Reduced bandwidth: a qualitative strengthening of twin-width in minor-closed classes (and beyond)

O-joung Kwon
(Incheon National University,
IBS Discrete Math Group, Korea)

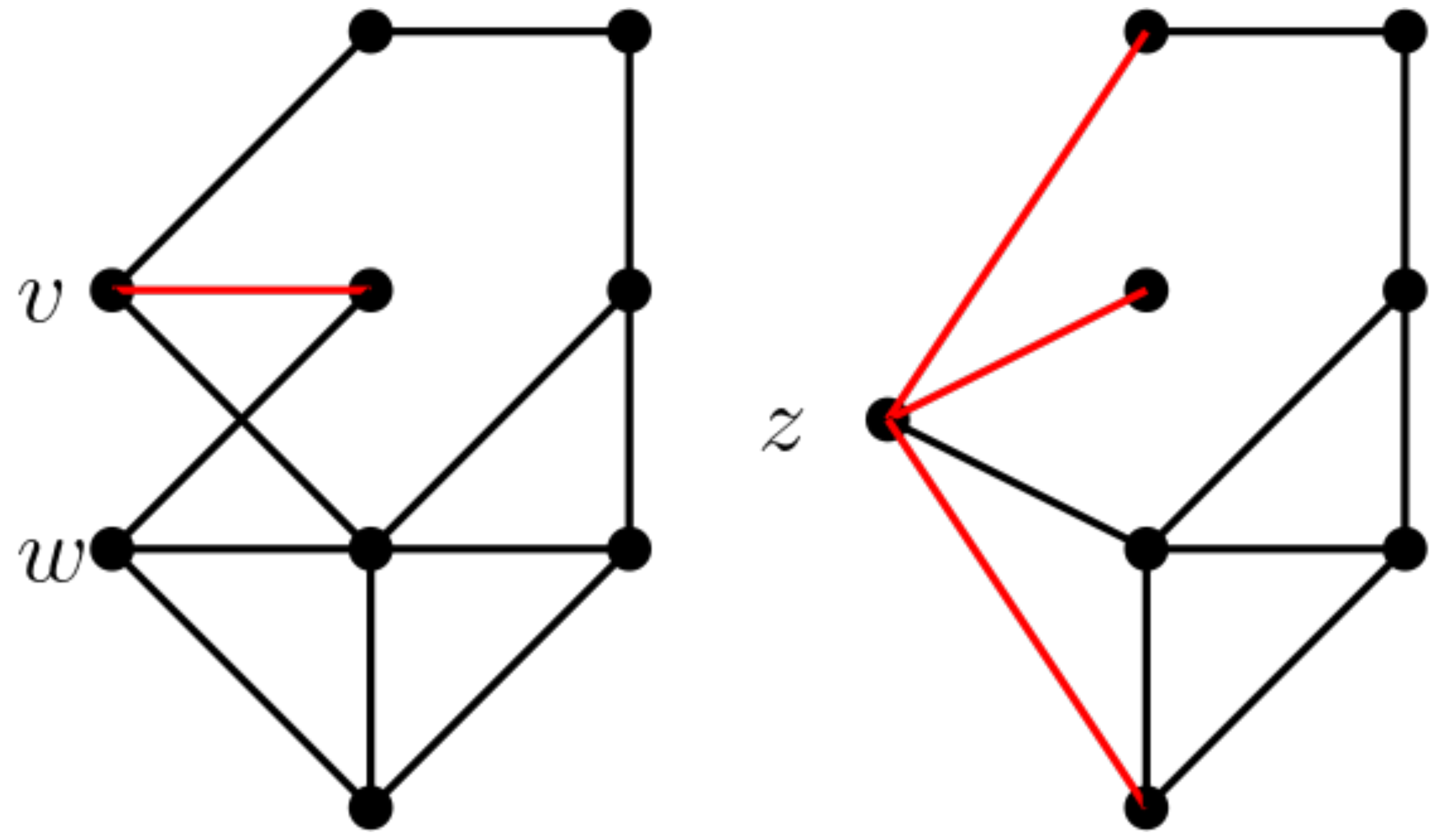
Joint with Édouard Bonnet (ENS Lyon, LIP)
and David Wood (Monash University)

2021 Banff international workshop on
"Graph Product Structure Theory"

Identifying vertices

- We consider graphs where some edges are colored red.

- When we identify two vertices v and w to z in a graph G,
  - all edges between z and $N(v) \triangle N(w)$ become red,
  - for $x \in N(v) \cap N(w)$,
    - > if at least one of vx and wx was red, then zx becomes red,
    - > otherwise, it becomes black.



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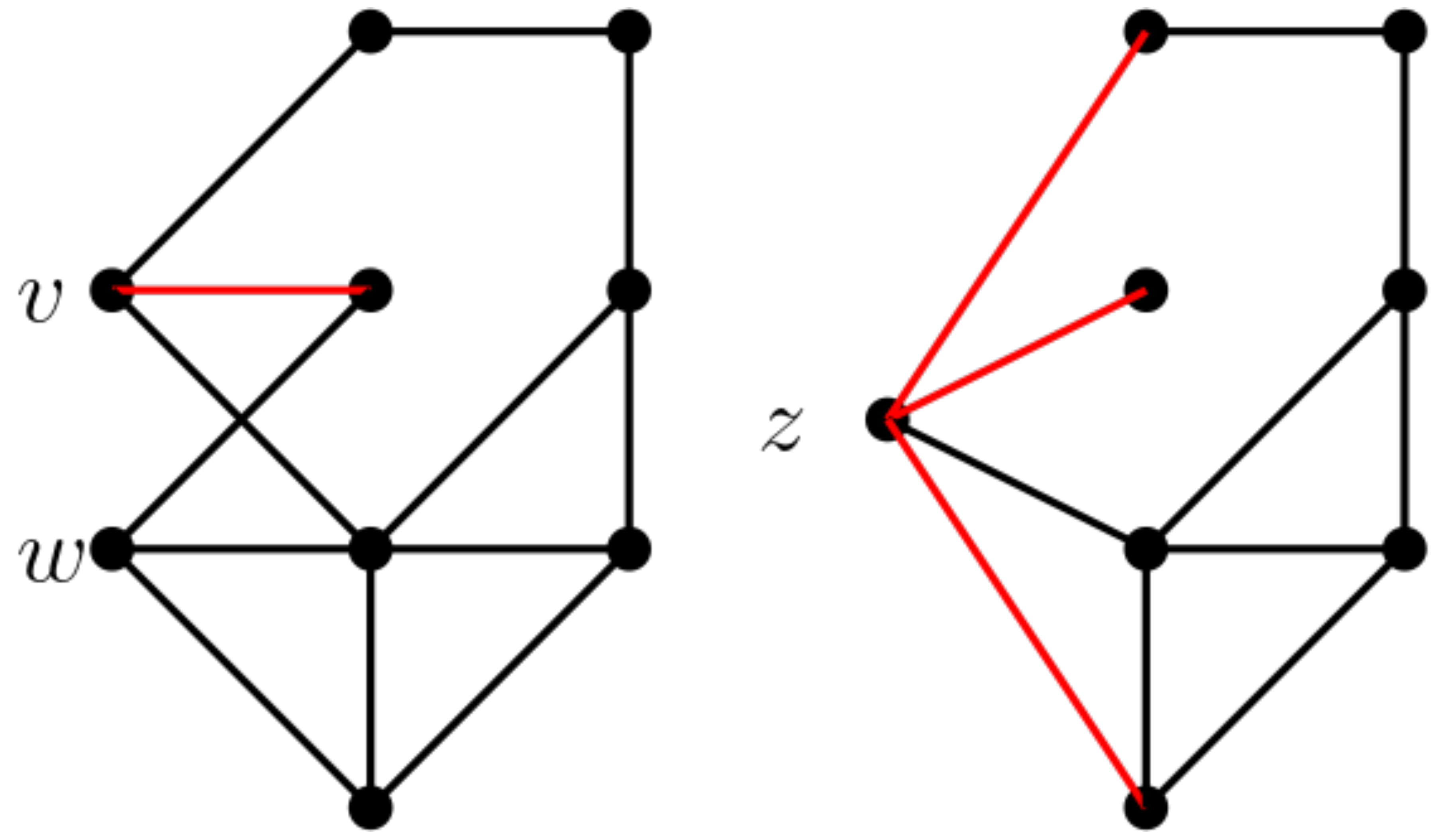
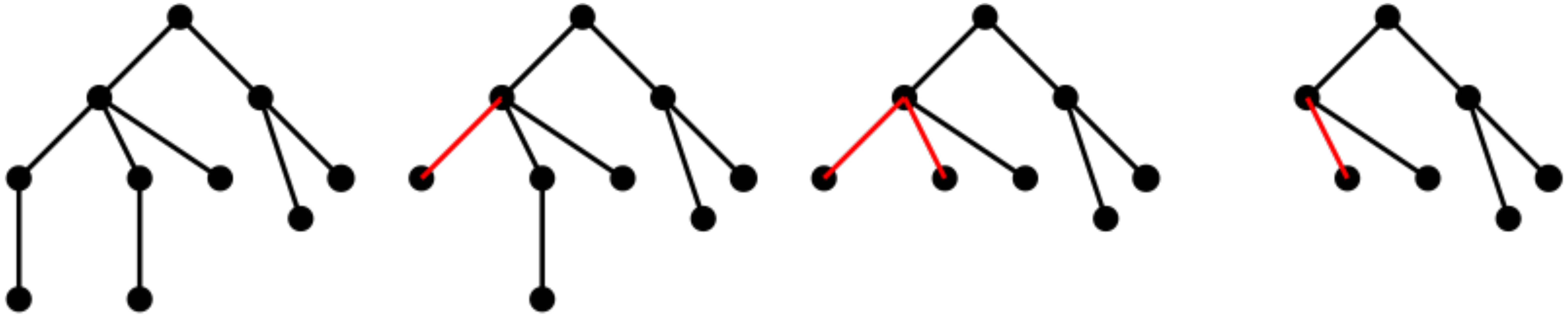
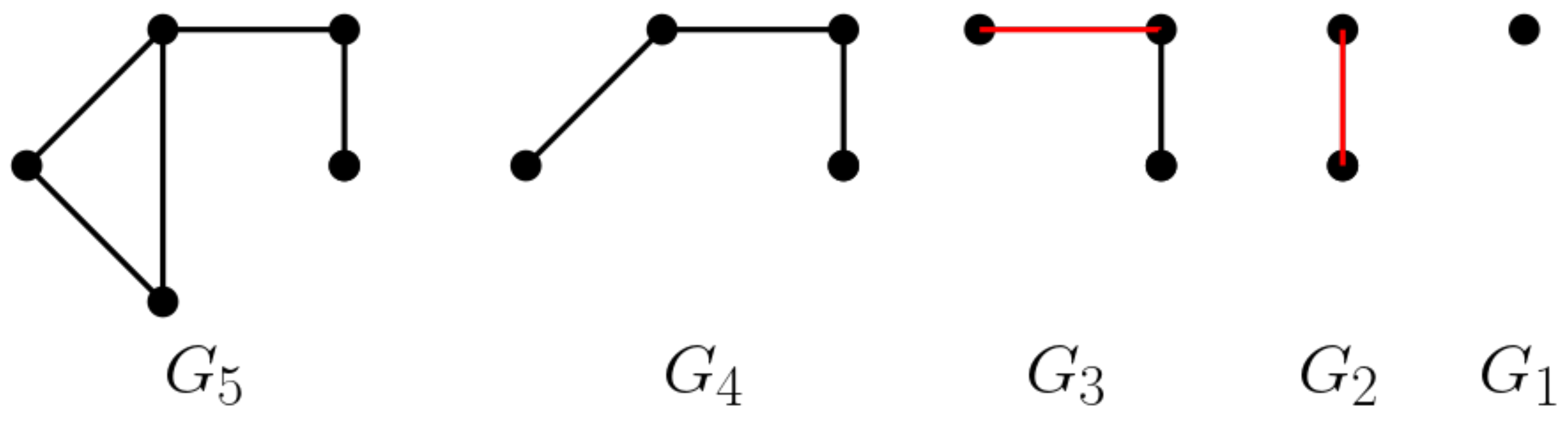


- General question: Can we recursively identify a given graph into one vertex without creating a vertex of large red degree?



