Linearizing Combinators

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Background Story

- Cartesian differential categories formalize differentiation in multivariable calculus of Euclidean spaces.

  R. Blute, R. Cockett, R.A.G. Seely, **Cartesian Differential Categories**

- Abelian functor calculus was developed by Johnson and McCarthy based on Goodwillie's functor calculus. Bauer, Johnson, Osborne, Riehl, and Tebbe (BJORT) constructed an Abelian functor calculus model of a Cartesian differential category.

  Bauer, K., Johnson, B., Osborne, C., Riehl, E. and Tebbe, A., **Directional derivatives and higher order chain rules for abelian functor calculus.**

  The differential combinator is defined using the *linearization* (or *linear approximation*) of functors.

- From the Cartesian differential category perspective, the BJORT construction is backwards!

- In any Cartesian differential category it is always possible to define the notion of a linear map and to linearize a map using the differential combinator. However, BJORT constructed their differential combinator using an already established notion of linear map and linearization.

- This made Robin and me very confused... But thanks to talking to with Kristine, Brenda and Sarah at the CMS Summer Meeting 2018, we set out the understand what was going on!

Story Today

- The goal is to reverse engineer BJORT's construction by abstracting the notion of linear approximation from the (Abelian) functor calculus.

- We introduce **linearizing combinators** on a Cartesian left additive category.
  - Every Cartesian differential category comes equipped with a canonical linearizing combinator obtained by differentiation at zero.
  - Conversely, a differential combinator can be constructed à la BJORT when one has a system of *partial* linearizing combinators.

Linearizing combinators provide an alternative axiomatization of Cartesian differential categories. This correspondence is the analogue on the monoidal side of the story:

$\otimes$-differential categories	Cartesian differential categories
Deriving transformations $\mathsf{d} : !A \otimes A \to !A$	Differential combinators D $\dfrac{f : A \to B}{\mathsf{D}[f] : A \times A \to B}$
Coderelictions $\eta : A \to !A$	Linearizing Combinators L $\dfrac{f : A \to B}{\mathsf{L}[f] : A \to B}$

Main Reference:

R. Cockett, J.-S. P. Lemay, **Linearizing Combinators**

Cartesian Left Additive Category - Definition

A **left additive category** is a category $\mathbb{X}$ where every homset is a commutative monoid, so we can add maps and have zero maps:

$$+ : \mathbb{X}(A, B) \times \mathbb{X}(A, B) \to \mathbb{X}(A, B) \qquad\qquad 0 \in \mathbb{X}(A, B)$$

such that composition preserves the addition in the following sense:

$$(f + g) \circ x = f \circ x + g \circ x \qquad\qquad 0 \circ x = 0$$

A map f is **additive** if $f \circ (x + y) = f \circ x + f \circ y$ and $f \circ 0 = 0$.

A **Cartesian left additive category** (CLAC) is a left additive category with finite products such that the projection maps $\pi_0 : A \times B \to A$ and $\pi_1 : A \times B \to B$ are additive.

Cartesian Differential Category - Definition

A **Cartesian differential category** (CDC) is a CLAC $\mathbb{X}$ equipped with a **differential combinator** D:

$$\frac{f : A \to B}{\mathsf{D}[f] : A \times A \to B}$$

To help us with the axioms, we will use the following notation/proto-term logic:

$$\mathsf{D}[f](a,b) := \frac{\mathsf{d}f(x)}{\mathsf{d}x}(a) \cdot b$$

Then D satisfies the following seven axioms:

[CD.1] $\frac{\mathsf{d}f(x)+g(x)}{\mathsf{d}x}(a) \cdot b = \frac{\mathsf{d}f(x)}{\mathsf{d}x}(a) \cdot b + \frac{\mathsf{d}g(x)}{\mathsf{d}x}(a) \cdot b$ and $\frac{\mathsf{d}0}{\mathsf{d}x}(a) \cdot b = 0$

[CD.2] $\frac{\mathsf{d}f(x)}{\mathsf{d}x}(a) \cdot (b+c) = \frac{\mathsf{d}f(x)}{\mathsf{d}x}(a) \cdot b + \frac{\mathsf{d}f(x)}{\mathsf{d}x}(a) \cdot c$ and $\frac{\mathsf{d}f(x)}{\mathsf{d}x}(a) \cdot 0 = 0$

