The (higher) topos classifying ∞-connected objects

André Joyal

UQÀM

Workshop on Tangent Categories and their Applications
(16/05/2021)

Joint work with

Mathieu Anel, Georg Biedermann and Eric Finster

Abstract

I will present an application of Goodwillie's calculus to higher topos theory. The (higher) topos which classifies ∞-connected objects is formally the "dual" of the (higher) logos $\mathcal{S}[U_\infty]$ freely generated by an ∞-connected object U_∞. The logos $\mathcal{S}[U_\infty]$ is a left exact *topological localization* of the logos $\mathcal{S}[U] = \mathit{Fun}[\mathrm{Fin}, \mathcal{S}]$ freely generated by an object U. We show that a functor $\mathrm{Fin} \to \mathcal{S}$ belongs to $\mathcal{S}[U_\infty]$ if and only if it is *crystallic* if and only if it is ∞-*excisive*. There is a morphism of logoi from $\mathcal{S}[U_\infty]$ to the category of (formal) Goodwillie towers of functors $\mathrm{Fin} \to \mathcal{S}$, but we do not know if it is an equivalence of categories.

In the first part of the talk, I will explain how logos theory can be used to prove the Klein-Rognes Chain rule.

Content

- The duality topos-logos;
- $\mathcal{S}[U]$ and the Klein-Rognes derivative;
- $\mathcal{S}[U_\infty]$, crystallic functors and ∞-excisive functors.

Conventions

For the sake of simplicity and clarity, we will drop the prefix ∞ when refering to ∞-topoi and $(\infty, 1)$-categories, and speak explicity of 1-topoi and 1-categories if the occasion arises. All limits and colimits are homotopical.

By the word "space" we mean an ∞-groupoid, and we denote the category of spaces by $\mathcal{S}$. We say that a map between two spaces is an *isomorphism*, if it is a homotopy equivalence.

Topoi vs logoi

We are adopting the view that topos theory is better understood by studying the opposite category of *logoi*.

Conventions [AJ]

1. We shall say that a topos $\mathcal{E}$ is a *logos*
2. A *morphism of logoi* is a functor $\phi : \mathcal{E} \to \mathcal{E}'$ which preserves colimits and finite limits (is cocontinuous and left-exact).

The category of topoi is the opposite of the category of logoi.

A logos is a ring-like object

commutative ring	*logos*
sum: $\sum_{i \in I} a_i$	*colimit:* $\varinjlim_{i \in I} A(i)$
product: $a \cdot b$	*finite limits:* $A \times_C B$
unit element: 1	*terminal object:* 1
distributive law: $a \cdot \sum_{i \in I} b_i = \sum_{i \in I} a \cdot b_i$	$\varinjlim_{i \in I} A \times_C B(i) = A \times_C \varinjlim_{i \in I} B(i)$

A morphism of logoi $\phi : \mathcal{E} \to \mathcal{F}$ preserves colimits (=sums) and finite limits ($=$ products).

Rings vs logoi

commutative ring	*logos*
initial ring: $\mathbb{Z}$	*initial logos:* $\mathcal{S}$
terminal ring: 0	*terminal logos:* 0
polynomial ring: $\mathbb{Z}[x]$	*free logos:* $\mathcal{S}[U]$
quotient: $R \to R/J$	*lex localization:* $\mathcal{E} \to \mathcal{E}[\Sigma^{-1}]_{lex}$
extension: $\mathbb{Z} \to \mathbb{Z}[a]$	*extension:* $\mathcal{S} \to \mathcal{S}[A]$

The polynomial ring $\mathbb{Z}[x]$

Recall that the polynomial ring $\mathbb{Z}[x]$ is freely generated the element $x \in \mathbb{Z}[x]$.

By definition, for every (commutative) ring R and every element $a \in R$ there exists a unique homomorphism of rings

$$ev_a : \mathbb{Z}[x] \to R$$

such that $ev_a(x) = a$.

By construction, for every $p(x) \in \mathbb{Z}[x]$

$$ev_a(p(x)) = p(a)$$

The "polynomial" logos $\mathcal{S}[U]$

Let Fin be the category of finite spaces and let $[\mathrm{Fin}, \mathcal{S}]$ be the category of functors $\mathrm{Fin} \to \mathcal{S}$.

