliser

$$\mathrm{Cent}_{\mathbf{G}}^{(<r)}(\gamma)^\circ := \mathrm{Cent}_{\mathbf{G}}(\gamma_i : 0 \leq i < r)^\circ$$

is well defined for every r

S 2021 lifts the connectedness restrictions, and defines $\mathrm{Cent}_{\mathbf{G}}^{(<r)}(\gamma)$ even for $\gamma \in G \setminus G^\circ$ *

* See also the general discussion of quasisemisimple groups of automorphisms in Adler–Lansky–S (in progress).

Given $\gamma = \prod_{i \geq 0} \gamma_i$, we put $\gamma_{<r} = \prod_{0 \leq i < r} \gamma_i$, $\gamma_{\geq r}$, $Y_{\geq r} = \log(\gamma_{\geq r})$, ...

The decomposition is not unique, but, for $\gamma \in G^\circ$, the connected centraliser

$$\mathrm{Cent}_{\mathbf{G}}^{(<r)}(\gamma)^\circ := \mathrm{Cent}_{\mathbf{G}}(\gamma_i : 0 \leq i < r)^\circ$$

is well defined for every r

S 2021 lifts the connectedness restrictions, and defines $\mathrm{Cent}_{\mathbf{G}}^{(<r)}(\gamma)$ even for $\gamma \in G \setminus G^\circ$ *

This decomposition is needed for fine control over the terms appearing in Frobenius-type formulæ for characters of compactly induced representations, such as supercuspidals

* See also the general discussion of quasisemisimple groups of automorphisms in Adler–Lansky–S (in progress).

How to parameterise representations? A common theme:

How to parameterise representations? A common theme:

- pretend to understand Abelian harmonic analysis (1-dimensional representations)

How to parameterise representations? A common theme:

- pretend to understand Abelian harmonic analysis (1-dimensional representations)
- describe higher- (even infinite-) dimensional representations of G in terms of 1-dimensional representations of subgroups of G.

How to parameterise representations? A common theme:

- pretend to understand Abelian harmonic analysis (1-dimensional representations)
- describe higher- (even infinite-) dimensional representations of G in terms of 1-dimensional representations of subgroups of G.

Examples:

- theory of the highest weight: pass from integral $\lambda \in \mathfrak{t}^*$ to the associated character $e^{i\lambda\circ\log}$ of T

How to parameterise representations? A common theme:

- pretend to understand Abelian harmonic analysis (1-dimensional representations)
- describe higher- (even infinite-) dimensional representations of G in terms of 1-dimensional representations of subgroups of G.

Examples:

- theory of the highest weight: pass from integral $\lambda \in \mathfrak{t}^*$ to the associated character $e^{i\lambda\circ\log}$ of T
- Harish-Chandra's parameterisation of discrete-series representations of real, semisimple Lie groups in terms of pairs (T, ϕ)

How to parameterise representations? A common theme:

- pretend to understand Abelian harmonic analysis (1-dimensional representations)
- describe higher- (even infinite-) dimensional representations of G in terms of 1-dimensional representations of subgroups of G.

Examples:

- theory of the highest weight: pass from integral $\lambda \in \mathfrak{t}^*$ to the associated character $e^{i\lambda\circ\log}$ of T
- Harish-Chandra's parameterisation of discrete-series representations of real, semisimple Lie groups in terms of pairs (T, ϕ)
- Corwin–Howe's parameterisation of representations of multiplicative groups of p-adic division algebras and general linear groups in terms of pairs $(E^\times, \phi)$

How to parameterise representations? A common theme:

- pretend to understand Abelian harmonic analysis (1-dimensional representations)
- describe higher- (even infinite-) dimensional representations of G in terms of 1-dimensional representations of subgroups of G.