[CD.3] $\frac{\mathsf{d}x}{\mathsf{d}x}(a) \cdot b = b$ and $\frac{\mathsf{d}\pi_i(x_0,x_1)}{\mathsf{d}(x_0,x_1)}(a_0, a_1) \cdot (b_0, b_1) = b_i$

[CD.4] $\frac{\mathsf{d}\langle f(x),g(x)\rangle}{\mathsf{d}x}(a) \cdot b = \left\langle \frac{\mathsf{d}f(x)}{\mathsf{d}x}(a) \cdot b, \frac{\mathsf{d}g(x)}{\mathsf{d}x}(a) \cdot b \right\rangle$

[CD.5] $\frac{\mathsf{d}g(f(x))}{\mathsf{d}x}(a) \cdot b = \frac{\mathsf{d}g(y)}{\mathsf{d}y}(f(a)) \cdot \left(\frac{\mathsf{d}f(x)}{\mathsf{d}x}(a) \cdot b \right)$

[CD.6] $\frac{\mathsf{d}\frac{\mathsf{d}f(x)}{\mathsf{d}x}(y) \cdot z}{\mathsf{d}(y,z)}(a, 0) \cdot (0, b) = \frac{\mathsf{d}f(x)}{\mathsf{d}x}(a) \cdot b$

[CD.7] $\frac{\mathsf{d}\frac{\mathsf{d}f(x)}{\mathsf{d}x}(y) \cdot z}{\mathsf{d}(y,z)}(a, b) \cdot (c, d) = \frac{\mathsf{d}\frac{\mathsf{d}f(x)}{\mathsf{d}x}(y) \cdot z}{\mathsf{d}(y,z)}(a, c) \cdot (b, d)$

Smooth Functions Example

Example

Define SMOOTH be the category whose objects are the Euclidean real vector spaces $\mathbb{R}^n$ and whose maps are $\mathcal{C}^\infty$ functions $F : \mathbb{R}^n \to \mathbb{R}^m$ between them. SMOOTH is a Cartesian differential category where the differential combinator is defined as the directional derivative of a smooth function. A smooth function $F : \mathbb{R}^n \to \mathbb{R}^m$ is in fact a tuple:

$$F = \langle f_1, \ldots, f_m \rangle$$

So for a smooth function $F : \mathbb{R}^n \to \mathbb{R}^m$, its derivative $\mathsf{D}[F] : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$ is then defined as:

$$\mathsf{D}[F](\vec{x}, \vec{y}) = \left\langle \sum_{i=1}^{n} \frac{\partial f_1}{\partial x_i}(\vec{x}) y_i, \ldots, \sum_{i=1}^{n} \frac{\partial f_m}{\partial x_i}(\vec{x}) y_i \right\rangle$$

Linear Maps

In a Cartesian differential category, there is a natural notion of **linear maps**. A map $f : A \to B$ is said to be D-linear if:

$$\mathsf{D}[f] := \; A \times A \xrightarrow{\quad \pi_1 \quad} A \xrightarrow{\quad f \quad} B \qquad\qquad \frac{\mathrm{d}f(x)}{\mathrm{d}x}(a) \cdot b = f(b)$$

Example

In $\mathsf{SMOOTH}_{\mathbb{R}}$, a smooth function $F : \mathbb{R}^n \to \mathbb{R}^m$ is D-linear if and only if it is $\mathbb{R}$-linear in the classical sense:

$$F(s\vec{x} + t\vec{y}) = sF(\vec{x}) + tF(\vec{y})$$

for all $s, t \in \mathbb{R}$ and $\vec{x}, \vec{y} \in \mathbb{R}^n$.

We would now like to have the ability of linearizing maps in a Cartesian differential category.

Calculus

- Given a smooth function $f : \mathbb{R} \to \mathbb{R}$, its linearization $\mathsf{L}[f] : \mathbb{R} \to \mathbb{R}$ is the best $\mathbb{R}$-linear function which is closest to f. This is given by the first degree term in its Maclaurin series expansion (i.e its Taylor series expansion at 0):

$$\mathsf{L}[f](x) = f'(0)x$$

- In terms of the differential combinator, its differential $\mathsf{D}[f] : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is defined as:

$$\mathsf{D}[f](x, y) = f'(x)y$$

  So $\mathsf{L}[f](x) = \mathsf{D}[f](0, x)$.

