The Shintani–Casselman–Shalika formula and its generalizations; harmonic analysis, L-functions, and geometry.

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Casselman @ 80, BIRS (online), Friday, November 19, 2021

G: connected reductive group over $\mathfrak{o} \subset F$, $\mathfrak{o} \twoheadrightarrow \mathbb{F}_q$.

$X^{\curvearrowleft G}$. Problem: Compute eigenfunctions of $\mathcal{H}(G, K)$ on $X = X(F)$ (where $G = G(F)$, $K = G(\mathfrak{o})$).

The problem shows up, e.g., when $X = H\backslash G$ has some multiplicity-1 property, and for an automorphic representation π of G over a global field k, the map $\pi \to C^\infty(X(\mathbb{A}))$ sending f to $g \mapsto \int_{[H]} f(hg)dh$ is of the form $\prod_v \Phi_v(g)$, with $\Phi_v \in C^\infty(X(k_v))$ an $\mathcal{H}(G(k_v), G(\mathfrak{o}_v))$-eigenfunction at almost every place.

In two 1980 papers in Compositio, Casselman and Casselman–Shalika introduced a new method to solve this problem, applied to the group case ($X = H$, $G = H \times H$) and to the Whittaker model ($X = N\backslash G$, with $C^\infty(X)$ replaced by $C^\infty(X, \mathcal{L}_\psi) = \mathrm{Ind}_N^G(\psi)$).

The goal of this talk is to revisit this method, in the light of subsequent developments.

What you need in order to follow this talk

$G \supset B, \;\; 1 \to N \to B \to A \to 1.$

$N\backslash G/K \leftrightarrow A/A(\mathfrak{o}) = \Lambda$ by $[\lambda(\varpi)] \leftrightarrow \lambda$.

L^2-normalized action of A on $N\backslash G$, $\;\; a \cdot f(Ng) = \delta^{-\frac{1}{2}}(a) f(Nag)$.

$\mathcal{S}(X) := C_c^\infty(X)$, basis for $\mathcal{S}(N\backslash G)^K$ consisting of

$$e^\lambda := \varpi^\lambda \cdot 1_{N\backslash NK} = q^{\langle \rho, \lambda \rangle} 1_{N\varpi^{-\lambda}K}.$$

$\mathrm{Hom}(\Lambda, \mathbb{C}^\times) = \check{A} =$ the Langlands dual torus of A.

For $\chi \in \check{A}$, the (A, χ)-eigenfunctions in $C^\infty(N\backslash G)$ to be denoted by $I(\chi)$ (normalized induction, unramified principal series).

Fixing suitable invariant measures throughout, identifying $C^\infty(X)$ as the smooth dual of $\mathcal{S}(X)$.

What you need in order to follow this talk

Mellin transforms $\mathcal{S}(N\backslash G) \to I(\chi)$,

$$\hat{f}(\chi, g) = \int_A (a \cdot f)(g)\chi^{-1}(a)da.$$

The Mellin transform of e^λ is λ understood as a character of $\check{A}$, also to be denoted e^λ.

In this notation, $(1 - e^\lambda)^{-1}$ means the function $\sum_{n\geq 0} e^{n\lambda}$, which has Mellin transform $(1 - e^\lambda(\chi))^{-1}$.

Basic example: $N\backslash \mathrm{SL}_2 = F^2 \smallsetminus \{0\}$, the function

$$1_{\mathfrak{o}^2} = \frac{1}{1 - q^{-1}e^\alpha} = \sum_{n\geq 0} 1_{\varpi^n \cdot (\mathfrak{o}^\times)^2}$$

This function is invariant under Fourier transform on (the symplectic space) F^2, which acts as $\hat{f}(\chi) \leftrightarrow \frac{1-q^{-1}e^{-\alpha}}{1-q^{-1}e^\alpha}\hat{f}(\chi^{-1})$

The Whittaker model

For simplicity, from now on, G is *split*. (CS formula applies to general unramified groups.)

Fix a maximal unipotent $N^- \subset G$ over $\mathfrak{o}$, and let $\psi : N^- \to \mathbb{C}^\times$ be a character whose restriction to any simple root space has conductor $\mathfrak{o}$. It defines the Whittaker model $\mathrm{Ind}_{N^-}^G(\psi) = C^\infty(N^-, \psi \backslash G)$.