Twin-width (Bonnet, Kim, Thomassé, Watrigant 2020)



- A trigraph is a graph whose edges are colored black or red.
- For a graph G, a sequence $G = G_n, G_{n-1}, \ldots, G_1$ of trigraphs is a reduction sequence if G_1 is a singleton graph.
- Twin-width of a graph G is the minimum k such that there is a reduction sequence $G = G_n, G_{n-1}, \ldots, G_1$ of G for which the maximum red degree of G_i is at most k.
- Cographs have twin-width 0 (Cographs on ≥ 2 vertices always have twins).

Reduced-f of a graph

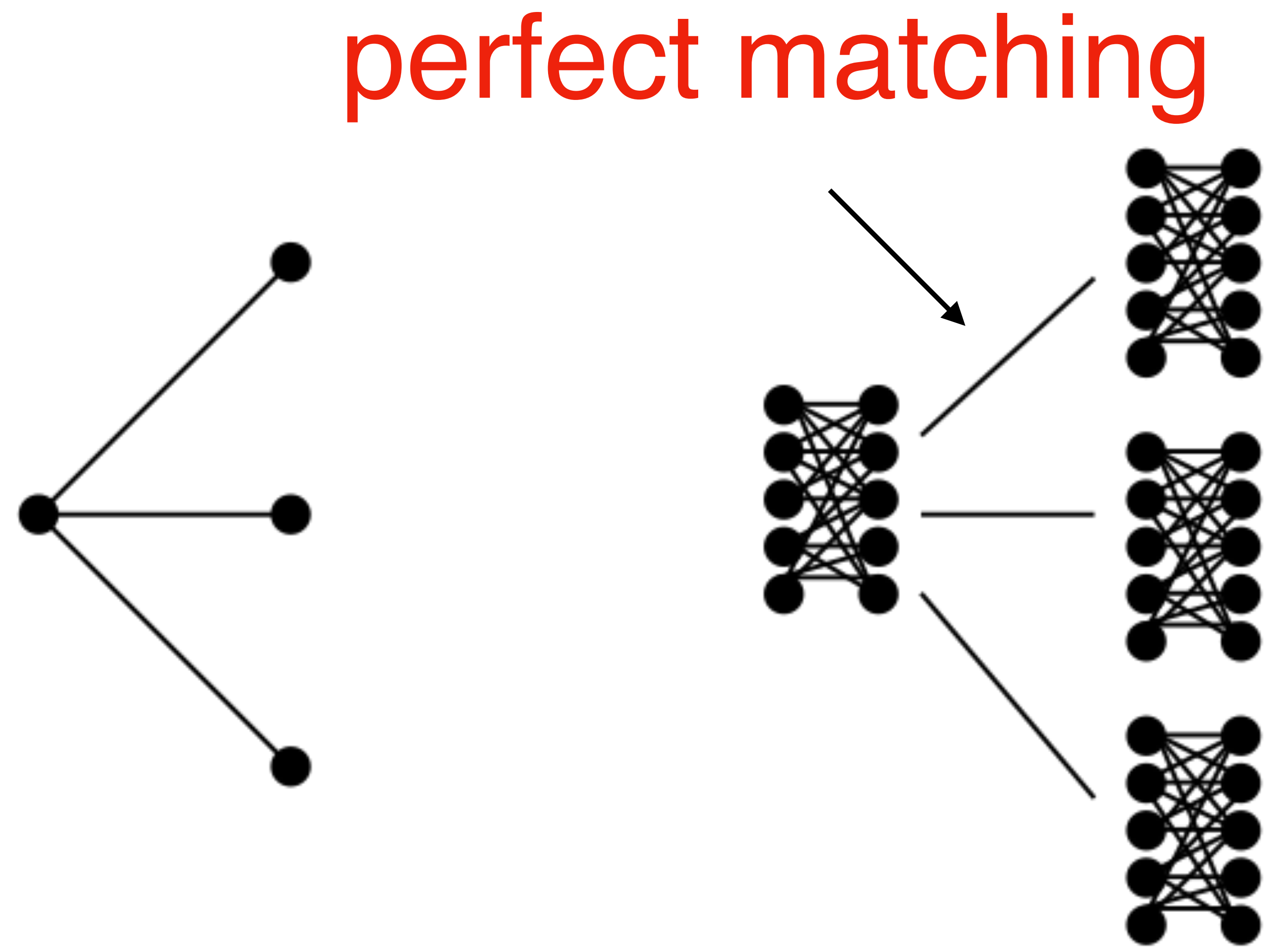
- We consider any natural graph parameter f
  (maximum degree, tree-width, band-width, component size, …)

- Reduced-f of a graph G is the minimum k such that there is a reduction sequence
  $G = G_n, G_{n-1}, \ldots, G_1$ of G for which $\max_{1 \le i \le k} f(G_i) \le k$.

- (Bonnet et al. 2020 TWW I) Reduced-maximum degree = twin-width
  (Bonnet et al. 2021 TWW VI) Reduced-component size ~ rank-width
  Reduced-number of edges ~ linear rank-width

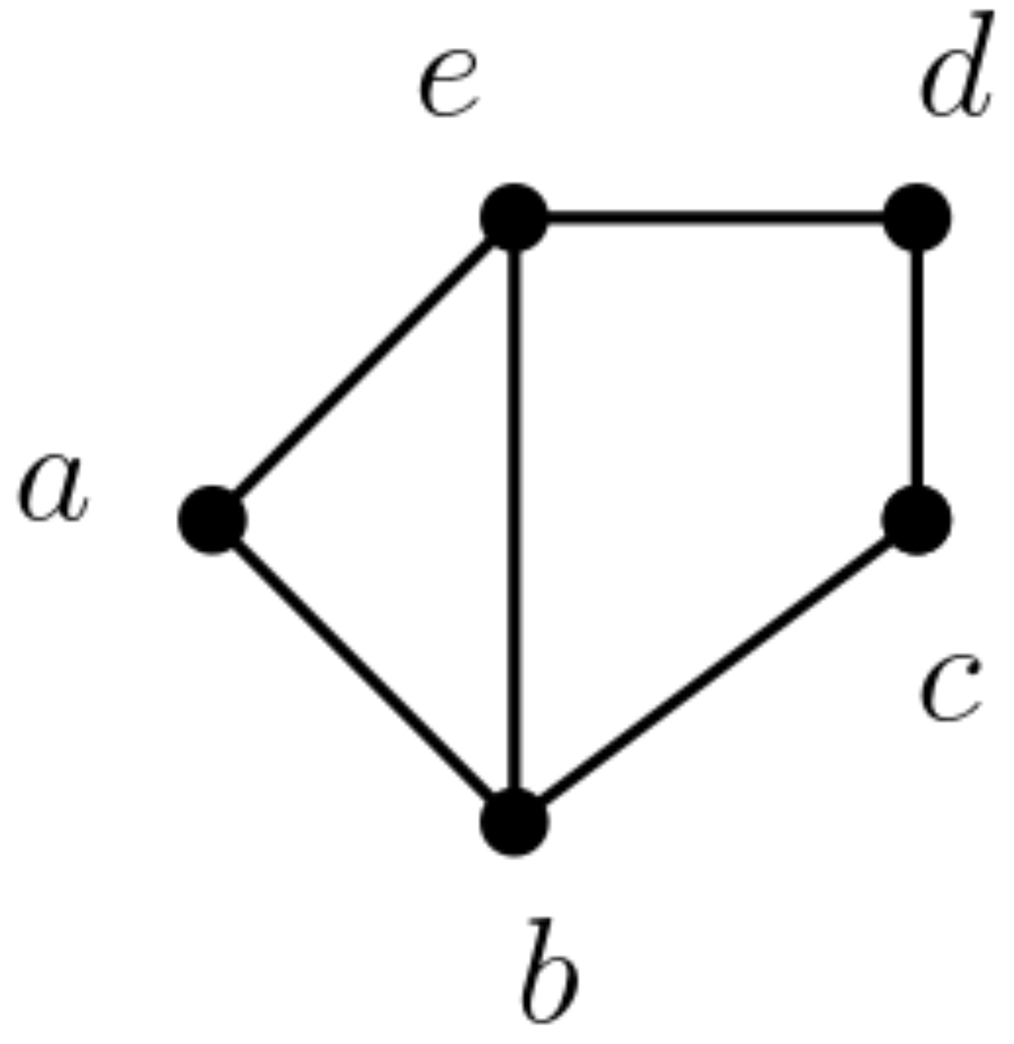
Reduced-f of a graph

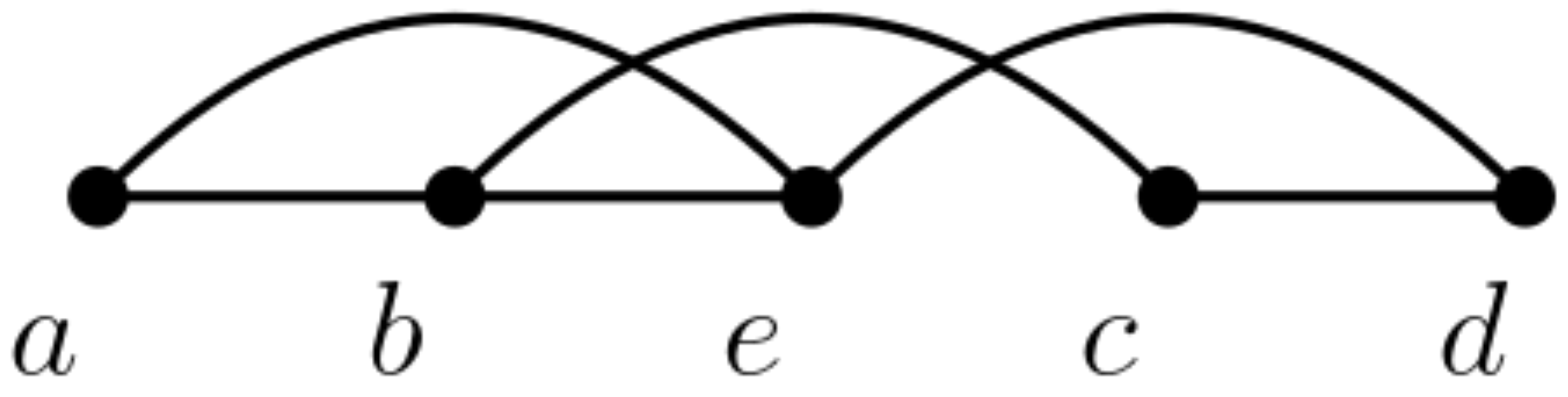
- If f is bounded on all stars, then reduced-f is bounded for all graphs.
- We may consider $\max\{f, \Delta\}$ as a function, where $\Delta(G)$ denotes max degree.
- Question: Are there differences between following classes?
  - graphs of bounded reduced-Δ
  - graphs of bounded reduced-max{treewidth,Δ}
  - graphs of bounded reduced-max{pathwidth,Δ}
  - graphs of bounded reduced-bandwidth
  - graphs of bounded reduced-component size



- Question: Do some known classes of bounded twin-width have actually bounded reduced-bandwidth?

Reduced-bandwidth of a graph





- Band-width of a graph G: minimum k such that there is a permutation $L : V(G) \to [n]$ where $|L(u) - L(v)| \leq k$ for every edge uv.
- If band-width is at most k, then maximum degree is at most $2k$.

Main results (product theorem + neigbhorhood complexity)

- Theorem (Bonnet, K, Wood 2021)
  Proper minor-closed classes have bounded reduced-bandwidth.
  Their r-powers also have bounded reduced-bandwidth.
- This strengthens the results in TWW I that proper minor-closed classes have bounded twin-width.

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- Theorem (Bonnet, K, Wood 2021)
  Planar graphs have reduced-bandwidth at most 466 and twin-width at most 583.
  By the result of (Morin 2021), we can produce in polynomial time.

  Graphs of Euler genus g have reduced-bandwidth at most 164g+468.
  Planar map graphs have reduced-bandwidth at most 10000.
- Previous bounds for planar graphs in TWW I/ TWW VI papers were $\geq 2^{1000}$.