Let $U : \mathrm{Fin} \to \mathcal{S}$ be the forgetful functor. Then for every $K \in \mathrm{Fin}$ we have $U^K = Map(K, -) : \mathrm{Fin} \to \mathcal{S}$.

By Yoneda, every functor $F : \mathrm{Fin} \to \mathcal{S}$ is a colimit of representables:

$$F = \int^{K \in \mathrm{Fin}} F(K) \times U^K$$

The "polynomial" logos $\mathcal{S}[U]$

Theorem

The logos $[\mathrm{Fin}, \mathcal{S}]$ *is freely generated by the functor* $U : \mathrm{Fin} \to \mathcal{S}$. *Thus,* $\mathcal{S}[U] = [\mathrm{Fin}, \mathcal{S}]$.

Proof.

See [AL], [Joh]. We must show that for every object A in a logos $\mathcal{E}$ there exists a unique morphism of logoi $ev_A : [\mathrm{Fin}, \mathcal{S}] \to \mathcal{E}$ such that $ev_A(U) = A$.

We must have $ev_A(U^K) = A^K$, since the functor ev_A preserves finite limits. Thus, for every $F \in \mathcal{S}[U]$ we must have

$$ev_A(F) = ev_A \int^{K \in \mathrm{Fin}} F(K) \times U^K = \int^{K \in \mathrm{Fin}} F(K) \times A^K$$

since the functor ev_A preserves colimits.

The rest of the proof is left to the reader! $\square$

Remarks

In algebra, a polynomial $p(x) \in \mathbb{Z}[x]$ defines a polynomial function $p : R \to R$ on any commutative ring R.

Similarly, every "polynomial" $F \in \mathcal{S}[U]$ defines an endofunctor $F : \mathcal{E} \to \mathcal{E}$ of any logos $\mathcal{E}$.

By construction,

$$FA = \int^{K \in \mathrm{Fin}} F(K) \times A^K \tag{1}$$

for every object $A \in \mathcal{E}$.

For example, the functor $U : \mathrm{Fin} \to \mathcal{S}$ induces the *identity functor* $\mathcal{E} \to \mathcal{E}$ of any topos $\mathcal{E}$.

Warning

The Yoneda formula

$$F = \int^{K \in \mathrm{Fin}} F(K) \times U^K \qquad (2)$$

for a functor $F : \mathrm{Fin} \to \mathcal{S}$ shows that a "polynomial" in $\mathcal{S}[U]$ is a kind of "linear combination" of representables U^K.

However, the coefficient of U^K in formula (2) is the *value* of F at K, it is *not* $F^{(K)}(0)/K!$ as one may expect in a Taylor expansion.

In other words, the formula (2) is useful for *extrapolating* or *extending* a functor $F : \mathrm{Fin} \to \mathcal{S}$.

The category $\mathcal{S}[U] = [\mathrm{Fin}, \mathcal{S}]$ is equivalent to the category $[\mathcal{S}, \mathcal{S}]^{\mathrm{fin}}$ of *finitary* functors $\mathcal{S} \to \mathcal{S}$.

Parametrised spectra

If R is a commutative ring, let us put $R[\epsilon] := R[x]/(x^2)$. The *first derivative* $p'(a)$ of a polynomial $p(x) \in \mathbb{Z}[x]$ at $a \in R$ can be defined by the formula

$$p(a+\epsilon) = p(a) + p'(a)\epsilon$$

There are good reasons to believe that $\mathcal{S}[\epsilon]$ is the category of *parametrised spectra* $P\mathcal{S}p$. [ABFJ2]. By definition,

$$P\mathcal{S}p = \int^{B \in \mathcal{S}} \mathcal{S}p^B$$

where $\mathcal{S}p$ is the category of spectra. By construction, $\mathcal{S}p^B$ is the category of spectra in $\mathcal{S}^B = \mathcal{S}/B$.

The category $P\mathcal{S}p$ is a logos! (Biedermann, Rezk, 2007).

The base functor $\beta : P\mathcal{S}p \to \mathcal{S}$ is a morphism of logoi.

