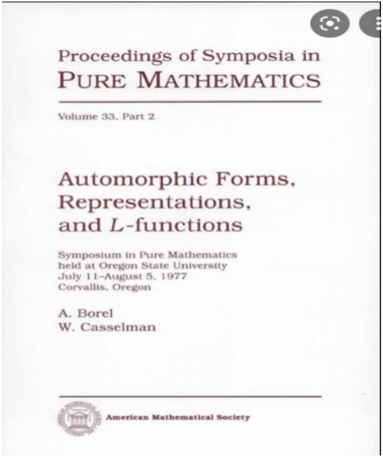
In Honor of Bill Casselman's 80th birthday

Basic Functions, Orbital Integrals, and Beyond Endoscopy (Online)

The Corvallis Proceedings



The IHES Summer School 2022

It has been almost 45 years since the influential summer school held in Corvallis, Oregon in 1977 brought together the leading experts of the Langlands program and defined the research agenda in this area for subsequent decades, at the same time inspiring and enabling several generations of young researchers to join in this exciting journey. This 3-week IHES summer school aims to do the same for the next phase of development in the Langlands program.

Arthur's Conjecture

- F a number field with adele ring $\mathbb{A}$
- G a classical group over F

Arthur's Conjecture

(a) Decomposition into near equivalence classes:

$$L^2(G(F)\backslash G(\mathbb{A})) = \bigoplus_{\psi} L^2_{\psi}$$

with indexing set $\{\psi\}$ given by the elliptic global A-parameters of G and L^2_{ψ} the associated near equivalence class.

(b) Decomposition of L^2_{ψ} in terms of local and global A-packets and the Arthur multiplicity formula.

In (a), the role of A-parameters is played by certain isobaric sums of cuspidal representations of GL's. So (a) amounts to showing weak lifting from G to GL and a description of the image of the lift.

Quasi-Split G

For quasi-split G, this conjecture has been shown:

- by Arthur for $\mathrm{Sp}(2n)$, $\mathrm{SO}(2n+1)$ and $\mathrm{O}(2n)$
- by Mok for $\mathrm{U}(n)$.

This is achieved by the stable trace formula for G and the stable twisted trace formula of GL.

When G is not quasi-split, the stable trace formula of G can be used to deduce Arthur's conjecture (via comparison with the quasi-split form). This was carried out:

- by Kaletha-Minguez-Shin-White for $\mathrm{U}(n)$;
- by Taibi for some cases of $\mathrm{O}(2n)$.

Goal of This Talk

I will discuss the thesis work of two of my students:

Rui Chen and Jialiang Zou

Using theta correspondence, they show how one can deduce Arthur's conjecture for non-quasi-split G from the quasi-split case in a rather efficient manner.

Theta Correspondence in Corvallis

About theta correspondence.....it was rumoured that Langlands is not a fan of it because it is viewed as an ad hoc tool which does not really fit into the Langlands program.

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The IHES summer school will discuss the relative Langlands program and explain how the theta correspondence fits into that broader framework.

Setup of Theta Correspondence

- Reductive dual pairs $G \times H$:

$$\mathrm{O}(2n) \times \mathrm{Sp}(2r), \quad \mathrm{O}(2n+1) \times \mathrm{Mp}(2r), \quad \mathrm{U}(n) \times \mathrm{U}(r).$$

- Weil representation Ω of $G \times H$.
- The local theta correspondence studies the spectral decomposition of the $G \times H$-module Ω:
  - smooth version: which $\pi \otimes \sigma \in \mathrm{Irr}(G \times H)$ is a quotient of Ω?
  - L^2-version:

$$\Omega = \int_{\widehat{G}} \pi \otimes \theta(\pi)\, d\mu_G$$

- The global theta correspondence gives a transfer of cuspidal automoprhic representations from G to H.

The Questions

The main questions studied in local theta correspondence are:

- Nonvanishing: is $\pi \in \mathrm{Irr}(G)$ a quotient of Ω?
- Howe duality theorem: if $\pi \otimes \sigma$ is a quotient of Ω, then π and σ determines each other; write $\sigma = \theta(\pi)$. So for $\pi \in \mathrm{Irr}(G)$, $\theta(\pi)$ is irreducible or 0.
- So we have:

  $$\theta : \mathrm{Irr}(G)_{\theta \neq 0} \longrightarrow \mathrm{Irr}(H)$$

  Describe the map θ concretely (e.g. in terms of LLC or other means).

This has in fact been achieved for p-adic F in my work with Savin, Ichino, Atobe and the work of Bakić-Hanzer (but these are irrelevant for this talk).

Stable Range (Local)

Assume that $G \times H = \mathrm{O}(2n) \times \mathrm{Sp}(2r)$ with $r > 2n$: this is the stable range.