Examples:

- theory of the highest weight: pass from integral $\lambda \in \mathfrak{t}^*$ to the associated character $e^{i\lambda\circ\log}$ of T
- Harish-Chandra's parameterisation of discrete-series representations of real, semisimple Lie groups in terms of pairs (T, ϕ)
- Corwin–Howe's parameterisation of representations of multiplicative groups of p-adic division algebras and general linear groups in terms of pairs $(E^\times, \phi)$

Other results, such as Moy's description of supercuspidal representations of $\mathrm{GSp}_4(k)$ and $\mathrm{U}(2,1)(k)$, suggested a general phenomenon

Systematically investigated by Adler, and then further generalised by Yu

Corwin–Howe dealt with *admissible* characters by factoring them into *generic* characters

Corwin–Howe dealt with *admissible* characters by factoring them into *generic* characters

Yu takes the factorisation to be the basic object: a Yu datum is a triple $(\vec{\mathbf{G}}, \pi_{-1}, \vec{\phi})$

Corwin–Howe dealt with *admissible* characters by factoring them into *generic* characters

Yu takes the factorisation to be the basic object: a Yu datum is a triple $(\vec{\mathbf{G}}, \pi_{-1}, \vec{\phi})$

Hakim–Murnaghan and Kaletha bring us back to the original perspective by showing that a Yu datum can be replaced by just $(\mathbf{G}^0, \pi_{-1} \prod_{j=0}^{\ell} \phi_j)$

Yu's parameterisation carries an implicit inductive structure: in the course of constructing a positive-depth, supercuspidal representation π of G, we construct a (usually) smaller-depth supercuspidal representation π' of a proper tame, twisted Levi subgroup G' of G

Yu's parameterisation carries an implicit inductive structure: in the course of constructing a positive-depth, supercuspidal representation π of G, we construct a (usually) smaller-depth supercuspidal representation π' of a proper tame, twisted Levi subgroup G' of G

Theorem (S 2018)

Let r be the depth of π. Then the coefficients $c_{\mathcal{O}}(\pi, \gamma_{<r})$ of a depth-r asymptotic expansion for Φ_π near $\gamma_{<r}$ of G' can be computed explicitly in terms of $c_{\mathcal{O}'}({\pi'}^g, \gamma_{<r})$.

Yu's parameterisation carries an implicit inductive structure: in the course of constructing a positive-depth, supercuspidal representation π of G, we construct a (usually) smaller-depth supercuspidal representation π' of a proper tame, twisted Levi subgroup G' of G

Theorem (S 2018)

Let r be the depth of π. Then the coefficients $c_{\mathcal{O}}(\pi, \gamma_{<r})$ of a depth-r asymptotic expansion for Φ_π near $\gamma_{<r}$ of G' can be computed explicitly in terms of $c_{\mathcal{O}'}({\pi'}^g, \gamma_{<r})$.

- "Explicit" is allowed to involve some quite elaborate ingredients, a full understanding of which requires a different linearisation of the Weil representation from the one used by Yu (see Jessica's talk)
- This recipe is not suitable for induction, because the depth of π' is usually strictly less than the depth of π

Plan of attack: Per Harish-Chandra, think simultaneously about Fourier transforms of orbital integrals on the Lie algebra, and characters on the group (as alluded to by Spencer yesterday)

1. Prove analogues of the character results of S 2018 for Fourier transforms of orbital integrals

Plan of attack: Per Harish-Chandra, think simultaneously about Fourier transforms of orbital integrals on the Lie algebra, and characters on the group (as alluded to by Spencer yesterday)

1. Prove analogues of the character results of S 2018 for Fourier transforms of orbital integrals
2. Inductively transfer the expansion of the character of π_0 near γ_0 to an expansion of the character of π' near γ_0, in terms of Fourier transforms of orbital integrals on $\mathrm{C}_{G'}(\gamma_0)^\circ$, evaluated at $Y_{>0}$

Plan of attack: Per Harish-Chandra, think simultaneously about Fourier transforms of orbital integrals on the Lie algebra, and characters on the group (as alluded to by Spencer yesterday)