Klein-Rognes derivative

Every "polynomial" $F \in \mathcal{S}[U]$ induces a functor $F : P\mathcal{S}p \to P\mathcal{S}p$ and the following square commutes

$$\begin{array}{ccc} P\mathcal{S}p & \xrightarrow{F} & P\mathcal{S}p \\ \beta\downarrow & & \downarrow\beta \\ \mathcal{S} & \xrightarrow{F} & \mathcal{S} \end{array}$$

since the functor β is a morphism of logoi.

It follows that the functor $F : P\mathcal{S}p \to P\mathcal{S}p$ induces a functor

$$F'(B) : \mathcal{S}p^B \to \mathcal{S}p^{FB}$$

for every $B \in \mathcal{S}$. The functor $F'(B)$ is stable.

It defines a matrix of spectra $D(F)(B) \in \mathcal{S}p^{B \times FB}$ called the *Klein-Rognes derivative* of F. See [KR]

The KR-Chain Rule for the first derivative

The composite $G \circ F$ of two "polynomials" F and $G \in \mathcal{S}[U]$ is defined by putting $G \circ F = G(F)$.

If $\mathcal{E}$ is a logos, then the following triangle commutes.

$$\mathcal{E} \xrightarrow{\;F\;} \mathcal{E} \xrightarrow{\;G\;} \mathcal{E}, \qquad \mathcal{E} \xrightarrow{\;G \circ F\;} \mathcal{E}$$

Consider the case where $\mathcal{E} = \mathcal{S}[\epsilon] := P\mathcal{S}p$. The following diagram commutes for every $B \in \mathcal{S}$.

$$\mathcal{S}p^{B} \xrightarrow{\;F'(B)\;} \mathcal{S}p^{FB} \xrightarrow{\;G'(FB)\;} \mathcal{S}p^{GFB}, \qquad \mathcal{S}p^{B} \xrightarrow{\;(G \circ F)'(B)\;} \mathcal{S}p^{GFB}$$

This proves the KR Chain rule.

The chain rule for higher derivatives was proved by Arone and Ching $[\mathrm{AC}]$.

The Goodwillie tower of a functor

It was proved by [GIII] that the subcategory of n-excisive functors $[\mathsf{Fin}, \mathcal{S}]^{(n)} \subset [\mathsf{Fin}, \mathcal{S}]$ is reflective and the reflector

$$P_n : [\mathsf{Fin}, \mathcal{S}] \to [\mathsf{Fin}, \mathcal{S}]^{(n)}$$

is left exact.

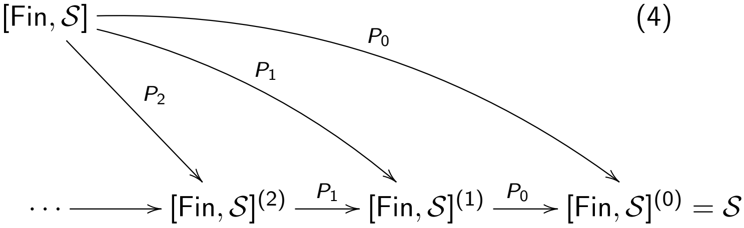
It follows that $[\mathsf{Fin}, \mathcal{S}]^{(n)}$ is a logos ! (Biedermann, Rezk).

The sequence of localizations (P_n) is decreasing.

To every $F \in [\mathsf{Fin}, \mathcal{S}]$ is associated a Goodwillie (Taylor) towers

$$\cdots \longrightarrow P_3 F \longrightarrow P_2 F \longrightarrow P_1 F \longrightarrow P_0 F$$

The category of formal towers



(4)

Let us put

$$[\mathrm{Fin}, \mathcal{S}]^{(\omega)} := \varprojlim_{n} [\mathrm{Fin}, \mathcal{S}]^{(n)}$$

An object of the category $[\mathrm{Fin}, \mathcal{S}]^{(\omega)}$ is a *formal (Goodwillie) tower*.

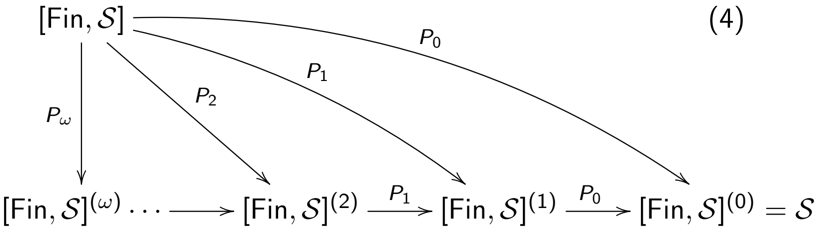
It is a decreasing sequence of functors $X_n \in [\mathrm{Fin}, \mathcal{S}]$, such that $P_n(X_{n+1}) = X_n$ for every $n \geq 0$.