- Theorem

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- (Product theorem (Ueckerdt, Wood, Yi 2021)) Every planar graph is a subgraph of $H \boxtimes P$ for some graph H of treewidth at most 6 and a path P.

- (Neighborhood complexity) For every vertex set S in a planar graph G, $|\{N(v) \cap S : v \in V(G) \setminus S\}| \leq 6|S| - 9$.

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- (Product theorem (Ueckerdt, Wood, Yi 2021)) Every planar graph is a subgraph of $H \boxtimes P$ for some graph H of treewidth at most 6 and a path $P = w_1 w_2 \cdots w_t$.
- (Neighborhood complexity) For every vertex set S in a planar graph G, $|\{N(v) \cap S : v \in V(G) \setminus S\}| \leq 6|S| - 9$.
- **Difficulty**: when you identify two vertices, planarity may be destroyed, and it is hard to find a natural sequence preserving planarity.
- **Idea**: we will not use planarity when constructing a reduction sequence.
- We can slightly **improve** bounds by looking at neighborhood complexity in the product structure carefully. But we do not know whether we can improve to ≤ 100.

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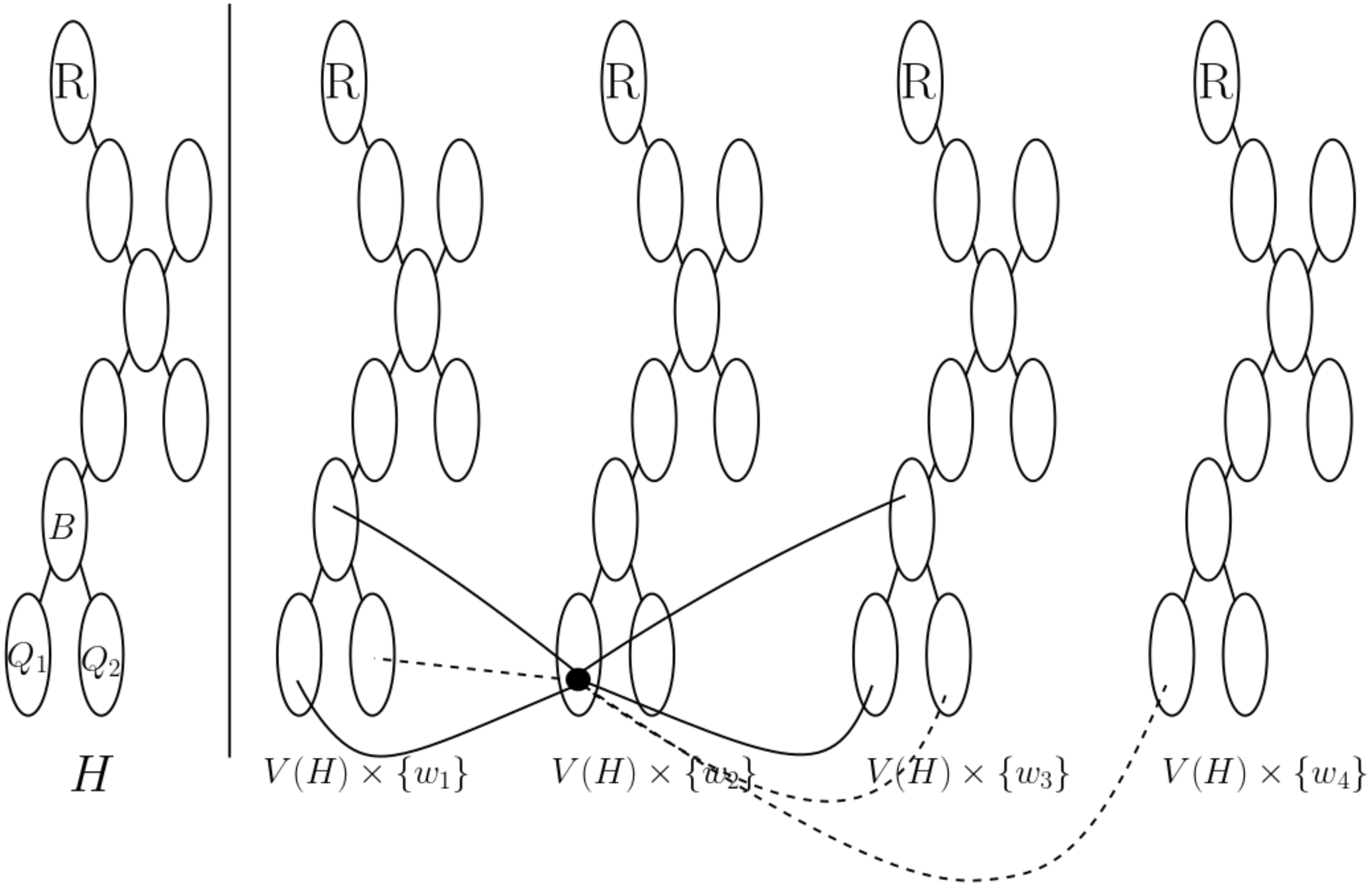

- Look at a vertex v in $(V(Q_1) \setminus V(B)) \times \{w_2\}$.
- Neighbors are contained in $(V(Q_1) \cup V(B)) \times \{w_1, w_2, w_3\}$

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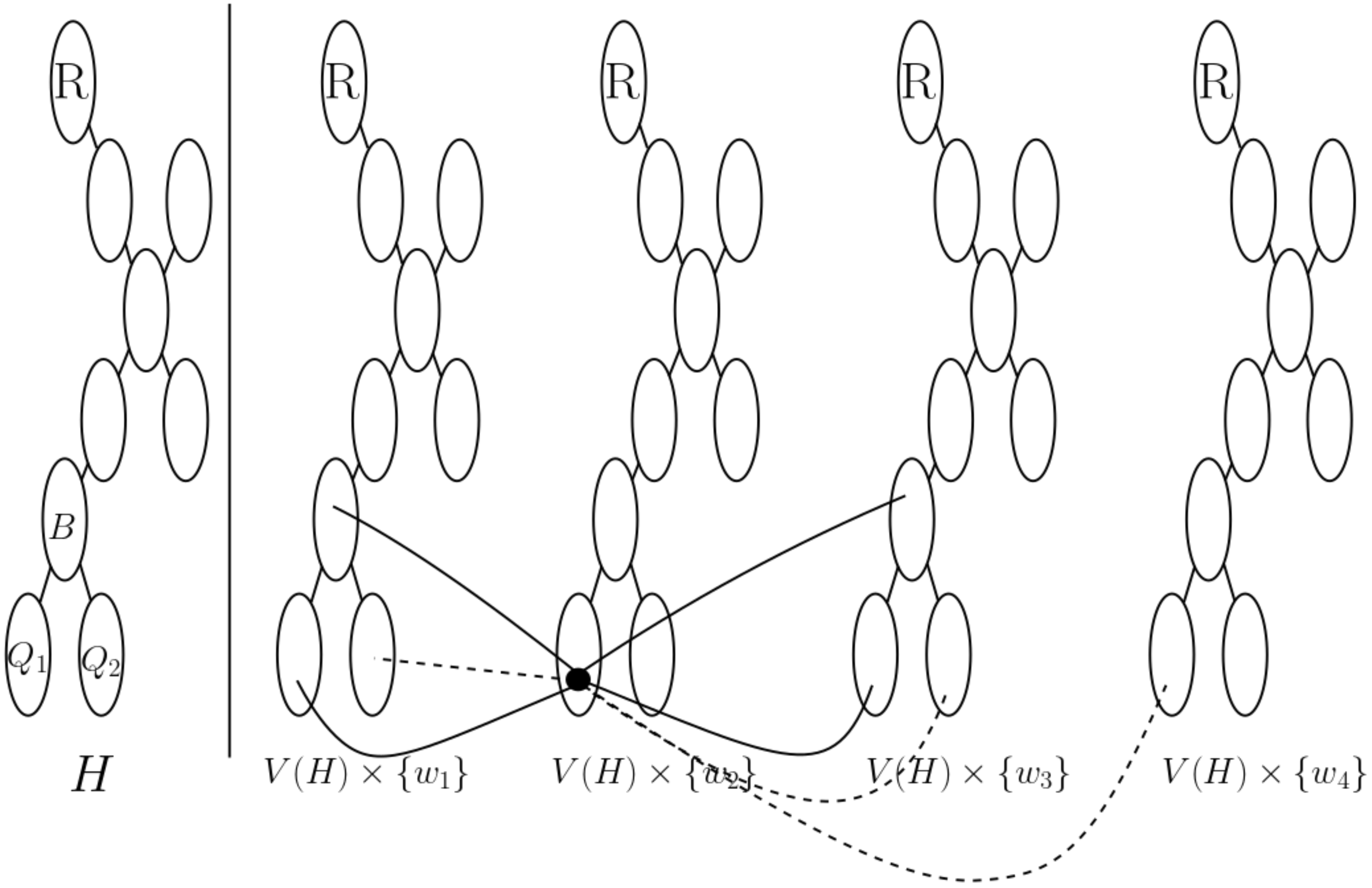