The (Shintani for GL_n, Casselman–Shalika for general G) formula for $\mathcal{H}(G, K)$-eigenfunctions on the Whittaker model

$$I(\chi)^K \ni \phi_{K,\chi} \mapsto \Omega_\chi \in C^\infty(N^-, \psi \backslash G)^K$$

states that (up to an arbitrary scalar)

$$q^{\langle \rho, \lambda \rangle} \Omega_\chi(\varpi^{-\lambda}) = \begin{cases} \mathrm{tr}(\chi, V_\lambda^\vee), & \text{if } \lambda \text{ is dominant,} \\ 0 & \text{otherwise,} \end{cases}$$

where $\mathrm{tr}(\chi, V_\lambda^\vee) =$ the trace of $\chi \in \check{A}$ on the dual of the irreducible $\check{G}$-module with highest weight λ.

The Whittaker model – dual formulation

For $\lambda \in \Lambda^+$ (dominant), let W_λ denote the "basic Whittaker function" with $W_\lambda|_{\varpi^{-\lambda}K} = q^{-\langle \rho, \lambda \rangle}$ and $W_\lambda = 0$ off $N^- \varpi^{-\lambda} K$.

Consider the Satake isomorphism $\mathcal{H}(G, K) \xrightarrow{\sim} \mathbb{C}[\mathrm{Rep}\check{G}]$, denoted $h_\lambda \leftrightarrow V_\lambda$.

Theorem

$$h_\lambda \cdot W_0 = W_\lambda.$$

Remark: If, instead, we replaced ψ by the trivial character, we would have the definition of the Satake isomorphism:

$$h_\lambda \cdot 1_{N^- K} = \operatorname{tr} V_\lambda,$$

where $\operatorname{tr} V_\lambda$ is understood as a function on $N^- \backslash G / K$ as explained previously (i.e., its Mellin transform is $\chi \mapsto \operatorname{tr} V_\lambda(\chi)$).

Proof that Theorem ⇒ CS formula.
$q^{\langle \rho, \lambda \rangle} \Omega_\chi(\varpi^{-\lambda}) = \langle \Omega_\chi, W_\lambda \rangle = \langle \Omega_\chi, h_\lambda \cdot W_0 \rangle = \left\langle h_\lambda^\vee \cdot \Omega_\chi, W_0 \right\rangle =$
$\operatorname{tr}(\chi, V_\lambda^\vee) \langle \Omega_\chi, W_0 \rangle = \operatorname{tr}(\chi, V_\lambda^\vee).$ □

Radon transforms

From now on, use $X = (N^-, \psi)\backslash G$.

Up to now, we have not fixed a morphism $I(\chi) \to C^\infty(X)$, but we can fix one (up to Haar measures) as the *adjoint of the χ^{-1}-Radon transform*

$$R_{\chi^{-1}} : \mathcal{S}(X) \to I(\chi^{-1}),$$

sending Φ to $g \mapsto \int_B \Phi(N^- bg)\chi\delta^{-\frac{1}{2}}(b)db$.

This is the χ^{-1}-Mellin transform composed with the Radon transform

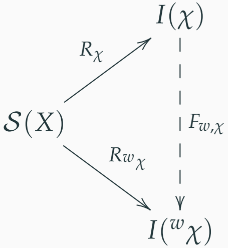
$$R : \mathcal{S}(X) \to \mathcal{S}^+(N\backslash G),$$

sending Φ to $g \mapsto \int_N \Phi(N^- ng)dn$. (Doesn't quite preserve compact support, but Mellin transform makes sense by meromorphic continuation.)

The problem of computing Ω_χ is equivalent to the problem of computing $R(W_\lambda)$ for all (dominant) λ.