Formal towers

The functor

$$P_\omega : [\mathrm{Fin}, \mathcal{S}] \to [\mathrm{Fin}, \mathcal{S}]^{(\omega)}$$

takes a functor F to its Goodwillie tower $P_\omega(F) := (P_n(F))$.



(4)

The functor P_ω has a right adjoint which takes a formal tower $(X_\star)$ to its limit $\varprojlim_n X_n$.

The category $[\mathrm{Fin}, \mathcal{S}]^{(\infty)}$

Lemma

A monomorphism $u : F \to G$ in $[\mathrm{Fin}, \mathcal{S}]$ is inverted by P_0 if and only it is inverted by P_ω.

If Λ is the class of monomorphisms inverted by P_0, let us put

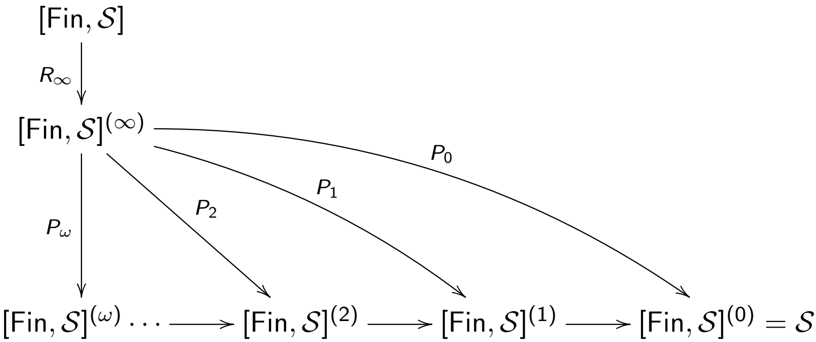
$$[\mathrm{Fin}, \mathcal{S}]^{(\infty)} := [\mathrm{Fin}, \mathcal{S}][\Lambda^{-1}]_{lex}$$

and let $R_\infty : [\mathrm{Fin}, \mathcal{S}] \to [\mathrm{Fin}, \mathcal{S}]^{(\infty)}$ be the localisation functor.

The functor P_ω factors through the localization R_∞.

$$\begin{array}{ccc} [\mathrm{Fin}, \mathcal{S}] & \xrightarrow{\quad P_\omega \quad} & [\mathrm{Fin}, \mathcal{S}]^{(\omega)} \\ & {\scriptstyle R_\infty} \searrow \qquad \nearrow {\scriptstyle P_\omega} & \\ & [\mathrm{Fin}, \mathcal{S}]^{(\infty)} & \end{array}$$

The category $[\mathsf{Fin}, \mathcal{S}]^{(\infty)}$



Is P_ω an equivalence ? (Probably not).

For the rest of the talk, we will study the category $[\mathsf{Fin}, \mathcal{S}]^{(\infty)}$.

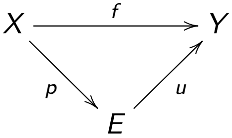
Surjective maps

We shall see that the logos $[\mathsf{Fin}, \mathcal{S}]^{(\infty)}$ is freely generated by an ∞-connected object. We need the notion of surjective map.

Let $\mathcal{E}$ be a logos.

A map $f : X \to Y$ is *surjective* if the base change functor $f^\star : \mathcal{E}_{/Y} \to \mathcal{E}_{/X}$ is conservative.

Every map $f : X \to Y$ admits a unique factorisation $f = up : X \to E \to Y$ with $p : X \to E$ a surjective map and $u : E \to Y$ a monomorphism [HTT].



The pair (E, u) is the *image* $Im(f)$ of the map f.

Connected objects

Let A be an object in a logos $\mathcal{E}$.

An object A is said to be (-1)-*connected* if the map $A \to 1$ is surjective.

An object A is said to be 0-*connected* if the maps $A \to 1$ and $A \to A \times A$ are surjective.