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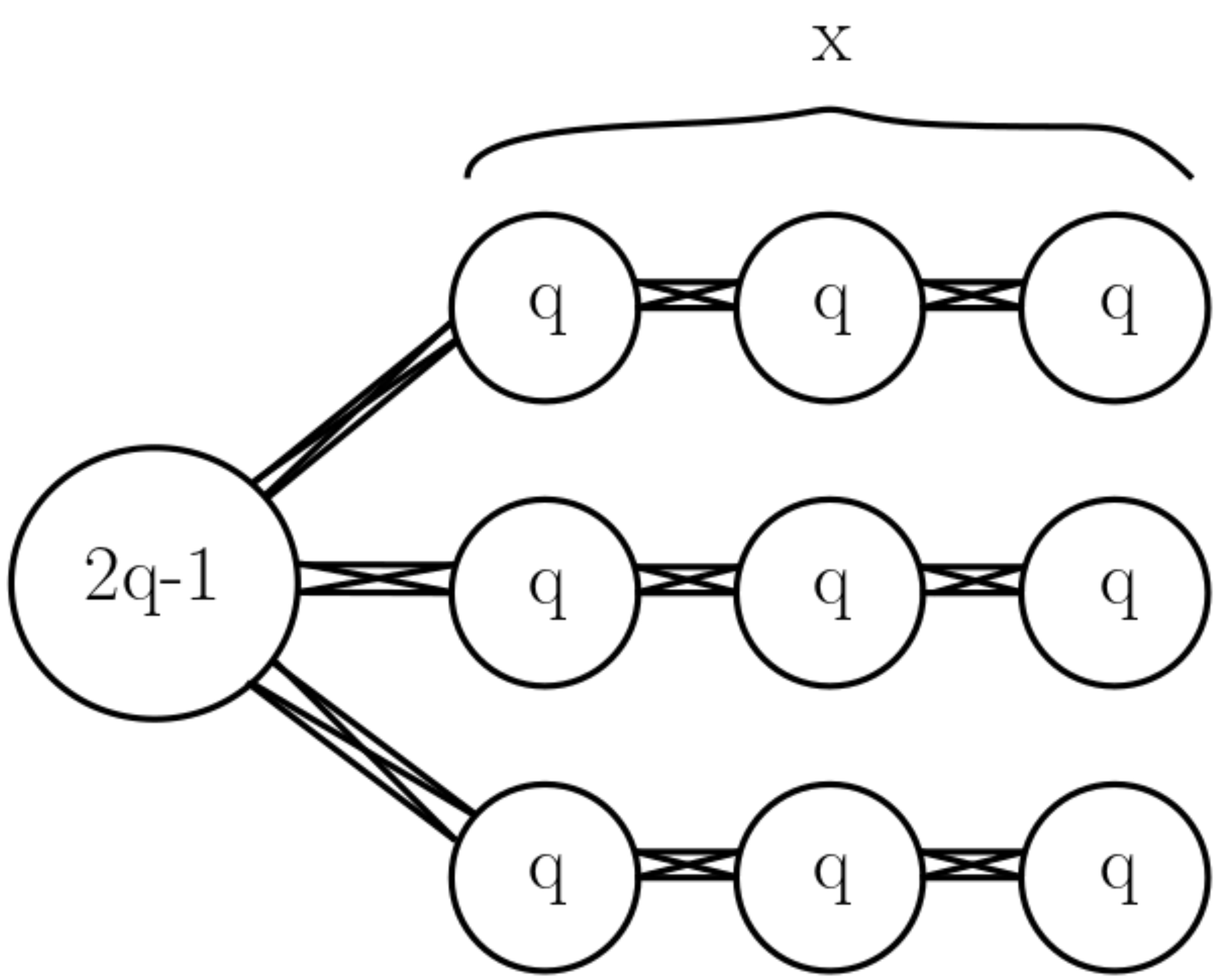
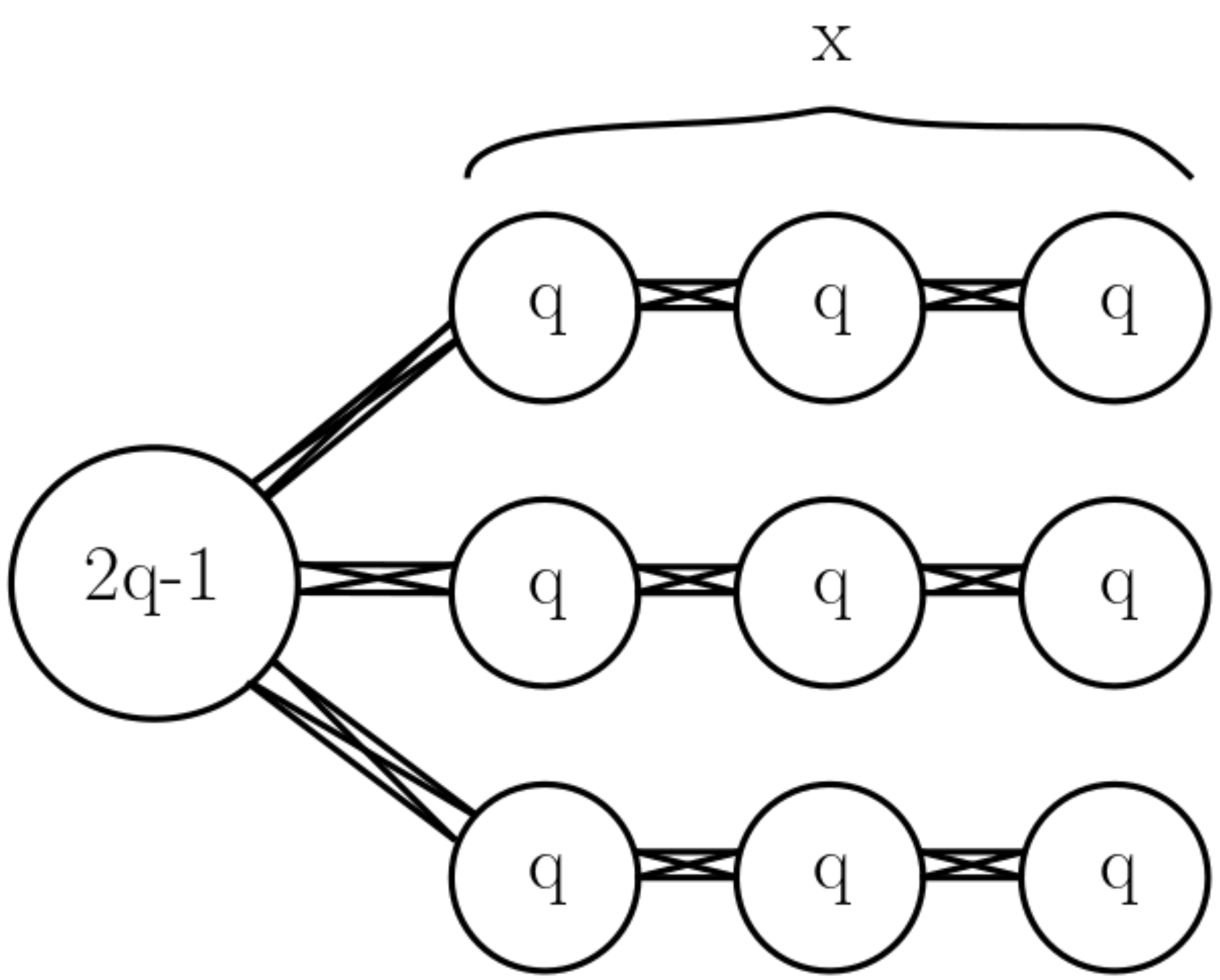
- Neighbors are contained in $(V(Q_1) \cup V(B)) \times \{w_1, w_2, w_3\}$
- We want to identify $((V(Q_1) \cup V(Q_2)) \setminus V(B)) \times V(P)$ so that no red edges incident with $V(B) \times V(P)$ are created.
- Idea: pick two vertices in the same silce that are twins to $V(B) \times V(P)$



- $S_{x,q}$ (each circle is a clique)
- $\Delta(S_{x,q}) \leq 5q - 2$ and bandw$(S_{x,q}) \leq 4q - 2$



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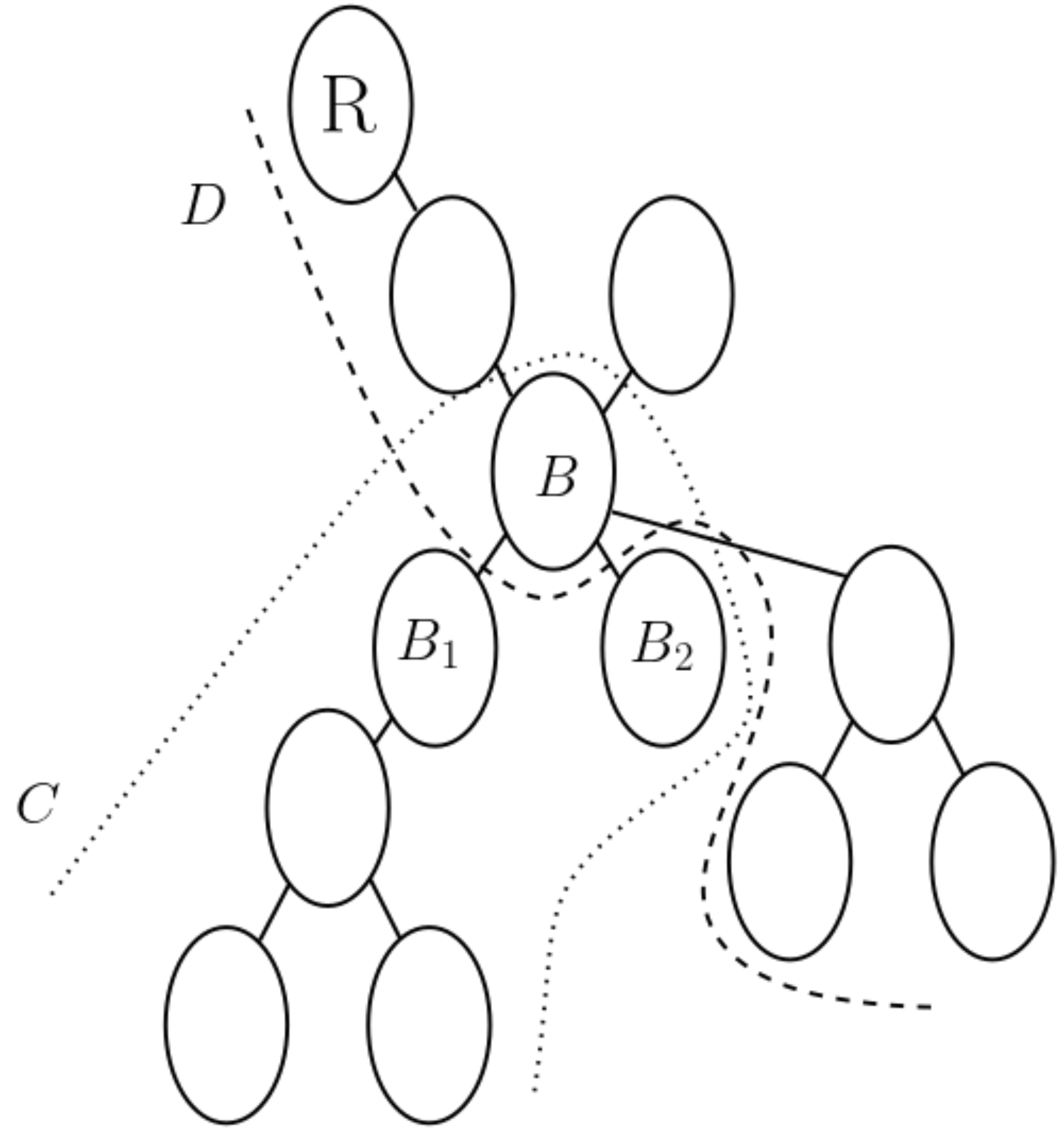
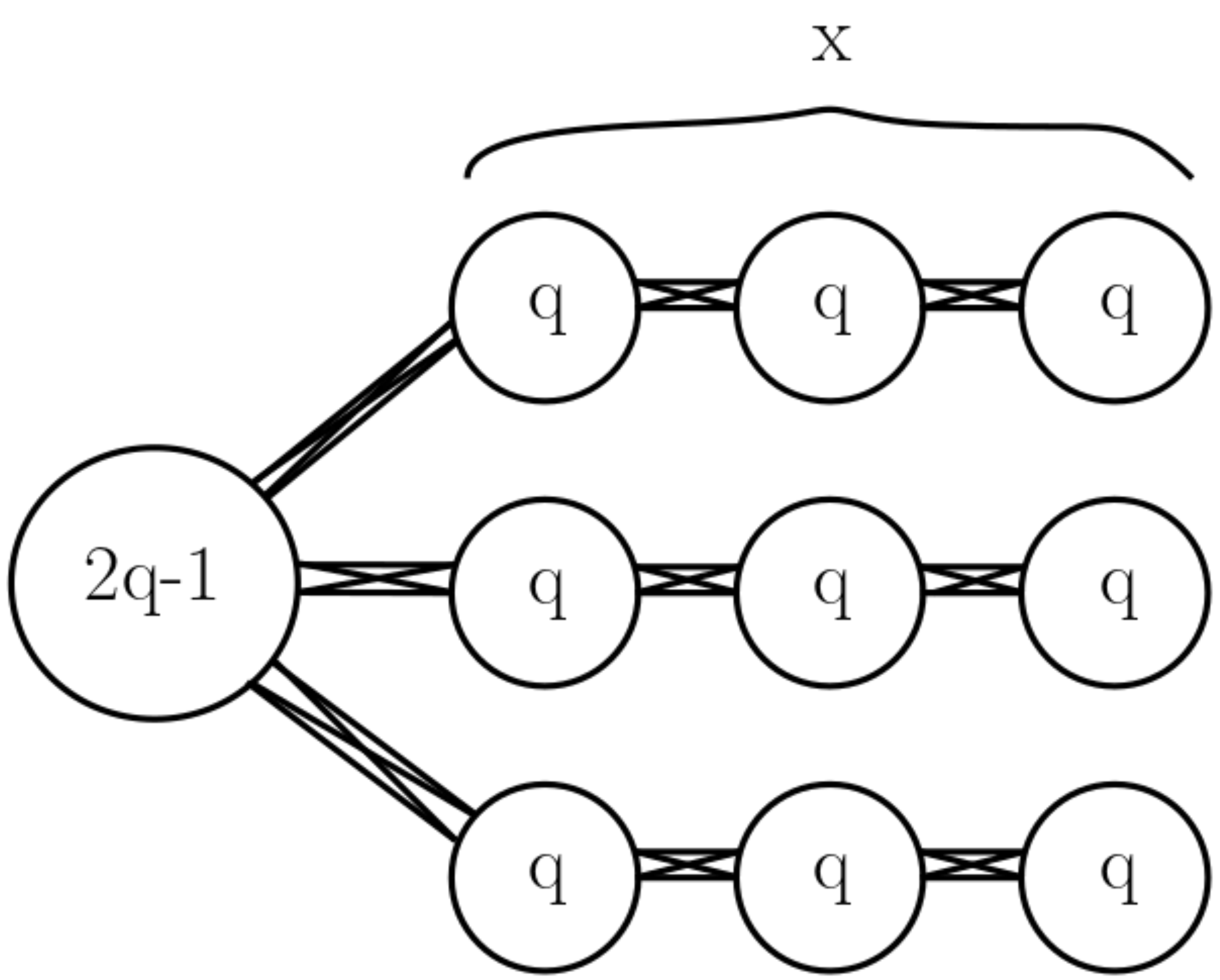