Functional equations

The idea (Idea 1) of Casselman was to use *functional equations*,



for some (meromorphic in w) family of intertwining operators F_w,

and (Idea 2), instead of the unramified functions, to compute $R(W_{J,\lambda})|_B$ for $W_{J,\lambda} =$ the Iwahori-invariant Whittaker function supported on $N^-\varpi^{-\lambda}B(\mathfrak{o})$:

$$R(W_{J,\lambda})|_B = 1_{N\varpi^{-\lambda}B(\mathfrak{o})}.$$

Dually, if $\phi_{J,\chi} \in I(\chi)$ is the image of 1_{NJ} under Mellin transform, this says that, for λ dominant,

$$R^*_{\chi^{-1}} \phi_{J,\chi}(\varpi^{-\lambda}) = q^{-\langle \rho, \lambda \rangle} \chi(\varpi^{-\lambda}).$$

If we can write a spherical vector in terms of the operators F_w^* applied to $\phi_{J,\chi}$,

$$\phi_{K,\chi} = \sum_W c_w(\chi) F_w^* \phi_{J,{}^w\chi},$$

this gives the desired formula

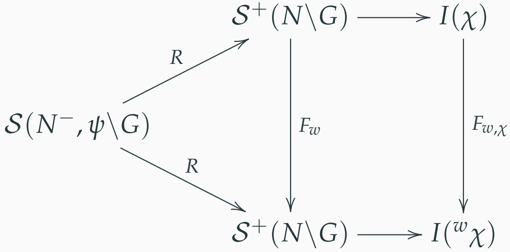
$$\Omega_\chi(\varpi^{-\lambda}) := R^*_{\chi^{-1}} \phi_{K,\chi}(\varpi^{-\lambda}) = \sum_W c_w(\chi) q^{-\langle \rho, \lambda \rangle} \chi(\varpi^{-\lambda}).$$

Remarkably, these look like eigenfunction for the torus A, although it doesn't act on the Whittaker model!

The same arguments work to compute $\mathcal{H}(G, K)$-eigenvectors $\Omega_\chi \in C^\infty(X)$ for every X with an open B-orbit (& good integral model). The only thing that changes are the intertwiners $F_{w,\chi}$, which we will now describe for the Whittaker model.

Fourier transforms

As was known to Gelfand and Kazhdan, the operators F_w that make the diagram above commute are the *Fourier transforms on the basic affine space*,



Assuming G simply connected, for every simple root α the fibers of $N\backslash G \to [P_\alpha, P_\alpha]\backslash G$ are $\simeq N_{\mathrm{SL}_2}\backslash \mathrm{SL}_2 \simeq F^2 \smallsetminus \{(0,0)\}$, and Fourier transform $F_{w_\alpha} : \mathcal{S}^+(N\backslash G) \to \mathcal{S}^+(N\backslash G)$ is defined fiberwise, with a symplectic structure on F^2 determined by the Whittaker character.

(This has been used by Nadya Gurevich and D. Kazhdan to extend the definition of Fourier transforms to the general quasisplit case.)

Digression: Bernstein–Casselman asymptotics

Casselman's theorem: For an admissible representation π of G, there is an invariant pairing of Jacquet modules $\pi_N \otimes \pi_{N^-} \to \mathbb{C}$, such that the matrix coefficients asymptotically (on $t \in A$ sufficiently dominant) satisfy

$$\langle \pi(t)v, \tilde{v} \rangle = \langle \pi_N(t)v_N, \tilde{v}_{N^-} \rangle .$$

Generalized by Bernstein to arbitrary smooth representations; can be formulated in terms of a $G \times G$-equivariant morphism $f \mapsto f_\emptyset : C^\infty(G) \to C^\infty(G_\emptyset)$, where the asymptotic cone is

$$G_\emptyset = A^{\mathrm{diag}}(N \times N^-)\backslash(G \times G).$$

This map is characterized by the property that $f = f_\emptyset$ when restricted to "a neighborhood of infinity" (e.g., evaluated on sufficiently dominant elements of $(T \times 1) \subset (G \times G)$.