- This construction can be done in any Cartesian differential category. We can use this to derive an abstract notion of a linearizing combinator, L, for arbitrary Cartesian left additive categories, which satisfies axioms which parallel those of the differential combinator.

Linearizing Combinator

A **linearizing combinator** L on a Cartesian left additive category $\mathbb{X}$ is a combinator:

$$\frac{f : A \to B}{\mathsf{L}[f] : A \to B}$$

Which we denote as follows in the term logic:

$$\mathsf{L}[f](a) := \frac{\ell f(x)}{\ell x} \cdot a$$

L.1 - Additivity of Combinator & L.2 - Additivity

- **[L.1]** Additivity of Combinator:

$$\frac{\ell f(x) + g(x)}{\ell x} \cdot a = \frac{\ell f(x)}{\ell x} \cdot a + \frac{\ell g(x)}{\ell x} \cdot a \qquad\qquad \frac{\ell 0}{\ell x} \cdot a = 0$$

- **[L.2]** Additivity: The linearization of a map is additive

$$\frac{\ell f(x)}{\ell x} \cdot (a + b) = \frac{\ell f(x)}{\ell x} \cdot a + \frac{\ell f(x)}{\ell x} \cdot b \qquad\qquad \frac{\ell f(x)}{\ell x} \cdot 0 = 0$$

- **[L.3]** Identities + Projections

$$\frac{\ell x}{\ell x} \cdot a = a \qquad\qquad \frac{\ell \pi_i(x_0, x_1)}{\ell(x_0, x_1)} \cdot (a_0, a_1) = \pi_i(a_0, a_1) = a_i$$

- **[L.4]** Pairings

$$\frac{\ell\langle f(x), g(x)\rangle}{\ell x} \cdot a = \left\langle \frac{\ell f(x)}{\ell x} \cdot a, \frac{\ell g(x)}{\ell x} \cdot a \right\rangle$$

- **[L.5]** Chain Rule:

$$\frac{\ell g\left(f(x)\right)}{\ell x} \cdot a = \frac{\ell g(f(0)+y)}{\ell y} \cdot \left(\frac{\ell f(x)}{\ell x} \cdot a\right)$$

The keen-eyed reader may have noticed that on the right hand side of **[L.5]**. In theory one could again apply **[L.5]** to the right hand side again. So **[L.5]** is indeed simplified as far as possible. That said, **[L.5]** does simplify when f or g is additive:

$$\frac{\ell g\left(f(x)\right)}{\ell x} \cdot a = \frac{\ell g(y)}{\ell y} \cdot \left(\frac{\ell f(x)}{\ell x} \cdot a\right)$$

- **[L.6]** Linearization is idempotent:

$$\frac{\ell \frac{\ell f(x)}{\ell x} \cdot y}{\ell y} \cdot a = \frac{\ell f(x)}{\ell x} \cdot a$$

Linearizing Combinator

A **linearizing combinator** L on a Cartesian left additive category $\mathbb{X}$ is a combinator:

$$\frac{f : A \to B}{\mathsf{L}[f] : A \to B}$$

Which we denote as follows in the term logic:

$$\mathsf{L}[f](a) := \frac{\ell f(x)}{\ell x} \cdot a$$

such that L satisfies **[L.1]** to **[L.6]**.

Remark

Those familiar with CDCs might point to the fact that differential combinator have **SEVEN** axioms while linearizing combinators have **SIX**... good observation! Specifically, the analogue of **[CD.7]** is not present... this will come up when we talk about partial linearization.

L-Linear

For a linearizing combinator, the analogues of linear maps are the maps for which the linearizing combinator does nothing. A map f is said to be L-**linear** if:

$$\mathsf{L}[f] = f \qquad\qquad \frac{\ell f(x)}{\ell x} \cdot a = f(a)$$

By **[L.6]**, for every map f, its linearization $\mathsf{L}[f]$ is L-linear.

