Divided power algebras with derivations

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(Classical) Divided power algebras

Let A be a commutative associative algebra over $\mathbb{F}$ of characteristic 0.

The family of operations $\gamma_n : x \in A \to \frac{x^n}{n!}$ satisfy, for $x, y \in A$, $\lambda \in \mathbb{F}$:

1) Symmetry : Commutativity of the multiplication
2) Monomiality : $\gamma_k(\lambda x) = \lambda^k \gamma_k(x)$.
3) Homogeneity : $\gamma_0(x) = 1$
4) Reduction Repetition : $\gamma_k(x)\gamma_h(x) = \binom{k+h}{k}\gamma_{k+h}(x)$,
5) Leibniz Rule : $\gamma_k(x+y) = \sum_{i+j=k} \gamma_i(x)\gamma_j(y)$,
6) Unitality : $\gamma_1(x) = x$,
7a) Composition 1 : $\gamma_k(xy) = k!\gamma_k(x)\gamma_k(y) = x^k\gamma_k(y) = \gamma_k(x)y^k$,
7b) Composition 2 : $\gamma_k(\gamma_h(x)) = \frac{(kh)!}{k!(h!)^k}\gamma_{kh}(x)$

Definition : A divided power algebra over any field is a commutative associative algebra with operations γ_n satisfying 1) to 6).
(Modulo some way to handle the unit)

First example : free divided power algebras

The free divided power algebra over a vector space V is the algebra of symmetric tensors :

$$\Gamma(V) = \bigoplus_n \left(V^{\otimes n}\right)^{\mathfrak{S}_n},$$

with shuffle product.

Explicitely, if $V = \langle x_1, \ldots, x_n \rangle$,

- As a vector space, $\Gamma(V) = \langle x_1^{[k_1]} \ldots x_n^{[k_n]}, k_1, \ldots, k_n \in \mathbb{N} \rangle$.
- The product is given by :
$x_i^{[k]} x_i^{[h]} = \binom{k+h}{k} x_i^{k+h}$ and $x_i^{[k]} x_j^{[h]} = x_j^{[h]} x_i^{[k]}$, and extended by bilinearity.
- The divided powers are given by :
$\gamma_k(x^{[h]}) = \frac{(kh)!}{k!(h!)^k} x^{[k+h]}$ and extended by Monomiality, Homogneity, Leibniz Rule, Unitality and Composition rule.

We think of $x^{[n]}$ as $\frac{x^n}{n!}$.

Second example : Formal Divided Power Series
(Hurwitz series)

Let A be a commutative associative algebra over $\mathbb{F}$. We can form the ring of formal divided power series over A :

$$HA = \prod_n A \otimes (\langle x \rangle^{\otimes n})^{\mathfrak{S}_n},$$

Whose elements are sequences $\sum_n a_n x^{[n]}$, with $(a_n)_n \in A^{\mathbb{N}}$.
This has an assoc. comm. product inherited from A and $\Gamma(\langle x \rangle)$.

In char. 0, classical derivation of polynomials yields :
$\partial\left(\frac{x^n}{n!}\right) = n\frac{x^{n-1}}{n!} = \frac{x^{n-1}}{(n-1)!}$
So, we set $\partial\,(a_0, a_1, a_2, \dots) = (a_1, a_2, \dots)$.
We also have 'integration' : $i_0\,(a_0, a_1, a_2, \dots) = i_0(0, a_0, a_1, \dots)$.

Proposition (Keigher–Pritchard, '00) : The operations γ_n, inductively defined by $\gamma_0(f) = (1_A, 0, \dots)$ and $\gamma_{n+1}(f) = i_0(\gamma_n(f)\partial(f))$, endow HA with a structure of divided power algebra.

Question : How should I define a derivation on a divided power algebra ?
If $d(ab) = d(a)b + ad(b)$, what is $d(\gamma_n(a))$?

On HA, one has the "Power Rule" :

$$\partial\left(\gamma_n\left(f\right)\right) = \gamma_{n-1}(f)\partial(f).$$

$\mathscr{C}$ is a symmetric monoidal category with countable coproducts.
Here, $\mathscr{C} = \mathbb{F}_{\text{vect}}$.

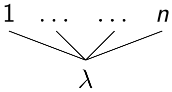
Symmetric sequence in $\mathscr{C}$: $\mathscr{M} = \{\mathfrak{S}_n \curvearrowright \mathscr{M}(n)\}_{n\in\mathbb{N}}$, $\mathscr{M}(n) \in ob(\mathscr{C})$.

$\mathscr{M} \underset{\Sigma}{\otimes} \mathscr{N}$: $(\mathscr{M} \underset{\Sigma}{\otimes} \mathscr{N})(n) = \bigoplus_{i+j=n} \mathrm{Ind}_{\mathfrak{S}_i\times\mathfrak{S}_j}^{\mathfrak{S}_n} \mathscr{M}(i) \otimes \mathscr{N}(j)$.

$\mathscr{M} \circ \mathscr{N}$: $(\mathscr{M} \circ \mathscr{N})(n) = \bigoplus_{k\in\mathbb{N}} \mathscr{M}(k) \otimes_{\mathfrak{S}_k} \mathscr{N}^{\underset{\Sigma}{\otimes} k}(n)$

An operad is a monoid in the category (Symmetric sequences, $\circ$, I).

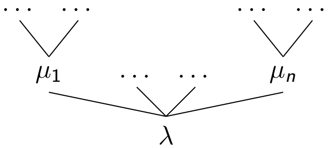
Operads



Let $\mathscr{P}$ be an operad.