Asymptotics for the Whittaker model

There is a similar map $W \mapsto W_\varnothing \colon C^\infty(N^-, \psi\backslash G) \to C^\infty(N^-\backslash G)$, and its restriction to compactly supported functions is related to Radon transforms by the diagram

$$\begin{array}{ccc} \mathcal{S}(N^-, \psi\backslash G) & \longrightarrow & \mathcal{S}^+(N^-\backslash G) \\ & {\scriptstyle R}\searrow \quad \swarrow{\scriptstyle R_\varnothing} & \\ & S^+(N\backslash G). & \end{array}$$

Casselman's Idea 1 + Idea 2 combine to give the following surprising corollary, for which I don't know a conceptual reason:

Proposition
For W unramified, the asymptotic equality holds on the entire dominant cone:

$$W(\varpi^{-\lambda}) = W_\varnothing(\varpi^{-\lambda}), \;\; \lambda \in \Lambda^+.$$

$$W(\varpi^{-\lambda}) = W_{\emptyset}(\varpi^{-\lambda}), \;\; \lambda \in \Lambda^{+}.$$

To prove the Theorem ($h_\lambda \cdot W_0 = W_\lambda$), now, it suffices to calculate RW_0, the Radon transform of the basic function. Indeed,

$$h_\lambda \cdot W_0(\varpi^{-\mu}) = h_\lambda \cdot R_{\emptyset}^{-1} \circ RW_0(\varpi^{-\mu}),$$

and the inversion $R_{\emptyset}^{-1}$ of Radon transform (standard intertwining operator) is well-known on spherical functions, while the action of h_λ is given by its Satake transform.

Given its invariance under Fourier transforms, and certain support restrictions, there are not many options for $R_{\emptyset}W_0$. (Recall Fourier: $\hat{f}(\chi) \leftrightarrow \frac{1-q^{-1}e^{-\alpha}}{1-q^{-1}e^{\alpha}} \hat{f}(\chi^{-1})$.)

Theorem
We have $R_{\emptyset}W_0 = \prod_{\check{\alpha}>0}(1 - q^{-1}e^{-\check{\alpha}})$,

and $W_{0,\emptyset} = R_{\emptyset}^{-1} \circ RW_0 = \prod_{\check{\alpha}>0}(1 - e^{-\check{\alpha}})$

Theorem
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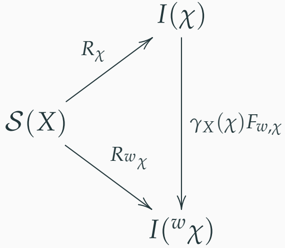
Corollary
We have
$h_\lambda \cdot W_0 = h_\lambda \cdot \prod_{\check{\alpha}>0}(1 - e^{-\check{\alpha}})|_{\Lambda^+} = e^{-\check{\rho}} \sum_W (-1)^w e^{\check{\rho}+\lambda}\big|_{\Lambda^+} = e^\lambda|_{\Lambda^+}$
$\Rightarrow h_\lambda \cdot W_0 = W_\lambda.$

Other spherical spaces

For more general spherical varieties X, the functional equations



involve multiples of the Fourier transforms by certain abelian gamma factors, corresponding to a representation $\check{A} \to \mathrm{GL}(V_X)$ that *determines the "L-function of the spherical variety."*

These gamma factors, in turn, modify the asymptotic formula

$$W_{0,\emptyset} = \prod_{\check{\alpha}>0}(1 - e^{-\check{\alpha}})$$

by an abelian local L-factor,

$$\Phi_{0,\emptyset} = L(\chi, V_X) \cdot \prod_{\check{\alpha}>0}(1 - e^{-\check{\alpha}}).$$

Example: In the group case, $X = H$, $\Phi_0 = 1_{H(\mathfrak{o})}$, and its asymptotics are

$$\Phi_{0,\emptyset} = \prod_{\check{\alpha}>0} \frac{1 - e^{-\check{\alpha}}}{1 - q^{-1}e^{-\check{\alpha}}},$$

the product ranging over positive coroots of H. This implies the *Macdonald formula for zonal spherical functions* (reproved by Casselman).

Here, $V_X = \check{\mathfrak{n}}_-$, and this formula implies the formula for the unramified Plancherel measure, $\frac{L(\chi,\check{\mathfrak{g}}/\check{\mathfrak{a}},1)}{L(\chi,\check{\mathfrak{g}}/\check{\mathfrak{a}},0)}$.