Linearizing Combinators from Differential Combinators

Proposition

Every Cartesian differential category, with differential combinator D*, admits a linearizing combinator* L_{D} *defined as follows for every map* $f : A \to B$*:*

$$\mathsf{L}_{\mathsf{D}}[f] := \mathsf{D}[f] \circ \langle 0, 1 \rangle \qquad \frac{\ell f(x)}{\ell x} \cdot a = \frac{\mathrm{d} f(x)}{\mathrm{d} x}(0) \cdot b$$

Furthermore,

(i) A map f *is* D*-linear if and only if* f *is* L_{D}*-linear.*

(ii) For every map f, $\mathsf{L}_{\mathsf{D}}[f]$ *is* D*-linear;*

Example

For SMOOTH, the linearizing combinator is defined as evaluating the directional derivative at zero in the first argument. Explicitly, for a smooth function $F : \mathbb{R}^n \to \mathbb{R}^m$, $F = \langle f_1, \dots, f_n \rangle$:

$$\mathsf{L}[F](\vec{x}) = \mathbf{J}(F)(\vec{0}) \cdot \vec{x} = \left\langle \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{0}) x_i, \dots, \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{0}) x_i \right\rangle$$

For example, consider $f(x, y) = e^x \cos(y)$. Its derivative is worked out to be $\mathsf{D}[f](x, y, z, w) = e^x \cos(y) z - e^x \sin(y) w$. Then evaluating at 0 in the first two arguments, we obtain that $\mathsf{L}[f](x, y) = e^0 \cos(0) x - e^0 \sin(0) y = x$.

Differential Combinators from Linearizing Combinators

Now for the other direction: to construct differential combinators from linearizing combinators.

Limits

- Consider the classical limit definition of the derivative of a smooth function $f : \mathbb{R} \to \mathbb{R}$:

$$\mathsf{D}[f](x, y) = \lim_{t \to 0} \frac{f(x + ty) - f(x)}{t}$$

Note that if we evaluate at $x = 0$, then we obtain an expression of $\mathsf{L}[f]$ in terms of a limit:

$$\mathsf{L}[f](y) = \mathsf{D}[f](0, y) = \lim_{t \to 0} \frac{f(ty) - f(0)}{t}$$

- For a fixed x, define $g_x : \mathbb{R} \to \mathbb{R}$ to be the smooth function defined as $g_x(y) = f(x + y)$.

$$\mathsf{D}[f](x, y) = \lim_{t \to 0} \frac{f(x + t \cdot y) - f(x)}{t} = \lim_{t \to 0} \frac{g_x(ty) - g_x(0)}{t} = \mathsf{L}[g_x](y)$$

So the derivative of f is the linearization of the function $g_x(y) = f(x + y)$ in the variable y.

- This is precisely how BJORT define their differential combinator. In fact, every differential combinator in a Cartesian differential category can be defined in this fashion.

Differential Combinators from Linearizing Combinators

Now for the other direction: to construct differential combinators from linearizing combinators.

- To do this we require the notion of partial linearization.
- However, while it is always possible to define partial differentiation from total differentiation...
- **In general it is not necessarily possible to define partial linearization from total linearization!**
- As such, we need to separately define the notion of linearizing combinators in contexts, which we call a *system* of linearizing combinators
- But first we need to discuss context... From a categorical perspective, a map in a fixed "context" C is interpreted as a map in the **simple slice category** over C.

Simple Slice Categories

Let $\mathbb{X}$ be a category with finite products. For each object C, the **simple slice category** over C is the category $\mathbb{X}[C]$ where:

- The objects are the objects of $\mathbb{X}$, $ob(\mathbb{X}[C]) := ob(\mathbb{X})$;
- The hom-sets are defined as $\mathbb{X}[C](A, B) := \mathbb{X}(C \times A, B)$, that is, a map from A to B in $\mathbb{X}[C]$ is a map $f : C \times A \to B$ in $\mathbb{X}(C \times A, B)$, and we say that f is in context C.
- The identity maps are the projection maps $\pi_1 : C \times A \to A$;
- The composition of maps $f : C \times A \to B$ and $g : C \times B \to D$ is the composition:

$$C \times A \xrightarrow{\langle \pi_0, f \rangle} C \times B \xrightarrow{g} D$$