- In this case, theta correspondence defines an injective map

$$\theta : \mathrm{Irr}(G) \longrightarrow \mathrm{Irr}(H)$$

- Indeed, it was shown by Jianshu Li that

$$\theta : \widehat{G} \longrightarrow \widehat{H}.$$

- Further, the image of θ is precisely the subset of $\widehat{H}$ consisting of those representations of rank $2n$.

For $H = \mathrm{Sp}(2r)$, $\sigma \in \mathrm{Irr}(H)$ has rank between 1 and r, and those of rank $< r$ are said to have low rank.

Adams Conjecture

In 1989, J. Adams suggested that it might be more natural to describe θ in terms of A-parameters rather than L-parameters.

Consider $G \times H = \mathrm{O}(2n) \times \mathrm{Sp}(2r)$ for example, so that

$$G^\vee = \mathrm{O}(2n) \quad \text{and} \quad H^\vee = \mathrm{SO}(2r+1).$$

One has a map

$$G^\vee \times \mathrm{SL}(2) \to \mathrm{O}(2n) \times \mathrm{SO}(2r-2n+1) \to \mathrm{O}(2r+1).$$

Conjecture: If Π_ψ^G is an A-packet of G with A-parameter ψ, then

$$\theta(\Pi_\psi^G) \subset \Pi_{\theta(\psi)}^H$$

with

$$\theta(\psi) = \chi_{\det(\psi)} \cdot (\psi + S_{2r-2n+1})\,.$$

Stable Range (Global)

Assume $G \times H$ is in stable range with G small.

Theorem (Jianshu Li)

Suppose that

$$L^2(G(F)\backslash G(\mathbb{A})) = \int_{\widehat{G(\mathbb{A})}} m(\pi) \cdot \pi \, d\mu(\pi).$$

Then

$$L^2(H(F)\backslash H(\mathbb{A})) \supset \int_{\widehat{G(\mathbb{A})}} m(\pi) \cdot \theta(\pi) \, d\mu(\pi).$$

This is the global analog of the local theorem that local theta lifting is nonvanishing in the stable range and takes unitary representations to unitary representations. A key point is that this theorem allows one to lift noncuspidal representations.

Jianshu Li's inequalities

For $\pi \in \widehat{G(\mathbb{A})}$, let

$$\begin{cases} m_{disc}(\pi) = \dim \mathrm{Hom}_{G(\mathbb{A})}(\pi, \mathcal{A}^2(G)) \\ m_{aut}(\pi) = \dim \mathrm{Hom}_{G(\mathbb{A})}(\pi, \mathcal{A}(G)). \end{cases}$$

Corollary

For $\pi \in \widehat{G(\mathbb{A})}$, one has:

$$m_{disc}(\pi) \leq m_{disc}(\theta(\pi)) \leq m_{aut}(\theta(\pi)) \leq m_{aut}(\pi).$$

The Main Idea

Given

- a non-quasi-split classical G,
- a local A-parameter ψ of G

one would like to construct the local A-packet Π_{ψ}^{G}.

Consider dual pair $G \times H$ in stable range with H quasi-split and sufficiently large. Now set

$$\Pi_{\psi}^{G} := \theta^{-1}(\Pi_{\theta(\psi)}^{H}) \subset \widehat{G}.$$

Indeed, any $\sigma \in \Pi_{\theta(\psi)}^{H}$ is unitary and one checks that it is of the appropriate rank. Hence by local results of Li, $\sigma = \theta(\pi)$ for some $\pi \in \widehat{G}$.

LLC

If the local A-parameter ψ is tempered, i.e. it is a tempered L-parameter, then the above construction produces the candidate tempered L-packet for G. To go from here to the LLC for G, there are of course things to check:

- elements of Π_ψ^G are tempered (resp. discrete series) if ψ is tempered (resp. d.s.);
- there is a natural inclusion $\Pi_\psi^G \hookrightarrow \mathrm{Irr}(S_\psi)$ with expected image;
- elements of different Π_ψ^G (as ψ varies over all tempered L-parameters) are disjoint;
- the union of the Π_ψ^G's exhaust $\widehat{G}_{temp}$.
- the set Π_ψ^G is independent of the choice of H.

Desiderata and Proof

In addition, one verifies the various desiderata of the LLC:

- compatibility of Plancherel measures or local factors;
- the local intertwining relation (LIR) describing the action of normalized intertwining operators on tempered generalized principal series.

These properties characterize the LLC for G uniquely.

The proof of the above results and properties require:

- detailed knowledge of local A-packets for quasi-split groups, especially those of Moeglin, Xu and Atobe on the explicit construction of local A-packets and the Jacquet modules of the reps in them;
- good control of the local theta correspondence;
- global arguments.