Digression: basic functions

Let V be a representation of $\check{H}$ on which the center of $\check{H}$ acts by a nontrivial character. The formal sum $L_V := \bigoplus_{n\geq 0} \operatorname{Sym}^n V$ corresponds under the Satake isomorphism to a series h_{LV} of elements in the Hecke algebra of H. Casselman asked for a calculation of this series, as a function on $K_H \backslash H / K_H = \Lambda_H^+$.

This is motivated by "Beyond Endoscopy," where one would like to feed L-functions into the trace formula, in the form of non-compactly supported test functions (of the form above).

Answer (S.; $\exists$ similar formula by W.W. Li):

$$h_{LV} = L_V \cdot \prod_{\check{\alpha}>0} \frac{1 - e^{-\check{\alpha}}}{1 - q^{-1} e^{-\check{\alpha}}} \bigg|_{\Lambda^+} .$$

Other spherical spaces (cont.)

Example: For the $G = \mathrm{GL}_n \times \mathrm{GL}_{n+1}$-Rankin–Selberg variety, $X = \mathrm{GL}_n^{\mathrm{diag}} \backslash G$, with $\Phi_0 = 1_{X(\mathfrak{o})}$, its asymptotics are

$$\Phi_{0,\emptyset} = \frac{\prod_{\check{\alpha}>0}(1 - e^{-\check{\alpha}})}{\prod_{\theta \in \Theta^+}(1 - q^{-\frac{1}{2}} e^{-\theta})},$$

where θ ranges over *half* the weights of the tensor product representation and its dual $\bigotimes \oplus \bigotimes^\vee : \check{G} \to \mathrm{GL}_{n(n+1)}$ (those with $\langle \rho, \theta \rangle > 0$).

Geometric meaning (joint w. Jonathan Wang)

Consider X with $\check{G}_X = \check{G}$ (a condition that implies that B acts with trivial generic stabilizers, such as in the Whittaker model and the Rankin–Selberg case).

As we have seen, the gist of the CS method is the computation of the functional equations satisfied by $\pi_! \Phi_0 = R\Phi_0|_B$, where $\pi : X \to X /\!\!/ N = \operatorname{spec} F[X]^N$.

Example: In the Whittaker case,

$$\pi_! \Phi_0 = \prod_{\check{\alpha}>0} (1 - q^{-1} e^{-\check{\alpha}}),$$

and in the Rankin–Selberg case

$$\pi_! \Phi_0 = \frac{\prod_{\check{\alpha}>0} (1 - q^{-1} e^{-\check{\alpha}})}{\prod_{\theta \in \Theta^+} (1 - q^{-\frac{1}{2}} e^{-\theta})}.$$

Geometric meaning (joint w. Jonathan Wang)

The basic function Φ_0 can be defined even when X is (affine and) singular, and is obtained by the sheaf-function dictionary from the *intersection complex of the arc space L^+X* (really, defined via finite-dimensional global models, [Bouthier–Ngô–S.]; here, we work in equal characteristic).

The Radon transform $\pi_!\Phi_0$ corresponds to the !-pushforward under $L^+X \to L^+(X /\!\!/ N)$.

The map π factors through the stack quotient $X \to X/N \to X /\!\!/ N$. We can "compactify" X/N by replacing it by $(X \times \overline{N\backslash G})/G$. If we replace the basic function of X by the basic function $\overline{\Phi_0}$ of $(X \times \overline{N\backslash G})/G$, we will have

$$\pi_!\overline{\Phi_0} = \frac{1}{\prod_{\theta\in\Theta^+}(1 - q^{-\frac{1}{2}}e^{-\theta})},$$

i.e., the factor $\prod_{\check{\alpha}>0}(1 - q^{-1}e^{-\check{\alpha}})$ disappears.

Geometric functional equations

The geometric interpretation of this formula involves:

- Perversity of the sheaves corresponding to $\pi_! \overline{\Phi_0}$.
- The fact that $\pi_! \overline{\Phi_0}$ has this form follows from factorization structures over a curve. The remaining problem is to determine the θ's.