For each map $h : C' \to C$ in $\mathbb{X}$, define the **substitution functor** $h^* : \mathbb{X}[C] \to \mathbb{X}[C']$ on objects as $h^*(A) := A$ and on maps as:

$$C \times A \xrightarrow{h \times 1} C' \times A \xrightarrow{f} B$$

System of Linearizing Combinators

A **system of linearizing combinators** on a Cartesian left additive category $\mathbb{X}$ is a family of linearizing combinators L^C:

$$\frac{f : C \times A \to B}{\mathsf{L}^C[f] : C \times A \to B}$$

where $\mathsf{L}^C[f]$ is the linearization of f in its second argument, which we denote in the term logic:

$$\mathsf{L}^C[f](c, a) := \frac{\ell f(c, x)}{\ell x} \cdot a$$

where L^C is a linearizing combinator for the simple slice category $\mathbb{X}[C]$:

[L.1] $\frac{\ell f(c,x)+g(c,x)}{\ell x} \cdot a = \frac{\ell f(c,x)}{\ell x} \cdot a + \frac{\ell g(c,x)}{\ell x} \cdot a$ and $\frac{\ell 0}{\ell x} \cdot a = 0$

[L.2] $\frac{\ell f(c,x)}{\ell x} \cdot (a + b) = \frac{\ell f(c,x)}{\ell x} \cdot a + \frac{\ell f(c,x)}{\ell x} \cdot b$ and $\frac{\ell f(c,x)}{\ell x} \cdot 0 = 0$

[L.3] $\frac{\ell \pi_1(c,x)}{\ell x} \cdot a = a$ and $\frac{\ell \pi_i(\pi_1(c,(x_0,x_1)))}{\ell (x_0,x_1)} \cdot (a_0, a_1) = \pi_i(a_0, a_1) = a_i$

[L.4] $\frac{\ell \langle f(c,x), g(c,x)\rangle}{\ell x} \cdot a = \left\langle \frac{\ell f(c,x)}{\ell x} \cdot a, \frac{\ell g(c,x)}{\ell x} \cdot a \right\rangle$

[L.5] $\frac{\ell g(c,f(c,x))}{\ell x} \cdot a = \frac{\ell g(c,f(c,0)+y)}{\ell y} \cdot \left(\frac{\ell f(c,x)}{\ell x} \cdot a \right)$

[L.6] $\frac{\ell \frac{\ell f(c,x)}{\ell x} \cdot y}{\ell y} \cdot a = \frac{\ell f(c,x)}{\ell x} \cdot a$

and such that the following two extra axioms hold:

L.7 - Symmetry & L.8 - Context Substitution

- **[L.7]** Symmetry Rule:

$$\frac{\frac{\ell f(c,x,y)}{\ell x} \cdot a}{\ell y} \cdot b = \frac{\frac{\ell f(c,x,y)}{\ell y} \cdot b}{\ell x} \cdot a$$

- **[L.8]** Context Substitution:

$$h^* \left(\frac{\ell f(c,x)}{\ell x} \cdot a \right) = \frac{\ell f(h(c),x)}{\ell x} \cdot a$$

This tells you how the linearizing combinators on the simple slices are all the same.

System of Linearizing Combinators

A **system of linearizing combinators** on a Cartesian left additive category $\mathbb{X}$ is a family of linearizing combinators L^C:

$$\frac{f : C \times A \to B}{\mathsf{L}^C[f] : C \times A \to B}$$

where $\mathsf{L}^C[f]$ is the linearization of f in its second argument, which we denote in the term logic:

$$\mathsf{L}^C[f](c, a) := \frac{\ell f(c, x)}{\ell x} \cdot a$$

where L^C is a linearizing combinator for the simple slice category $\mathbb{X}[C]$ and such that **[L.7]** and **[L.8]** hold.