Example: Mp($2n$) and SO($2n+1$)

$$\text{LLC for split } \mathrm{SO}(2n+1)$$
$$\downarrow$$
$$\text{LLC for } \mathrm{Mp}(2n)$$
$$\downarrow$$
$$\text{LLC for nonsplit } \mathrm{SO}(2n+1)$$

This approach differs from what I had done with G. Savin. There, we assume the LLC for both split and nonsplit SO($2n+1$) and then deduce the LLC for Mp($2n$) by theta correspondence in the equal rank setting (the LLC for nonsplit SO($2n+1$) was claimed/shown by Moeglin-Renard, via simple stable TF).

Assigning A-parameters

Theorem (Chen-Zou)

Let G be a non-quasi-split classical group over a number field F. Then

$$L^2_{disc}(G(F)\backslash G(\mathbb{A})) = \bigoplus_{\psi} L^2_{\psi}$$

where ψ runs over elliptic A-parameters for G and L^2_{ψ} is the full near equivalence class determined by ψ.

This amount to showing: for any $\pi \subset L^2_{dic}$, π has a weak lifting to GL whose image is given by an elliptic A-parameter for G.

Applying Li's inequalities

The proof of the previous theorem proceeds by:

- Apply Li's inequality to $G \times H$ (with H quasi-spit and large):

  $$m_{disc}(\pi) \leq m_{disc}(\theta(\pi)) \leq m_{aut}(\theta(\pi)) \leq m_{aut}(\pi).$$

  to deduce:

  $$m_{disc}(\pi) > 0 \Longrightarrow m_{disc}(\theta(\pi)) > 0.$$

  so that $\theta(\pi)$ has an associated elliptic A-parameter Ψ.
- Using poles of standard L-functions, show that

  $$\Psi = \psi + S_k$$

  for ψ an elliptic A-parameter for G.

Explicating L^2_ψ

It remains to explicate L^2_ψ for each ψ, using the local A-packets constructed before. Li's results imply: if

$$L^2_\psi = \bigoplus_\pi m(\pi) \cdot \pi$$

then

$$\bigoplus_\pi m(\pi) \cdot \theta(\pi) \subset L^2_{\theta(\psi)}.$$

The module structure of $L^2_{\theta(\psi)}$ is given byu the Arthur multiplicity formuls for H. If the above containment is an equality, then we can transport this structure back by θ to deduce the AMF for L^2_ψ.

Hence we need to show: for $\sigma \subset L^2_{\theta(\psi)}$,

$$m_{disc}(\theta^{-1}(\sigma)) = m_{disc}(\sigma).$$

When does Equality hold?

Consider Jianshu Li's inequalities:

$$m_{disc}(\pi) \leq m_{disc}(\theta(\pi)) \leq m_{aut}(\theta(\pi)) \leq m_{aut}(\pi).$$

The desired equality would hold if one can show:

$$m_{disc}(\pi) = m_{aut}(\pi).$$

This is known to hold under one of the following conditions:

- π belongs to a tempered (or generic) A-packet Π_ψ; in this case, one shows that $m_{cusp}(\pi) = m_{aut}(\pi)$.
- G has F-rank ≤ 1.

In these cases, one has a precise description of L^2_ψ. In particular, when G has F-rank ≤ 1, Arthur's conjecture holds.

A Question

- In local representation theory, we have Casselman's criterion for square-integrability as well as for temperedness.
- In the global setting, one has the analogous criterion for square-integrability: an automorphic form f lies in $L^2(G(F)\backslash G(\mathbb{A}))$ if and only if the cuspidal exponents of f are strictly negative linear combinations of the simple roots.

Question: Is there a global analog of Casselman's temperedness criterion which detects if an automorphic (sub)representation is weakly contained in $L^2(G(F)\backslash G(\mathbb{A}))$ through its cuspidal exponents?

An affirmative answer to this question would go a long way towards establishing equality in Li's inequalities.

What Else is Missing?

What about the (twisted) endoscopic character identities?

- the technique of theta correspondence does not make use of characters and therefore say nothing about the endoscopic character identities. To show these character identities, one would still need to resort to the stable trace formula. However, since one is interested in a local result, it should suffice to use a simple version of the STF. Moreover, one is applying the simple STF with essentially full control of the spectral side. For $\mathrm{Mp}(2n)$, this was carried out in Caihua Luo's thesis.
- T. Prezebinda shows how the characters of $\theta(\pi)$ and π are related in the stable range (via Cauchy-Harish-Chandra integrals). This type of character identities relates the character of individual representations, and not just the stable sum of characters. Perhaps one can say that such character identities go beyond endoscopy?

Returning to the IHES Summer School

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Happy Birthday, Bill!