- (k, q)-rooted decomposition = internal bags have size $\leq k + 1$, leaf bags have size $\leq q$
- rooted separation is a separation (C, D) as in the picture



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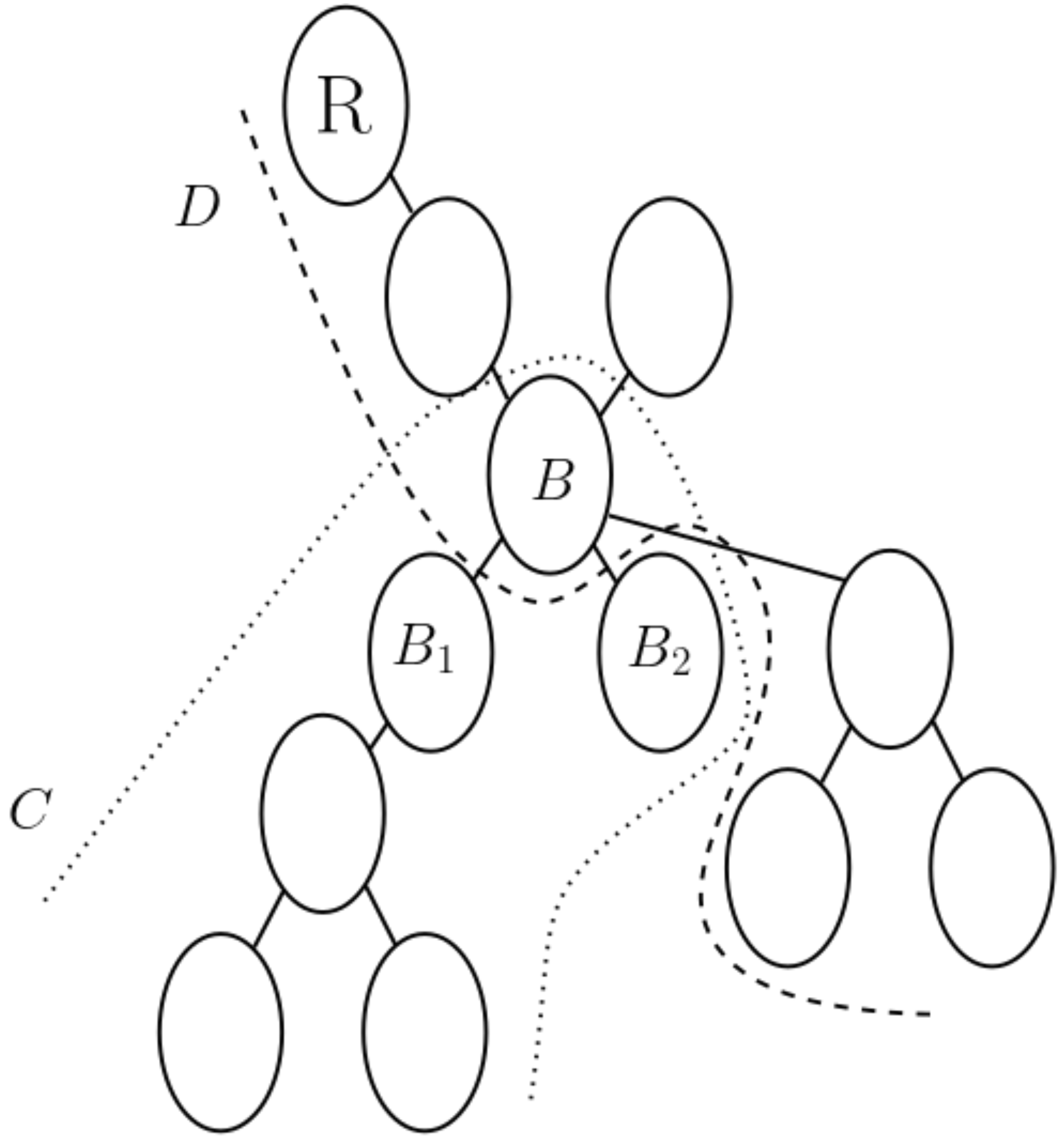

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- rooted separation is a separation (C,D) as in the picture
- A parameter f is good if it is closed under subgraph / disjoint union

- Let f be a good parameter and $g : \mathbb{N} \to \mathbb{R}$ be a function where $f(S_{x,q}) \leq g(q)$ for all q. Let $(\mathcal{T}, \mathcal{B})$ be a (k, q)-rooted tree-decomposition of H and let F be a trigraph with $V(F) \subseteq V(H \boxtimes P)$ such that

  1) (red edge condition) for every red edge vw, there is a leaf bag B with parent B' so that $v, w \in (B \backslash B') \times V(P)$,

  2) (separation condition)
  for every rooted separation (C, D) of H and $z \in V(P)$,
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  3) (neighborhood condition)
  for every vertex $v \in (V(H) \times \{z\}) \cap V(F)$ for some $z \in V(P)$,
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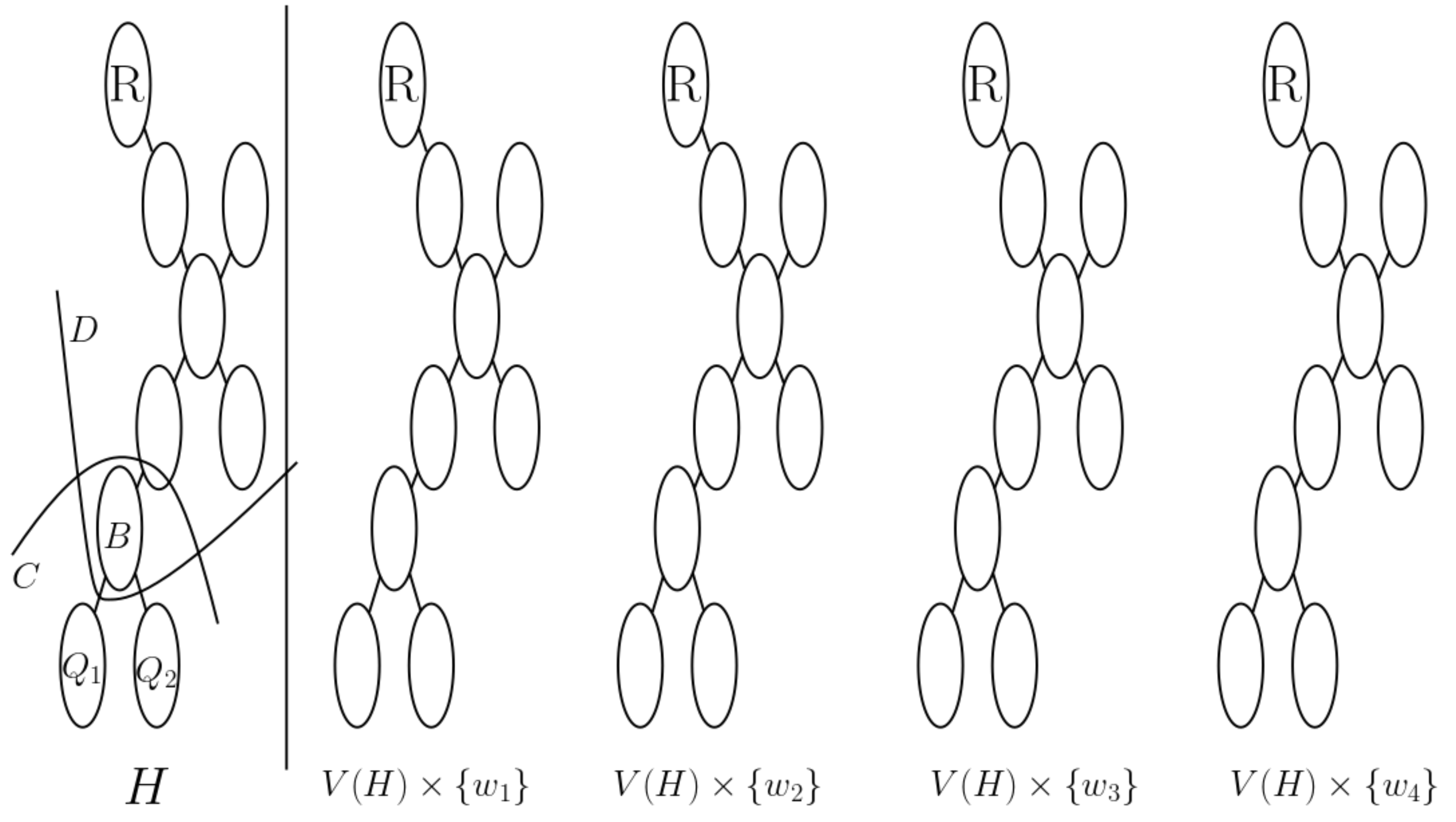
  Then reduced-f of F is at most $g(q)$.



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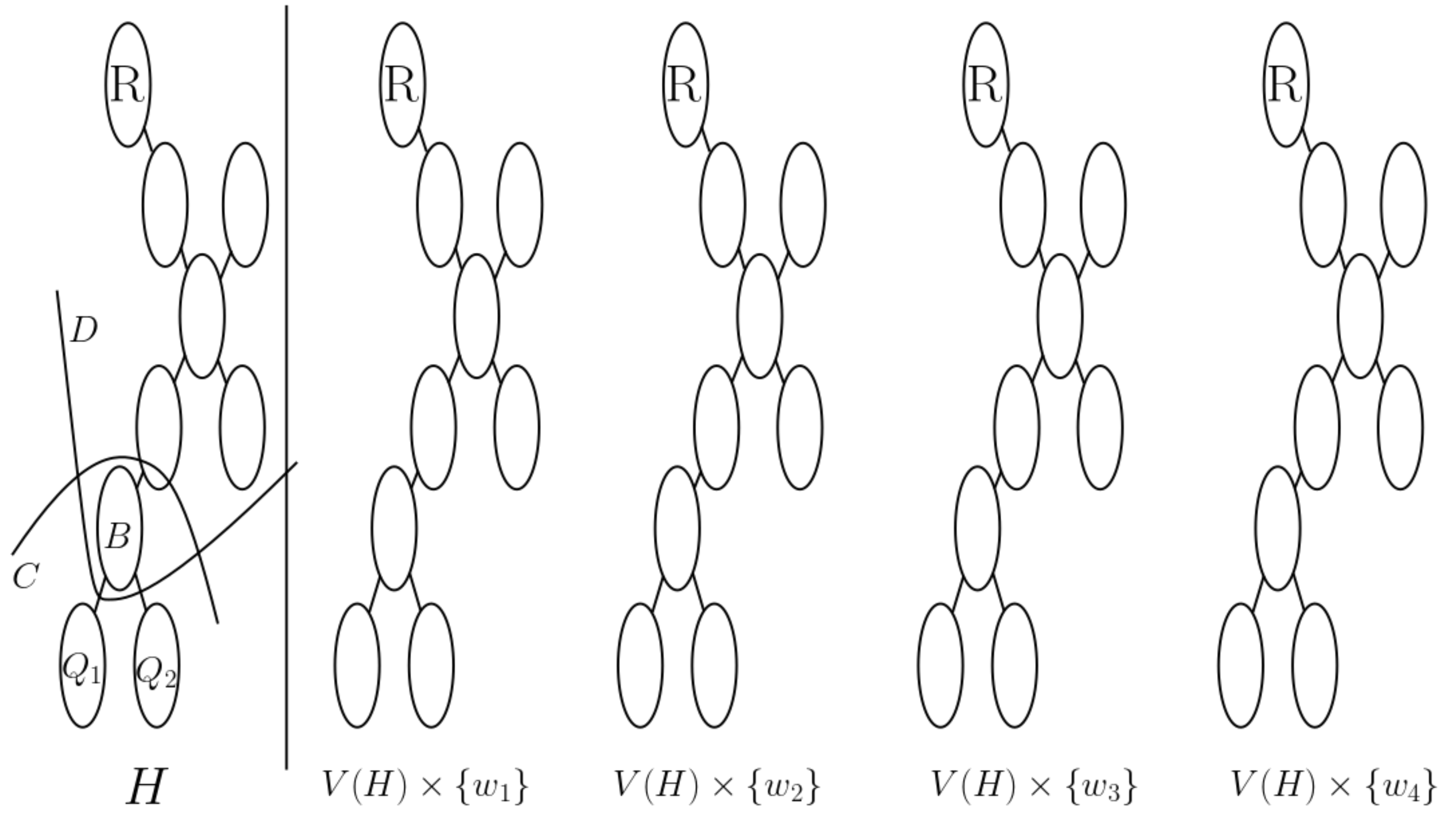

- For planar graphs, we can take k=6 and q=6*(7*3)-9=117.