1. Prove analogues of the character results of S 2018 for Fourier transforms of orbital integrals
2. Inductively transfer the expansion of the character of π_0 near γ_0 to an expansion of the character of π' near γ_0, in terms of Fourier transforms of orbital integrals on $C_{G'}(\gamma_0)^\circ$, evaluated at $Y_{>0}$
3. Expand each Fourier transform of an orbital integral on $C_{G'}(\gamma_0)^\circ$ near $Y_{<r}$ in terms of Fourier transforms of orbital integrals on $C_{G'}(\gamma_{<r})^\circ$, evaluated at $Y_{\geq r}$

Plan of attack: Per Harish-Chandra, think simultaneously about Fourier transforms of orbital integrals on the Lie algebra, and characters on the group (as alluded to by Spencer yesterday)

1. Prove analogues of the character results of S 2018 for Fourier transforms of orbital integrals
2. Inductively transfer the expansion of the character of π_0 near γ_0 to an expansion of the character of π' near γ_0, in terms of Fourier transforms of orbital integrals on $\mathrm{C}_{G'}(\gamma_0)^\circ$, evaluated at $Y_{>0}$
3. Expand each Fourier transform of an orbital integral on $\mathrm{C}_{G'}(\gamma_0)^\circ$ near $Y_{<r}$ in terms of Fourier transforms of orbital integrals on $\mathrm{C}_{G'}(\gamma_{<r})^\circ$, evaluated at $Y_{\geq r}$
4. Combine to obtain an expansion of the character of π' near $\gamma_{<r}$ in terms of Fourier transforms of orbital integrals on $\mathrm{C}_{G'}(\gamma_{<r})^\circ$

Plan of attack: Per Harish-Chandra, think simultaneously about Fourier transforms of orbital integrals on the Lie algebra, and characters on the group (as alluded to by Spencer yesterday)

1. Prove analogues of the character results of S 2018 for Fourier transforms of orbital integrals
2. Inductively transfer the expansion of the character of π_0 near γ_0 to an expansion of the character of π' near γ_0, in terms of Fourier transforms of orbital integrals on $\mathrm{C}_{G'}(\gamma_0)^\circ$, evaluated at $Y_{>0}$
3. Expand each Fourier transform of an orbital integral on $\mathrm{C}_{G'}(\gamma_0)^\circ$ near $Y_{<r}$ in terms of Fourier transforms of orbital integrals on $\mathrm{C}_{G'}(\gamma_{<r})^\circ$, evaluated at $Y_{\geq r}$
4. Combine to obtain an expansion of the character of π' near $\gamma_{<r}$ in terms of Fourier transforms of orbital integrals on $\mathrm{C}_{G'}(\gamma_{<r})^\circ$
5. Use S 2018 to expand the character of π near $\gamma_{<r}$ in terms of Fourier transforms of orbital integrals on $\mathrm{C}_G(\gamma_{<r})^\circ$

Plan of attack: Per Harish-Chandra, think simultaneously about Fourier transforms of orbital integrals on the Lie algebra, and characters on the group (as alluded to by Spencer yesterday)

1. Prove analogues of the character results of S 2018 for Fourier transforms of orbital integrals
2. Inductively transfer the expansion of the character of π_0 near γ_0 to an expansion of the character of π' near γ_0, in terms of Fourier transforms of orbital integrals on $\mathrm{C}_{G'}(\gamma_0)^\circ$, evaluated at $Y_{>0}$
3. Expand each Fourier transform of an orbital integral on $\mathrm{C}_{G'}(\gamma_0)^\circ$ near $Y_{<r}$ in terms of Fourier transforms of orbital integrals on $\mathrm{C}_{G'}(\gamma_{<r})^\circ$, evaluated at $Y_{\geq r}$
4. Combine to obtain an expansion of the character of π' near $\gamma_{<r}$ in terms of Fourier transforms of orbital integrals on $\mathrm{C}_{G'}(\gamma_{<r})^\circ$
5. Use S 2018 to expand the character of π near $\gamma_{<r}$ in terms of Fourier transforms of orbital integrals on $\mathrm{C}_G(\gamma_{<r})^\circ$
6. "Collapse" the expansion of the character of π near $\gamma_{<r}$ into an expansion near γ_0