In general, an object A is said to be n-*connected* if the diagonal map $A \to A^{S^i}$ is surjective for every $-1 \leq i \leq n$.

An object A is said to be ∞-*connected* if it is n-connected for all integer n.

An ∞-connected space is contractible (Whitehead theorem).

An ∞-connected object in a logos may not be contractible. For example, in the logos of parametrised spectra $P\mathcal{S}p$, every spectrum is ∞-connected.

Crystallic functors

Let $\mathsf{C}_n \subset \mathsf{Fin}$ be the category of finite *n-connected* spaces.

Consider the decreasing chain of sub-categories

$$\mathsf{Fin} \supset \mathsf{C}_{-1} \supset \mathsf{C}_0 \supset \mathsf{C}_1 \supset \cdots$$

Definition

We say that a functor $F : \mathsf{Fin} \to \mathcal{S}$ is *crystallic* if it is the right Kan extension of its restriction $F|\mathsf{C}_n$ for **every** $n \geq -1$.

A crystallic functor $F : \mathsf{Fin} \to \mathcal{S}$ is determined by its values on the subcategory $\mathsf{C}_n \subset \mathsf{Fin}$ for *any* n.

Remarks on crystallic functors

Let us say that a functor F defined on $\mathsf{C}_n \subset \mathsf{Fin}$ is *crystallic* if it is the right Kan extension of its restriction $F|\mathsf{C}_m$ for **every** $m > n$.

If $[\mathsf{C}_n, \mathcal{S}]^{(\infty)}$ denotes the category of crystallic functors $\mathsf{C}_n \to \mathcal{S}$, then the restriction functor

$$[\mathsf{Fin}, \mathcal{S}]^{(\infty)} \to [\mathsf{C}_n, \mathcal{S}]^{(\infty)}$$

is an equivalence of categories for every $n \geq -1$.

$\mathcal{S}[U_\infty]$

We shall denote the logos freely generated by a ∞-connected object U_∞ by $\mathcal{S}[U_\infty]$.

The logos $\mathcal{S}[U_\infty]$ is a left exact localisation of the logos $\mathcal{S}[U]$. Hence the category $\mathcal{S}[U_\infty]$ is a reflective sub-category of the category $\mathcal{S}[U] = [\mathrm{Fin}, \mathcal{S}]$.

Theorem

A functor $F : \mathrm{Fin} \to \mathcal{S}$ belongs to $\mathcal{S}[U_\infty]$ if and only if it is crystallic. Moreover, $\mathcal{S}[U_\infty] = [\mathrm{Fin}, \mathcal{S}]^{(\infty)}$.

It follows that the limit $\varprojlim_n X_n$ of any formal Goodwillie tower $(X_\star)$ is crystallic.

In particular, the limit $P_\infty F$ of the Goodwillie tower $(P_n F)$ of any functor $F : \mathrm{Fin} \to \mathcal{S}$ is crystallic.

Analytic functors are crystallic

Theorem

Every n-analytic functor $\mathsf{C}_n \to \mathcal{S}$ is crystallic. In particular, the functor $U_1 : \mathsf{C}_1 \to \mathcal{S}$ is crystallic.

Proof.

The limit $P_\infty F$ of the Goodwillie tower $(P_n F)$ of any functor $F : \mathsf{Fin} \to \mathcal{S}$ is crystallic. But if F is n-analytic, then $P_\infty F(X) = F(X)$ for $X \in \mathsf{C}_n$ by [GII]. Hence the functor $F : \mathsf{C}_n \to \mathcal{S}$ is crystallic. In particular, the functor $U_1 : \mathsf{C}_1 \to \mathcal{S}$ is cristallic, since U_1 is 1-analytic by [AK]. □

It follows that the functor $U_\infty : \mathsf{Fin} \to \mathcal{S}$ is the right Kan extension of the functor $U_1 : \mathsf{C}_1 \to \mathcal{S}$ along the inclusion $\mathsf{C}_1 \subset \mathsf{Fin}$.

Cech groupoid of a map

Recall that the *Cech groupoid* of a map $u : A \to B$ in a logos is the simplicial object $C_\bullet(u)$ defined by putting

$$C_n(u) = A \times_B A \times_B \cdots \times_B A$$

(factors indexed by $i \in [n]$) for every $n \geq 0$.