- bandw$(S_{x,q}) \leq 4q - 2$ and $\Delta(S_{x,q}) \leq 5q - 2$.

- So, reduced-bandw ≤ 466 and twin-width ≤ 583.

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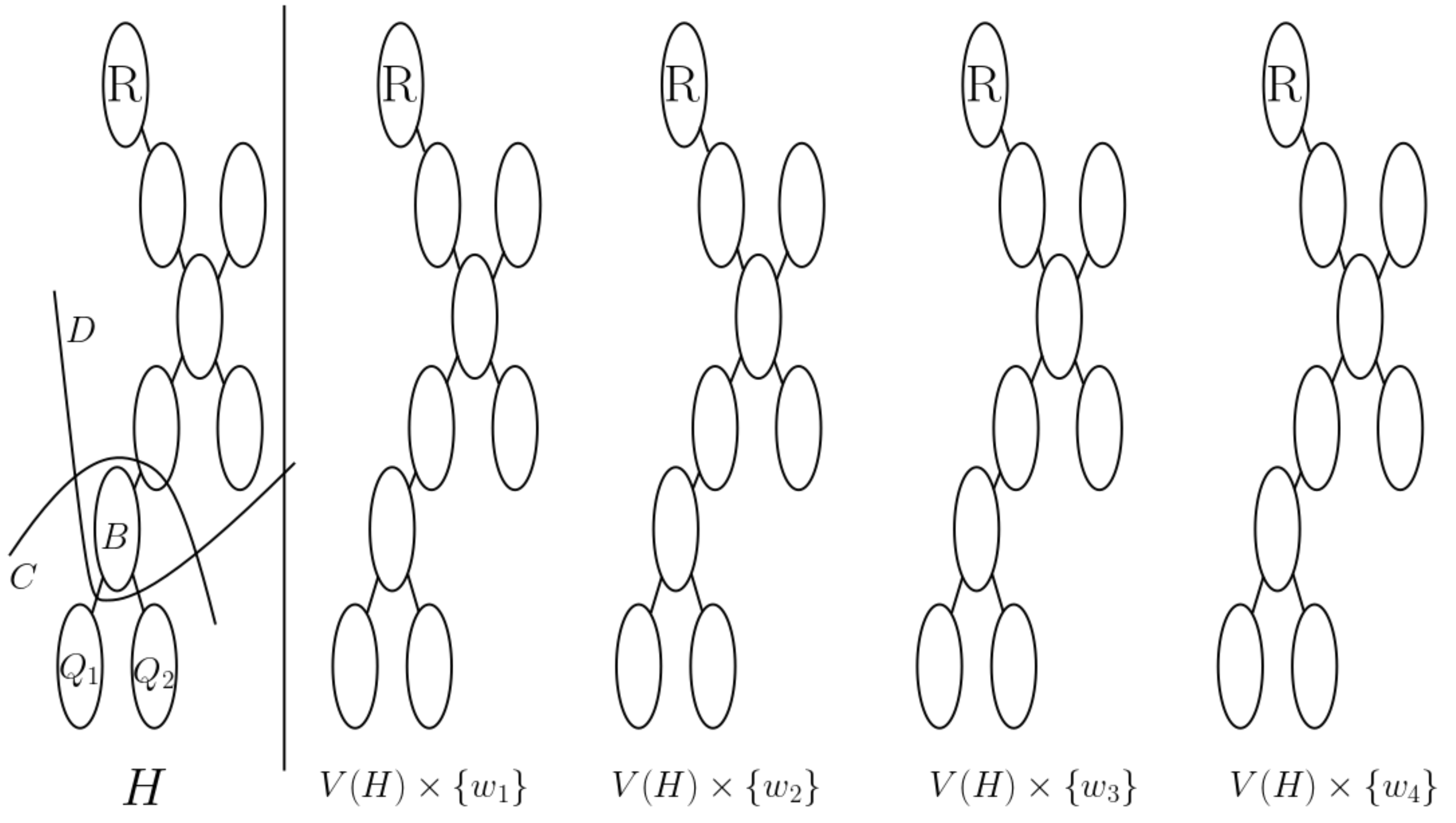

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- bandw$(S_{x,q}) \leq 4q - 2$ and $\Delta(S_{x,q}) \leq 5q - 2$.
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- Can be applied to any class with product structure!

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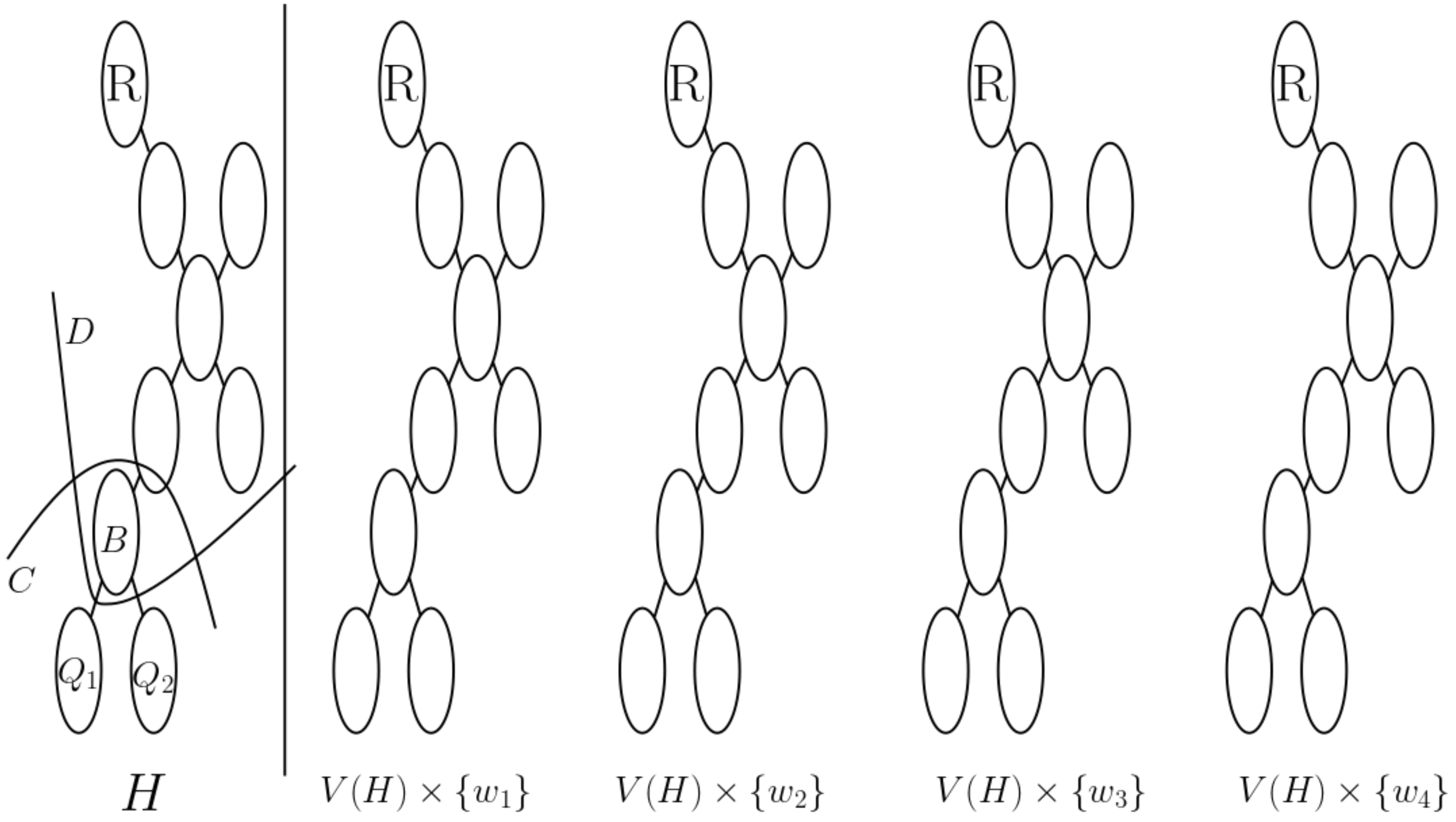