The tricky part is the “collapse” step 6, which requires a delicate understanding of the relationship among the various expansions

Making this relationship precise is notationally awkward, but here goes:

The tricky part is the "collapse" step 6, which requires a delicate understanding of the relationship among the various expansions

Making this relationship precise is notationally awkward, but here goes:

Theorem (S 2018, as re-phrased in S 2021)

If

$$\Phi_{\pi'^g}(\gamma_{<r} \cdot \exp(Y'_{\geq r})) \quad \textit{equals} \quad \sum_{\mathcal{O}' \in \mathcal{O}^{H'^{\circ}_g}(\mathcal{U}'^*_g)} \widetilde{\mathfrak{G}}_{G'_g/H'_g}(\mathcal{O}', \gamma_{<r}) c_{\mathcal{O}'}(\pi'^g, \gamma_{<r}) \widehat{O}^{H'^{\circ}_g}_{\mathcal{O}'}(Y'_{\geq r}),$$

then

$$\Phi_{\pi}(\gamma_{<r} \cdot \exp(Y_{\geq r})) \quad \textit{equals} \quad \sum_{\substack{g \in G' \backslash G / H^\circ \\ \Gamma_{\pi,g} \in \operatorname{Lie}^*(H)}} \sum_{\mathcal{O}'} \widetilde{\mathfrak{G}}_{G/H}(\mathcal{O}', \gamma_{<r}) c_{\mathcal{O}'}(\pi'^g, \gamma_{<r}) \widehat{O}^{H^\circ}_{\mathcal{O}'}(Y_{\geq r}).$$

Making this relationship precise is notationally awkward, but here goes:

Theorem (S 2021)

If

$$\widehat{O}^{G_g^0}_{\xi_g}(\gamma_{\le 0} + Y^0_{>0}) \quad \textit{equals}$$

$$\sum_{\substack{g^0 \in G_g^0 \backslash G_g^0 / J_g^{0\,\circ} \\ \Gamma_{\xi, gg^0} \in \mathrm{Lie}^*(J_g^0)}} \sum_{\mathcal{O}^0 \in \mathcal{O}^{J_g^{0\,\circ}}(\Gamma_{\xi, gg^0})} \mathfrak{G}_{G_g^0/J_g^0}(\mathcal{O}^0, \gamma_{\le 0}) c_{\mathcal{O}^0}(\xi_{gg^0}, \gamma_{\le 0}) \widehat{O}^{J_g^{0\,\circ}}_{\mathcal{O}^0}(Y^0_{>0}),$$

then

$$\widehat{O}^G_\xi(\gamma_{\le 0} + Y_{>0}) \quad \textit{equals}$$

$$\sum_{\substack{g \in G^0 \backslash G / J^\circ \\ \Gamma_{\xi, g} \in \mathrm{Lie}^*(J)}} \sum_{\mathcal{O}^0} \mathfrak{G}_{G/J}(\mathcal{O}^0, \gamma_{\le 0}) c_{\mathcal{O}^0}(\xi_g, \gamma_{\le 0}) \widehat{O}^{J^\circ}_{\mathcal{O}^0}(Y_{>0}).$$

$$\widehat{O}^G_\xi(\gamma) = \sum_{\substack{g \in G^0 \backslash G / J^\circ \\ \Gamma_{\xi,g} \in \mathrm{Lie}^*(J)}} \sum_{\mathcal{O}^0} \mathfrak{G}_{G/J}(\mathcal{O}^0, \gamma_{\le 0}) c_{\mathcal{O}^0}(\xi_g, \gamma_{\le 0}) \widehat{O}^{J^\circ}_{\mathcal{O}^0}(Y_{>0})$$