Since $\mathbb{X}[\top] \cong \mathbb{X}$, then $\mathbb{X}$ has a linearizing combinator L defined as follows for a map $f : A \to B$:

$$\mathsf{L}[f] = A \xrightarrow{\langle 0,1 \rangle} \top \times A \xrightarrow{\mathsf{L}^\top[f \circ \pi_1]} B$$

Example of partial linearization

Consider the polynomial function

$$f(x, y) = xy + 2xy^3 + 3x + 4y$$

- The total linearization of f, that is, linearizing f jointly in x and y is the polynomial:

$$\mathsf{L}[f](x, y) = 3x + 4y$$

- Linearizing in terms of x while keeping y in context picks out the terms with $deg(x) = 1$:

$$\mathsf{L}^y[f] = xy + 2xy^3 + 3x$$

  which is now linear in x

- Linearizing in terms of y while keeping x in context picks out the terms with $deg(y) = 1$:

$$\mathsf{L}^x[f] = xy + 4y$$

  which this time is linear in y

- Linearizing $xy + 2xy^3 + 3x$ in terms of y or linearizing $xy + 4y$ in terms of x both results in:

$$\mathsf{L}^x[\mathsf{L}^y[f]] = \mathsf{L}^y[\mathsf{L}^x[f]] = xy$$

  which this time is bilinear in x and y

System of Linearizing Combinators from Differential Combinators

If $\mathbb{X}$ is a CDC, then every simple slice category $\mathbb{X}[C]$ is a CDC via partial differentiation, so the differential combinator D^C is defined as follows on a map $f : C \times A \to B$:

$$\mathsf{D}^C[f] := \ (C \times A) \times A \xrightarrow{(1\times 1)\times\langle 0,1\rangle} (C \times A) \times (C \times A) \xrightarrow{\mathsf{D}[f]} B$$

$$\mathsf{D}^C[f](c, a, b) := \frac{\mathsf{d}f(c, x)}{\mathsf{d}x}(a) \cdot b := \frac{\mathsf{d}f(z, x)}{\mathsf{d}(z, x)}(c, a) \cdot (0, b)$$

Proposition

Every Cartesian differential category $\mathbb{X}$, with differential combinator D, admits a system of linearizing combinators where the linearizing combinators $\mathsf{L}_{\mathsf{D}^C}$ for the simple slice categories are defined as in the previous slides. In the term logic:

$$\frac{\ell f(c, x)}{\ell x} \cdot a = \frac{\mathsf{d}f(c, x)}{\mathsf{d}x}(0) \cdot b$$

Differential Combinators from System of Linearizing Combinators

Recall that we saw that the differential of f is equal to the linearization of $f(x+y)$ in the y.

In a Cartesian left additive category, define the map $\oplus_A : A \times A \to A$ as $\oplus_A = \pi_0 + \pi_1$

$$\oplus_A(x, y) = x + y$$

Proposition

Every Cartesian left additive category $\mathbb{X}$ *with a system of linearizing combinators* L^C *is a Cartesian differential category with differential combinator* D_L *defined as follows on a map* $f : A \to B$*:*

$$\mathsf{D}_\mathsf{L}[f] := \mathsf{L}^A[f \circ \oplus_A] \qquad \frac{\mathrm{d}f(x)}{\mathrm{d}x}(a) \cdot b = \frac{\ell f(a+x)}{\ell x} \cdot b$$

Equivalence

Theorem

For a Cartesian left additive category $\mathbb{X}$, there is a bijective correspondence between:

(i) Differential combinators;

(ii) Systems of linearizing combinators.

Therefore, a Cartesian differential category is precisely a Cartesian left additive category equipped with a system of linearizing combinators.

Counter-Example

Here's an example which has a total linearization but does not have partial linearization.

Example

Define $\mathcal{C}^1$-DIFF be the category whose objects are the Euclidean real vector spaces $\mathbb{R}^n$ and whose maps are $\mathcal{C}^1$ functions $F : \mathbb{R}^n \to \mathbb{R}^m$ between them. $\mathcal{C}^1$-DIFF is a Cartesian left additive category.

$\mathcal{C}^1$-DIFF has a (total) linearizing combinator L defined in the same way as the linearizing combinator in SMOOTH, that is, for a $\mathcal{C}^1$ function $F = \langle f_1, \dots, f_m \rangle$:

$$\mathsf{L}[F](\vec{x}) = \left\langle \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{0}) x_i, \dots, \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{0}) x_i \right\rangle$$

If $\mathcal{C}^1$-DIFF had partial linearization then $\mathcal{C}^1$-DIFF would also have a differential combinator, but this can't be since the derivative of $\mathcal{C}^1$ functions are not necessarily $\mathcal{C}^\infty$ functions.