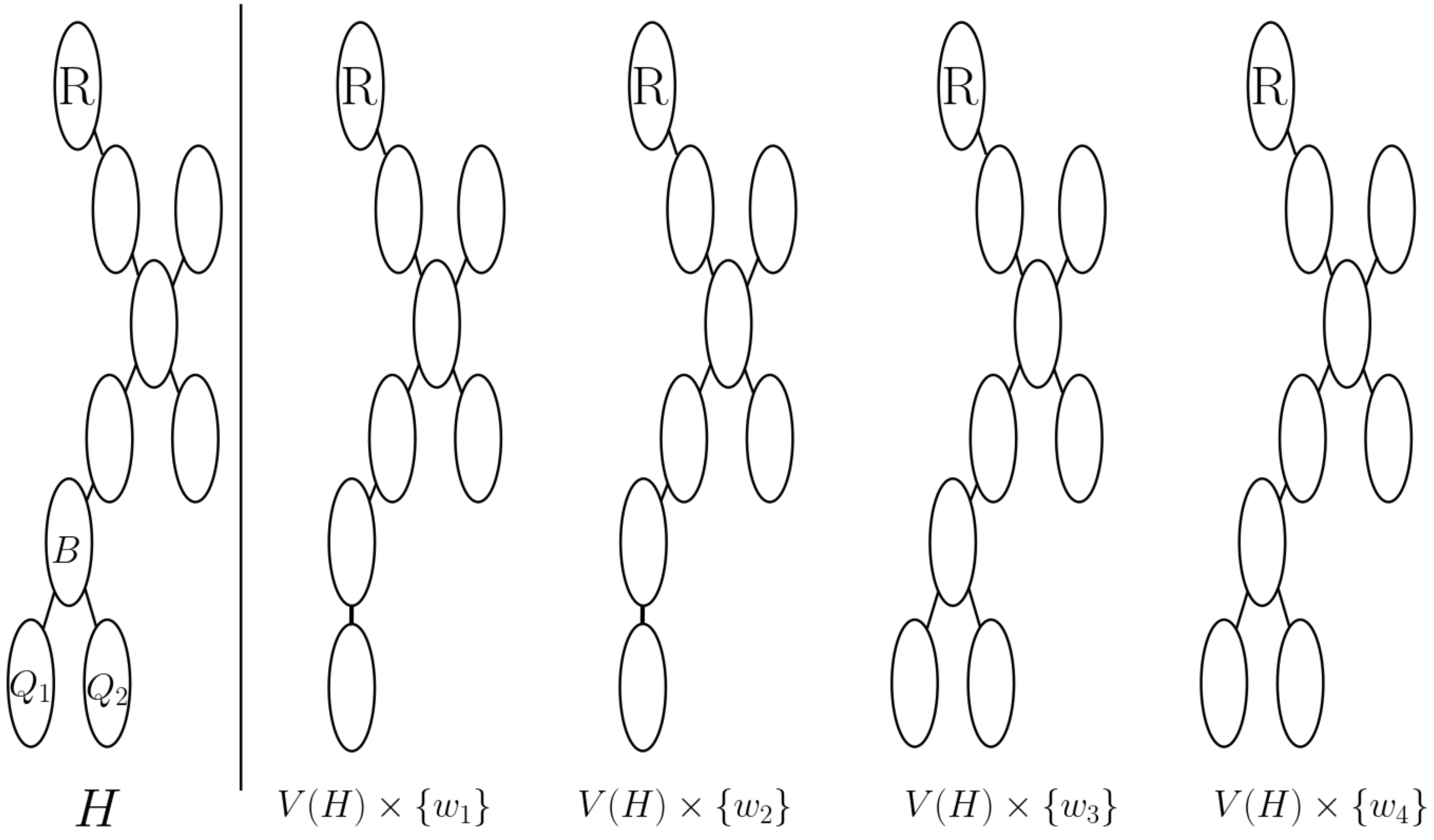

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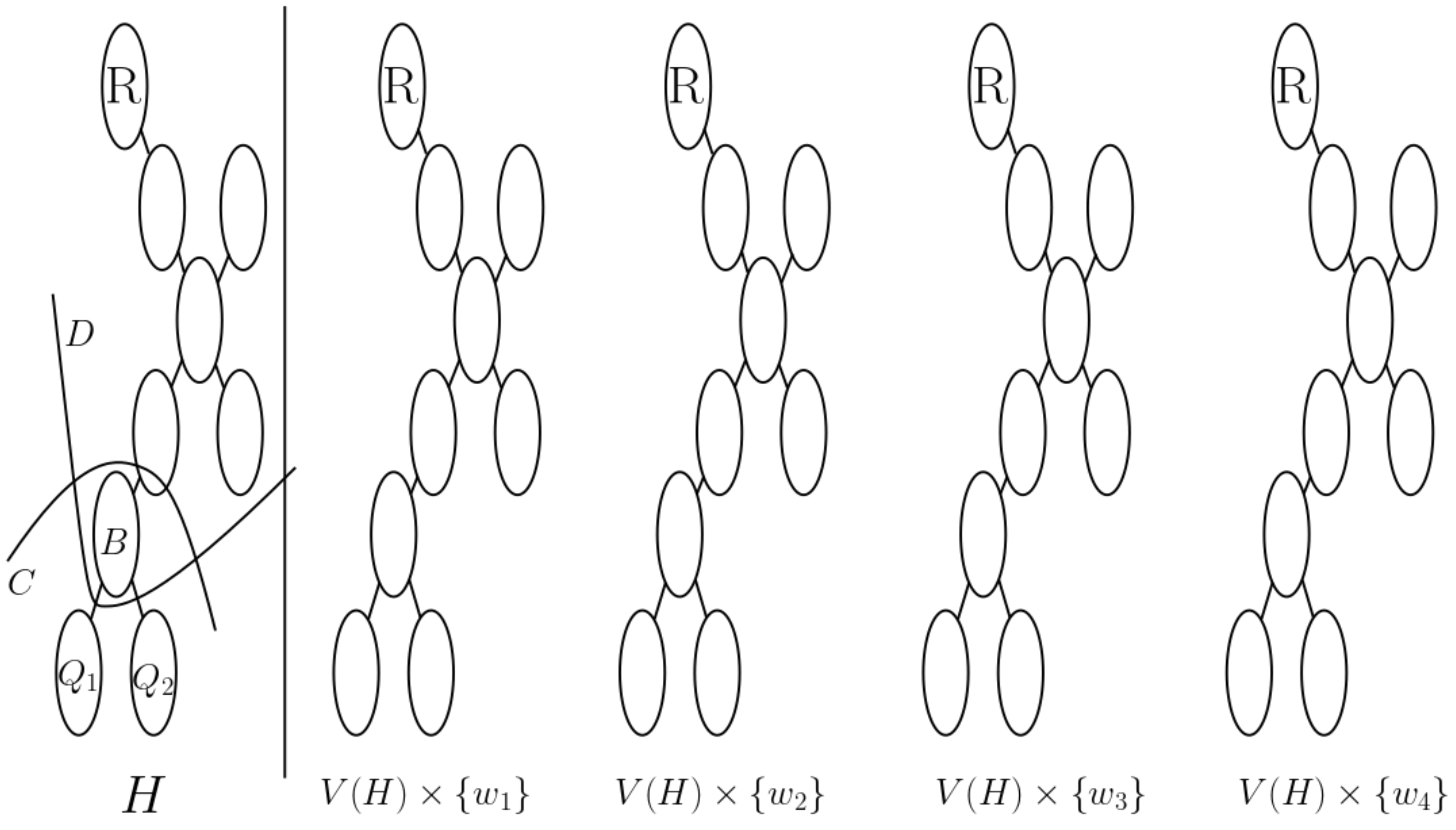

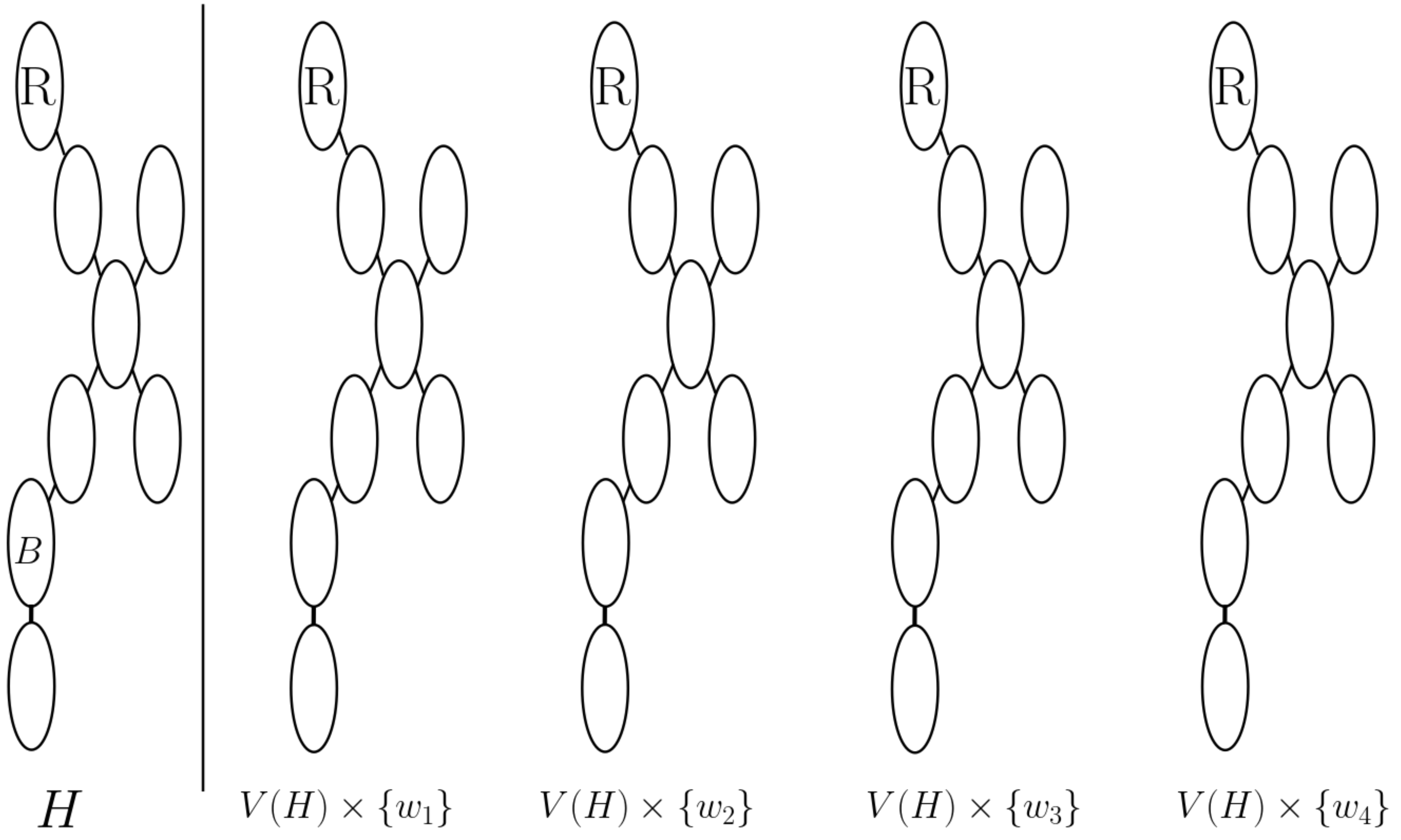

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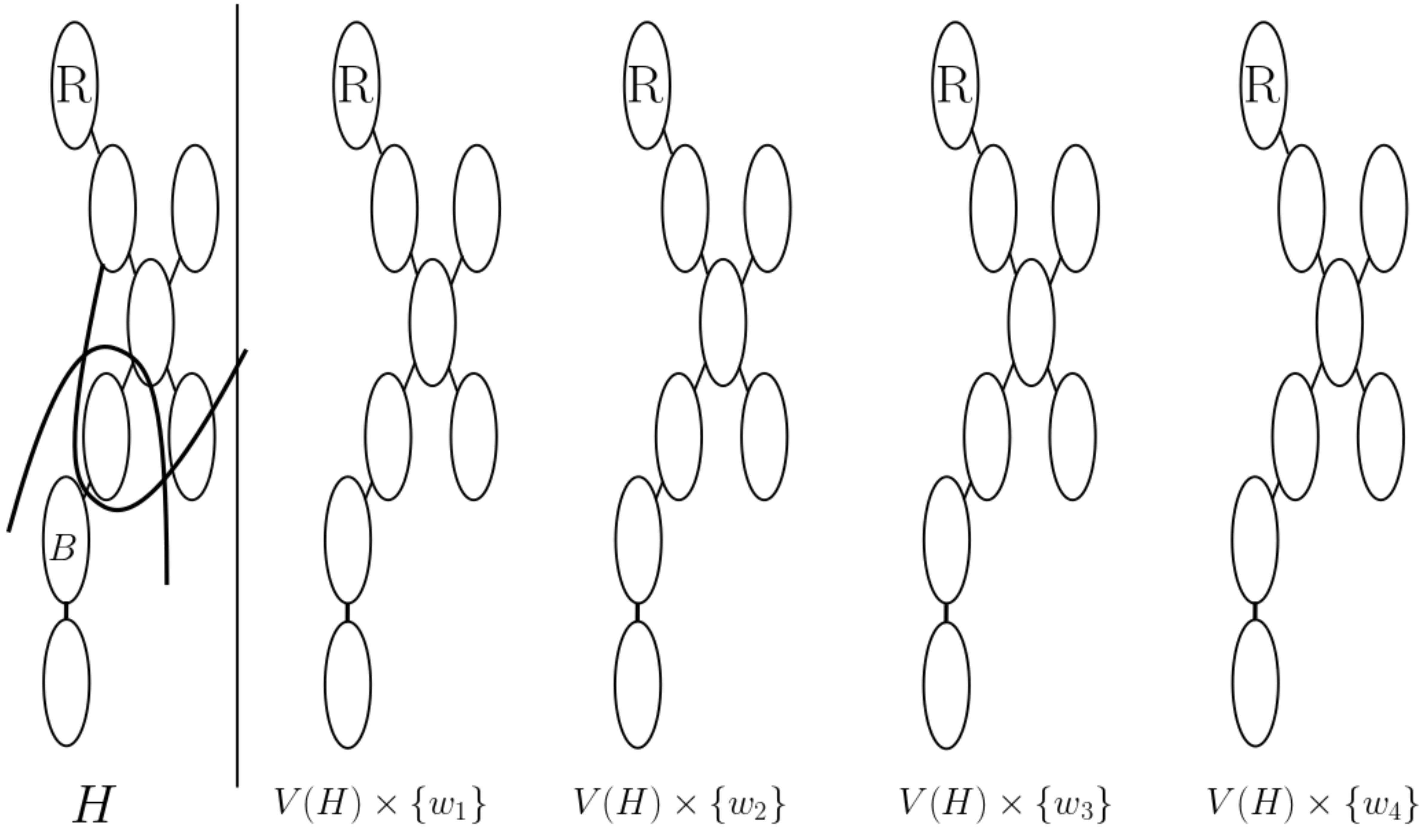