and

$$\Phi_\pi(\gamma) = \sum_{\substack{g \in G' \backslash G / H^\circ \\ \Gamma_{\pi,g} \in \mathrm{Lie}^*(H)}} \sum_{\mathcal{O}'} \widetilde{\mathfrak{G}}_{G/H}(\mathcal{O}', \gamma_{<r}) c_{\mathcal{O}'}(\pi'^{g}, \gamma_{<r}) \widehat{O}^{H^\circ}_{\mathcal{O}'}(Y_{\ge r}).$$

$\mathfrak{G}$ is a Gauss sum; $\widetilde{\mathfrak{G}}$ is essentially the same (upon using the exponential map to move between $\mathrm{Lie}(G)_{>0}$ and $G_{>0}$), but also has a contribution from the modified Weil representation that Fintzen–Kaletha–S use in Yu's construction, which depends on the topologically semisimple part γ_0 (see Jessica's talk)

$$\widehat{O}^G_\xi(\gamma) = \sum_{\substack{g \in G^0\backslash G/J^\circ \\ \Gamma_{\xi,g}\in\mathrm{Lie}^*(J)}} \sum_{\mathcal{O}^0} \mathfrak{G}_{G/J}(\mathcal{O}^0, \gamma_{\le 0}) c_{\mathcal{O}^0}(\xi_g, \gamma_{\le 0}) \widehat{O}^{J^\circ}_{\mathcal{O}^0}(Y_{>0})$$

and

$$\Phi_\pi(\gamma) = \sum_{\substack{g \in G'\backslash G/H^\circ \\ \Gamma_{\pi,g}\in\mathrm{Lie}^*(H)}} \sum_{\mathcal{O}'} \widetilde{\mathfrak{G}}_{G/H}(\mathcal{O}', \gamma_{<r}) c_{\mathcal{O}'}(\pi'^g, \gamma_{<r}) \widehat{O}^{H^\circ}_{\mathcal{O}'}(Y_{\ge r}).$$

$\mathfrak{G}$ is a Gauss sum; $\widetilde{\mathfrak{G}}$ is essentially the same (upon using the exponential map to move between $\mathrm{Lie}(G)_{>0}$ and $G_{>0}$), but also has a contribution from the modified Weil representation that Fintzen–Kaletha–S use in Yu's construction, which depends on the topologically semisimple part γ_0 (see Jessica's talk)

This causes no problems, since, in the "re-expansion" step 3, we have already passed to the topologically unipotent element $\gamma_{>0} = \exp(Y_{>0})$, for which the two quantities agree

Theorem (S 2021)

If

$$\Phi_{\pi_0^g}(\gamma_0 \cdot \exp(Y^0_{>0})) \quad \textit{equals}$$

$$\sum_{\substack{g^0 \in G^0_g \backslash G^0_g / J^{0\,\circ}_g \\ \Gamma_{gg^0} \in \mathrm{Lie}^*(J^0_g)}} \sum_{\mathcal{O}^0 \in \mathcal{O}^{J^{0\,\circ}_g}(\Gamma_{\pi, gg^0})} \widetilde{\mathfrak{G}}_{G^0_g / J^0_g}(\mathcal{O}^0, \gamma_0) c_{\mathcal{O}^0}(\pi^g_0, \gamma_0) \widehat{O}^{J^{0\,\circ}_g}_{\mathcal{O}^0}(Y^0_{>0}),$$

then

$$\Phi_\pi(\gamma_0 \cdot \exp(Y_{>0})) \quad \textit{equals}$$

$$\sum_{\substack{g \in G^0 \backslash G / J^\circ \\ \Gamma_g \in \mathrm{Lie}^*(J)}} \sum_{\mathcal{O}^0} \widetilde{\mathfrak{G}}_{G/J}(\mathcal{O}^0, \gamma_0) c_{\mathcal{O}^0}(\pi^g_0, \gamma_0) \widehat{O}^{J^\circ}_{\mathcal{O}^0}(Y_{>0}).$$