The functional equations for X, now, amount to the statement:

For every simple root α, we have

$$\frac{(\pi_! \overline{\Phi_0})^{w_\alpha}}{\pi_! \overline{\Phi_0}} = \frac{(1 - q^{-\frac{1}{2}} e^{-\theta_1})(1 - q^{-\frac{1}{2}} e^{-\theta_2})}{(1 - q^{-\frac{1}{2}} e^{\theta_1})(1 - q^{-\frac{1}{2}} e^{\theta_2})}$$

as functions on $\check{A}$, where $\theta_1, \theta_2 \in \Lambda$ are the valuations induced by the B-stable prime divisors ("colors") contained in $X^\circ P_\alpha$.

(This applies to cases such as the Rankin–Selberg variety, where $X^\circ P_\alpha / \mathcal{R}(P_\alpha) \simeq \mathbb{G}_m \backslash \mathrm{PGL}_2$; in the Whittaker case, these factors are trivial.)

The half-crystal of a spherical variety

To understand the θ's, we define the "half-crystal of a spherical variety" in terms of (a global model of) L^+X/L^+B.

Definition
The half-crystal of a spherical variety X is a set $\mathfrak{B}_+ = \bigsqcup_{\theta\in\Lambda} \mathfrak{B}_\theta$, where $\mathfrak{B}_\theta$ denotes the components "of maximal possible dimension" (= those which contribute an irreducible perverse sheaf to $\pi_! \overline{\Phi_0}$) in the preimage of $\varpi^{-\theta} A(\mathfrak{o})$ in $L^+X = X(\mathfrak{o})$.

Theorem (S.–Wang)
There is an embedding of $X^\circ(F)/B(\mathfrak{o})$ into the affine Grassmannian, with the preimage of $\varpi^{-\theta}A(\mathfrak{o})$ belonging the semiinfinite orbit $N(F)\varpi^\theta G(\mathfrak{o})/G(\mathfrak{o})$. For every simple root α, intersection of the closure with $N(F)\varpi^{\theta-\alpha}G(\mathfrak{o})/G(\mathfrak{o})$ gives rise to a weight-lowering operator $f_\alpha : \mathfrak{B}_\theta \to \mathfrak{B}_{\theta-\alpha} \sqcup \{0\}$, with $f_\alpha(b) = 0$ only if $\langle\theta, \alpha\rangle < 0$ or α is a color in $X^\circ P_\alpha$, and for every $b \in \mathfrak{B}^+$ there is a series of weight-lowering operators lowering it to $\mathfrak{B}_\theta$, for θ a color.
The set $\mathfrak{B}_+ \sqcup \mathfrak{B}_-$ (where $\mathfrak{B}_-$ is a copy of $\mathfrak{B}_-$ lying over the opposite weights $-\theta$), has the structure of a seminormal crystal over $\check{\mathfrak{g}}$.

Thus, the weights θ that appear can be read off from the colors of the spherical variety.

In minuscule cases, we can also identify the multiplicities, showing that this is the crystal associated to a $\check{G}$-representation. For example,

Theorem (S.–Jonathan Wang)
For

$$X = \text{the affine closure of } \mathbb{G}_m^{\text{diag}} N_0 \backslash \mathrm{GL}_2^n,$$

where

$$N_0 = \left\{ \begin{pmatrix} 1 & x_1 \\ & 1 \end{pmatrix} \times \begin{pmatrix} 1 & x_2 \\ & 1 \end{pmatrix} \times \cdots \times \begin{pmatrix} 1 & x_n \\ & 1 \end{pmatrix} \middle| x_1 + x_2 + \cdots + x_n = 0 \right\},$$

with $\Phi_0 =$ the IC function of L^+X, we have

$$\pi_! \Phi_0 = \frac{\prod_{\check{\alpha}>0}(1 - q^{-1} e^{-\alpha})}{\prod_{\theta>0}(1 - q^{-\frac{1}{2}} e^{-\theta})},$$

where θ ranges over "half" the weights of the n-fold tensor product representation and its dual, $\bigotimes \oplus \bigotimes^\vee : \mathrm{GL}_2^n \xrightarrow{\bigotimes} \mathrm{GL}_{2^{n+1}}$.

Happy Birthday, Bill! Many happy returns and travels!