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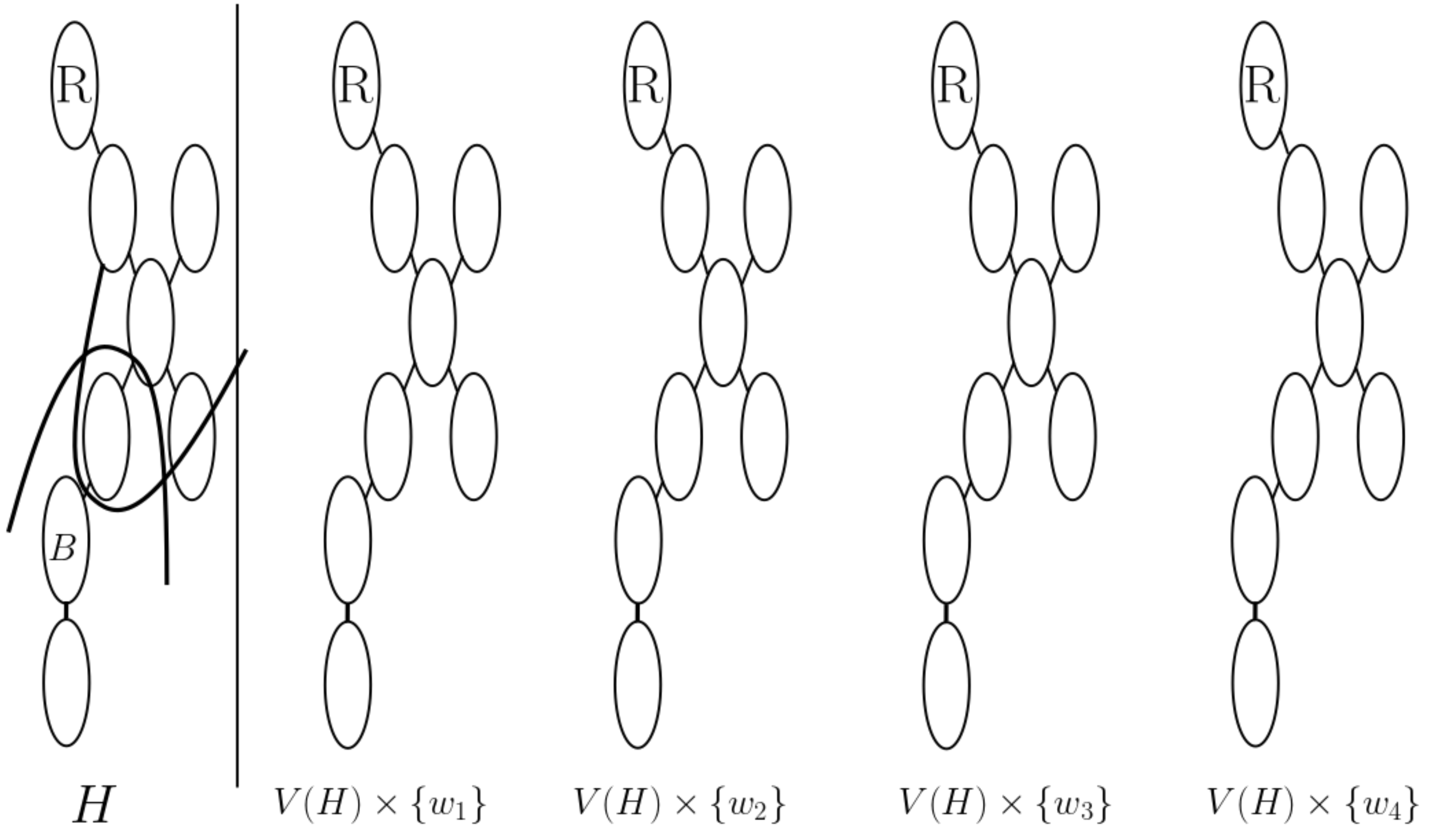

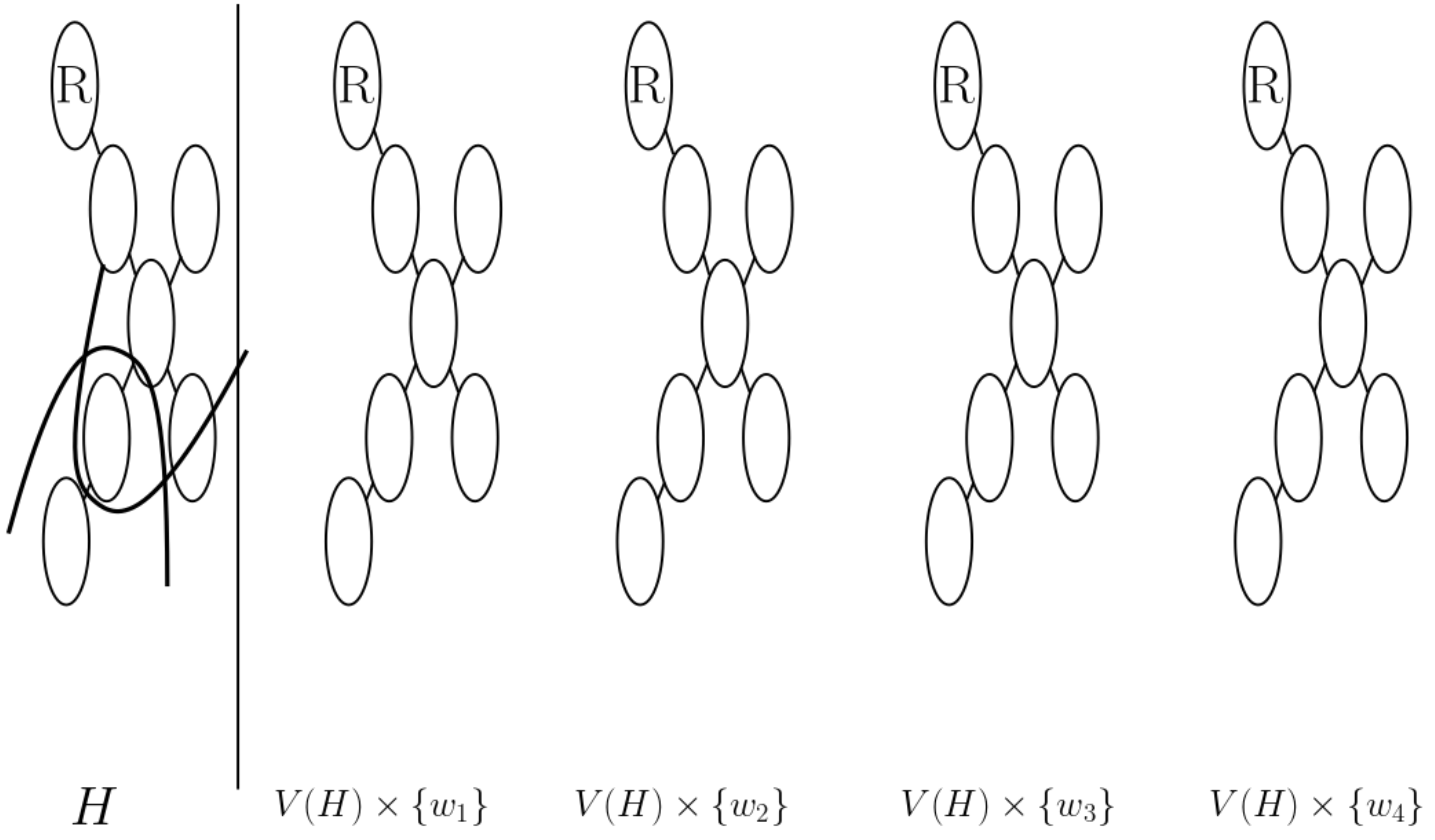

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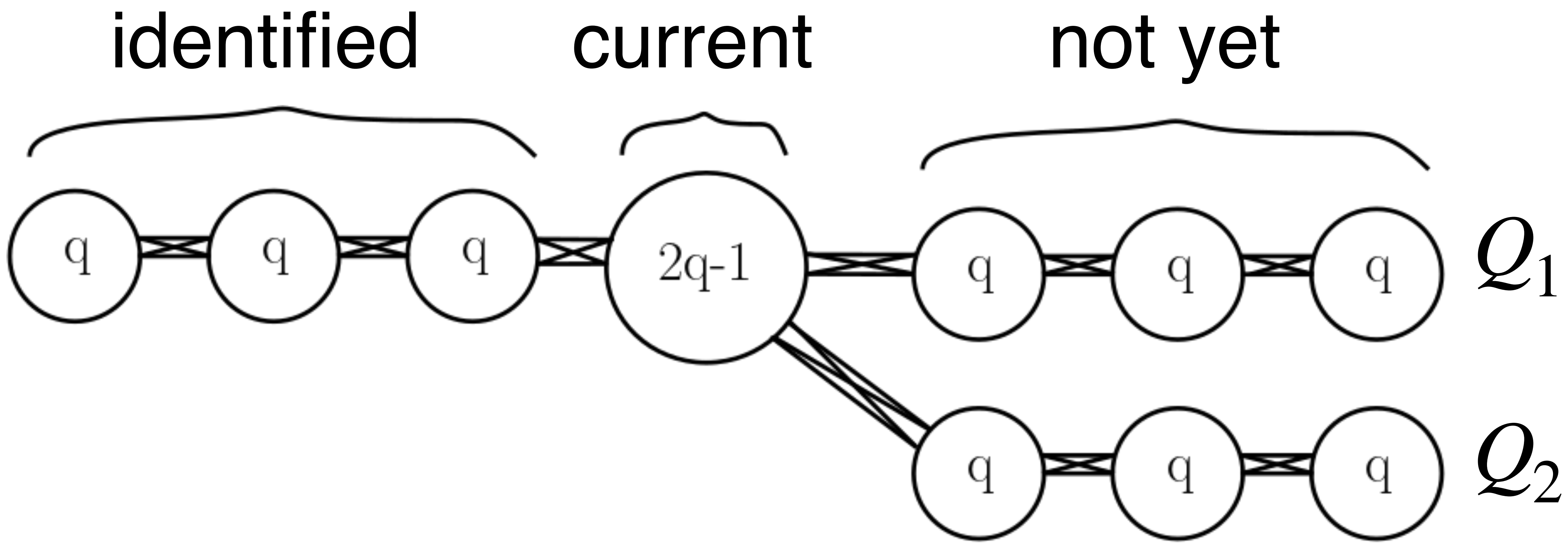

- One can see that each red component is always a subgraph of $S_{|V(P)|,q}$.

- Since f is closed under subgraph / disjoint union, red graph has f-value $\leq g(q)$. So, reduced-f of F is at most $g(q)$.

r-powers

- For r-th powers :

  1) we consider the r-th power of $S_{x,q}$ instead of $S_{x,q}$

  2) (neighbourhood condition) $N_P[z]$ is replaced with $N_P^r[z]$

  3) (separation condition) We can use linear bounds on distance-r profiles by Eickmeyer et al. (2017) (or simply $(r+1)^{|S|}$)

  4) For a map graph G and vertex set S, we prove that $|\{N(v) \cap S : v \in V(G) \backslash S\}| \leq \max\{2^{10}, 37|S| - 81\}$ (where we apply $|S| = 35$)

X-minor free graphs

- (Dujmović, Joret, Micek, Morin, Ueckerdt, and Wood 2020)
  For every graph X, there exists $k, a \in \mathbb{N}$ such that every X-minor-free graph G has a tree-decomposition in which every torso is a subgraph of $(H \boxtimes P) + K_a$ for some graph H of treewidth at most k and some path P.

- We consider the neighborhood complexity to bags in $H \boxtimes P$ together with K_a.
- We obtain a reduction sequence from bottom to top in the tree-decomposition, so that during the sequence, we do not create red edge to above bags.
- We need to extend the lemma to deal with information from below subtrees.

Conclusion

- Proper minor-closed classes and their r-powers have bounded reduced-bandwidth.

Question :

- Is it true that planar graphs have reduced-bandwidth / twin-width at most 10?

- We write $f_1 \prec f_2$ if there is a function ϕ such that for every graph G, $f_1(G) \leq \phi(f_2(G))$.
  Is there a parameter f such that
  - planar graphs have bounded reduced-f and $f \prec$ bandwidth but bandwidth $\nprec f$?

- Is there a natural parameter tied to reduced-bandwidth?

- Is there an interesting application of reduced-bandwidth?