Our results on the Lie algebra may be viewed as *completely* reducing the behaviour of Fourier transforms of orbital integrals to that of Fourier transforms of nilpotent orbital integrals (subject to tameness conditions; see Fintzen)

Our results on the Lie algebra may be viewed as *completely* reducing the behaviour of Fourier transforms of orbital integrals to that of Fourier transforms of nilpotent orbital integrals (subject to tameness conditions; see Fintzen)

For example, they recover a version of the Shalika germ expansion with an explicit domain of validity, centred at any semisimple element

Our results on the Lie algebra may be viewed as *completely* reducing the behaviour of Fourier transforms of orbital integrals to that of Fourier transforms of nilpotent orbital integrals (subject to tameness conditions; see Fintzen)

For example, they recover a version of the Shalika germ expansion with an explicit domain of validity, centred at any semisimple element

If we are dealing with semisimple orbital integrals, then we may also recover a quantitative version of a result of Waldspurger:

Corollary (Waldspurger 1995 and S 2021)

If $\gamma_{\le 0} \in \mathrm{Lie}(G)$ *is regular semisimple, then*

$$\widehat{O}^G_\xi(\gamma_{\le 0} + Y_{>0}) \quad \textit{equals} \quad \sum_{\substack{g \in G^0 \backslash G / J^0 \\ \Gamma_{\xi,g} \in \mathrm{Lie}^*(J)}} \mathfrak{G}_{G/G^0_g}(\Gamma_{\xi,g}, \gamma_{\le 0}) \Lambda_{\xi_g}(\gamma_{\le 0} + Y_{>0}).$$

Our results on the group specialise to those of Kim–Murnaghan (when $\gamma = 1$) and to (the DeBacker and Adler–Korman quantitative versions of) the Harish-Chandra–Howe local character expansion (when $\Gamma_\pi = 0$)

Our results on the group specialise to those of Kim–Murnaghan (when $\gamma = 1$) and to (the DeBacker and Adler–Korman quantitative versions of) the Harish-Chandra–Howe local character expansion (when $\Gamma_\pi = 0$)

They completely reduce the computation of arbitrary supercuspidal representations to those of depth-zero supercuspidal representations

Our results on the group specialise to those of Kim–Murnaghan (when $\gamma = 1$) and to (the DeBacker and Adler–Korman quantitative versions of) the Harish-Chandra–Howe local character expansion (when $\Gamma_\pi = 0$)

They completely reduce the computation of arbitrary supercuspidal representations to those of depth-zero supercuspidal representations (which the work of DeBacker–Tsai suggests is the really hard part . . .)

Our results on the group specialise to those of Kim–Murnaghan (when $\gamma = 1$) and to (the DeBacker and Adler–Korman quantitative versions of) the Harish-Chandra–Howe local character expansion (when $\Gamma_\pi = 0$)

They completely reduce the computation of arbitrary supercuspidal representations to those of depth-zero supercuspidal representations (which the work of DeBacker–Tsai suggests is the really hard part ...)

They specialise to a result of Kaletha, which I regard as the group analogue of Waldspurger's result on Fourier transforms of orbital integrals:

Corollary (Kaletha 2019 and S 2021)

If γ_0 is regular semisimple, then

$$\Phi_\pi(\gamma_0 \cdot \exp(Y)) \quad \textit{equals} \sum_{\substack{g \in G^0 \backslash G / J^0 \\ \Gamma_{\pi,g} \in \operatorname{Lie}^*(J)}} \widetilde{\mathfrak{G}}_{G/G^0_g}(\Gamma_{\pi,g}, \gamma) \Phi_{\pi_0}(\gamma) \Lambda_{\Gamma_{\pi,g}}(Y).$$