Consider the function $f(x) = x^{\frac{3}{2}}$, which is a $\mathcal{C}^1$ function since its derivative $f'(x) = \frac{3}{2} x^{\frac{1}{2}}$ exists and is continuous. If partial linearization was possible, then we would be able to define $\mathsf{D}[f]$ as:

$$\mathsf{D}[f](x, y) = \mathsf{L}[z \mapsto f(x+z)](y) = \frac{3}{2}(x+y)^{\frac{1}{2}}$$

However, this linearization is not define when $x + y < 0$ and therefore is not a map in $\mathcal{C}^1$-DIFF. Therefore $\mathcal{C}^1$-DIFF does not have a system of linearizing combinators.

From total linearization to partial linearization

We would like to define partial linearization from total linearization.

- As previously discussed, in general this is not necessarily possible...

- However in the setting of a Cartesian closed category, it is possible to construct a system of linearizing combinators from a linearizing combinator on the base category.

$$\dfrac{\dfrac{\dfrac{C \times A \xrightarrow{\quad f \quad} B}{\text{Curry} \qquad A \xrightarrow{\quad \lambda(f) \quad} [C, A]}}{\text{Linearize} \qquad A \xrightarrow{\quad \mathsf{L}[\lambda(f)] \quad} [C, A]}}{\text{Uncurry} \qquad \mathsf{L}^{C}[f] := \; C \times A \xrightarrow{\quad \lambda^{-1}(\mathsf{L}[\lambda(f)]) \quad} [C, A]}$$

- We do however need to assume some compatibility between L and the closed structure. In this case, we call this an **exponential** linearizing combinator.

Equivalence

Theorem

For a Cartesian closed left additive category $\mathbb{X}$, there are bijective correspondences between:

(i) Differential combinators D *on* $\mathbb{X}$ *which satisfy* **[CD.λ]**: $\mathsf{D}[\lambda(f)] = \lambda\left(\mathsf{D}^C[f]\right)$

(ii) Systems of linearizing combinators L^C *on* $\mathbb{X}$ *which satisfy* **[L.λ]**: $\mathsf{L}[\lambda(f)] = \lambda\left(\mathsf{L}^C[f]\right)$

(iii) Exponentiable linearizing combinators L *on* $\mathbb{X}$.

Therefore, a Cartesian closed differential category is precisely a Cartesian closed left additive category equipped with a exponentiable linearizing combinator or equivalently a Cartesian closed left additive category equipped with a closed system of linearizing combinators.

Concluding Remarks

- The main purpose of this project was to establish n alternative axiomatization for Cartesian differential categories using a system of linearizing combinators. However, the weakness of this alternative axiomatization should not be overlooked. The problem is that one needs to assume partial linearization at the outset: this is a significant requirement. In this regard the total differential combinator has a clear advantage.

- That said, linearization can exist for functions which are *not* infinitely differentiable! This suggests that linearization could play a significant role in providing a broader categorical approach for non-smooth analysis.

- Is the BJORT model Cartesian closed and is the linearization combinator an exponentiable linearizing combinator?

- The correspondence is also capture on the monoidal side:

$\otimes$-differential categories	Cartesian differential categories
Deriving transformations $\mathsf{d} : !A \otimes A \rightarrow !A$	Differential combinators D $\dfrac{f : A \rightarrow B}{\mathsf{D}[f] : A \times A \rightarrow B}$
Coderelictions $\eta : A \rightarrow !A$	Linearizing Combinators L $\dfrac{f : A \rightarrow B}{\mathsf{L}[f] : A \rightarrow B}$

  Linearizing combinators should also provide equivalent axiomatizations for generalizations of Cartesian differential categories including generalized Cartesian differential categories, differential restriction categories, and even tangent categories.

Merci!

Hope you enjoyed it!
Thanks for listening!
Merci!

R. Cockett, J.-S. P. Lemay, **Linearizing Combinators**