$\mathscr{P}(n)$ = arity n operations

$\mathfrak{S}_n \curvearrowright \mathscr{P}(n)$ *"permuting inputs"*



Total composition : $\lambda(\mu_1, \dots, \mu_n)$

$1_{\mathscr{P}} \in \mathscr{P}(1)$

Partial composition : $\lambda \in \mathscr{P}(n), \mu \in \mathscr{P}(m) \rightsquigarrow \lambda \circ_i \mu \in \mathscr{P}(n+m-1)$

$+$ compatibility relations.

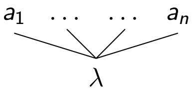
Free algebra functor

$S(\mathscr{P}, -)$: "free $\mathscr{P}$-algebra" functor.

$$S(\mathscr{P}, V) = \mathscr{P} \circ V = \bigoplus_{n \geq 0} \mathscr{P}(n) \otimes_{\mathfrak{S}_n} V^{\otimes n}.$$

$S(\mathscr{P}, -) : \mathbb{F}_{\text{vect}} \to \mathbb{F}_{\text{vect}}$ is a monad $\rightsquigarrow \mathscr{P}_{\text{alg}} : (A \in \mathbb{F}_{\text{vect}}, ev)$,

$$\begin{aligned} ev : \quad S(\mathscr{P}, A) &\to A \\ (\lambda; a_1, \ldots, a_n) &\mapsto \lambda(a_1, \ldots, a_n) \end{aligned} \quad + \text{compatibilities.}$$



A is a left module over the monoid $\mathscr{P}$.

Free divided power algebra over an operad

Define :

$$(\mathscr{M}\tilde{\circ}\mathscr{N})(n) = \bigoplus_{k\in\mathbb{N}} \left(\mathscr{M}(k)\otimes\mathscr{N}^{\otimes_{\mathfrak{S}}k}(n)\right)^{\mathfrak{S}_k},$$

$\Gamma(\mathscr{P},-)$: "free divided power $\mathscr{P}$-algebra" functor,

$$\Gamma(\mathscr{P},V) = \mathscr{P}\tilde{\circ}V = \bigoplus_{n\geq 0} \left(\mathscr{P}(n)\otimes V^{\otimes n}\right)^{\mathfrak{S}_k}.$$

$\Gamma(\mathscr{P},-) : \mathbb{F}_{\text{vect}} \to \mathbb{F}_{\text{vect}}$ is a monad $\rightsquigarrow$ "divided $\mathscr{P}$-alg." $= \Gamma(\mathscr{P})_{\text{alg}}$.

Motivating example : $\Gamma(\text{Com},V) = \Gamma(V)$.

General characterisation of $\Gamma(\mathscr{P})$-algebras

Theorem (I.)

A $\Gamma(\mathscr{P})$-algebra is a vector space A equipped with operations $\beta_{x,\underline{r}} : A^{\times p} \to A$, linear in x, satisfying relations of the type : Symmetry, Homogeneity, Monomiality, Reduction/Repetition, Leibniz Rule, Unitality, Composition.

In other words, $\Gamma(\mathscr{P})$-algebras are indeed vector spaces with monomial operations (and relations).

Distributive laws, Divided powers, Derivations

Definition : For $\mathscr{P},\mathscr{Q}$ two operads, a distributive law is a map $\lambda:\mathscr{Q}\circ\mathscr{P}\to\mathscr{P}\circ\mathscr{Q}$ that endows $\mathscr{P}\circ\mathscr{Q}$ with an operad structure.

Proposition (I.)

If $\mathscr{P},\mathscr{Q}$ are two operads with a distributive law $\Lambda:\mathscr{Q}\circ\mathscr{P}\to\mathscr{P}\circ\mathscr{Q}$, then $\Gamma(\mathscr{P}\circ\mathscr{Q})$ can be obtained from $\Gamma(\mathscr{P},-)\circ\Gamma(\mathscr{Q},-)$ by a distributive law

$$\tilde{\Lambda}:\Gamma(\mathscr{Q},-)\circ\Gamma(\mathscr{P},-)\to\Gamma(\mathscr{P},-)\circ\Gamma(\mathscr{Q},-).$$

The expression of $\tilde{\Lambda}$ from that of Λ can be a bit tricky.

Let $\mathrm{D}=\mathbb{F}[d]$ in arity 1. For all $\mathscr{P}$, $\mu\in\mathscr{P}(m)$,

$$(d;\mu)\mapsto\sum_{\underline{q}\in\mathrm{Comp}_m(j)}\binom{j}{q_1,\dots,q_m}(\mu;d^{q_1},\dots,d^{q_m})$$

The algebras over $\mathscr{P}\circ\mathrm{D}$ are $\mathscr{P}$-algebra with derivations, ex, $\mathrm{Der}_{\mathrm{Com}}=\mathrm{Com}\circ\mathrm{D}$.

Divided power algebra with derivative

Theorem (I.)

A divided power Com ∘ D*-algebra (with the preceding distributive law) is a divided power algebra A endowed with a linear map* $d : A \to A$ *satisfying :*

a) d *is a derivation* ($d(ab) = d(a)b + ad(b)$),

b) $d(\gamma_n(a)) = d(a)\gamma_{n-1}(a)$.

For any $f \in HA$ and $n \in \mathbf{N}$, we define $f^{[n]} \in HA$, called the nth *divided power of* f, inductively by $f^{[0]} = 1_{HA}$, and if $n \geq 1$, $f^{[n]} = i_0(f^{[n-1]}\partial_A(f))$. It is immediate from the definition that the following "power rule" holds, i.e., for any $n \geq 1$, $\partial_A(f^{[n]}) = f^{[n-1]}\partial_A(f)$.

Keigher Pritchard, Hurwitz Series (1998)

Thank you for your attention, and thanks to the organisers for a wonderful conference.