The simplicial object $C_\bullet(u)$ is naturally augmented by the map $u : A \to B$.

$$B \xleftarrow{u} A \leftleftarrows A \times_B A \,\overset{\leftarrow}{\leftleftarrows}\, A \times_B A \times_B A \cdots$$

Recall that the realisation $|X_\bullet|$ of a simplicial object $X_\bullet$ in a logos is defined to be its colimit. From the augmentation $C_\bullet(u) \to B$ we obtain a factorisation $A \to |C_\bullet(u)| \to B$ of the map $u : A \to B$.

The map $A \to |C_\bullet(u)|$ is surjective and the map $|C_\bullet(u)| \to B$ is a monomorphism.

Cech cogroupoid of a map

The *Cech cogroupoid* of a map $u : A \to B$ in a logos is the cosimplicial object $C^\bullet(u)$ defined by putting

$$C^n(u) = B \sqcup_A B \sqcup_A \cdots \sqcup_A B$$

(factors indexed by $i \in [n]$) for every $n \geq 0$.

The cosimplicial object $C^\bullet(u)$ is coaugmented by the map $u : A \to B = C^0(u)$.

$$A \xrightarrow{u} B \rightrightarrows B \sqcup_A B \Rrightarrow B \sqcup_A B \sqcup_A B \cdots$$

Recall that the homotopy limit of a cosimplicial object $X^\bullet$ is said to be its *totalisation* and is denoted $Tot(X^\bullet)$.

n-excisive functors revisited

The *partial n-totalisation* $Tot_n(X^\bullet)$ of a cosimplicial object $X^\bullet$ is defined to be its limit over the subcategory of Δ spanned by the posets $[k]$ with $k \leq n$.

Recall that a functor $F : \mathrm{Fin} \to \mathcal{S}$ is said to be *n-excisive* if it takes every strongly cocartesian $(n+1)$-cube to a cartesian cube.

Lemma

A functor $F : \mathrm{Fin} \to \mathcal{S}$ is n-excisive if and only the canonical map

$$FA \to Tot_n(C^\bullet(u))$$

is an isomorphism for every map $u : A \to B$ in Fin.

Proof.

By [Sin], the partial totalisation $Tot_n(X^\bullet)$ is the limit of the cubical diagram $S \mapsto X^{|S|}$ indexed by non-empty subsets $S \subseteq [n]$. □

∞-excisive functors

Definition

We say that a functor $F : \mathrm{Fin} \to \mathcal{S}$ is a ∞-*excisive* if the canonical map

$$FA \to Tot(C^{\bullet}(u))$$

is an isomorphism for every map $u : A \to B$ in Fin.

Theorem

A functor $F : \mathrm{Fin} \to \mathcal{S}$ is ∞-excisive if and only if it is crystallic.

Sketch of proof: For every map $u : A \to B$ in Fin, let us denote the image of the map $U^u : U^B \to U^A$ by $J(u) \subseteq U^A$. By definition, a functor $F : \mathrm{Fin} \to \mathcal{S}$ is ∞-excisive if an only if it is local with respect to the set Σ of inclusions $J(u) \subseteq U^A$. On the other hand F is crystallic if and only of it is local with respect to the class Λ of maps inverted by the functor P_0. It suffices to show that Σ and Λ generates the same lex-localisation....

Thank you for your attention!

Bibliography (1)

- [AK] G. Arone and M. Kankaanrinta, *A functorial model for iterated Snaith Splitting with applications to calculus of functors* (1997) .
- [AL] M. Anel and D. Lejay, *Exponentiable ∞-topoi*.
- [ABFJ1] M. Anel, G. Biedermann, E. Finster and A. Joyal, *Higher sheaves and left exact localisation of ∞-topoi*. (2020)
- [ABFJ2] M. Anel, G. Biedermann, E. Finster and A. Joyal, *Nilpotence towers for Goodwillie calculus*. (2021?).
- [AJ] M. Anel and A. Joyal, *Topo-logie*, New Spaces in Mathematics. (2021)
- [AC] G. Arone and M. Ching, *Operads and Chain rules for the calculus of functors*. (2009)
- [BBC] K. Bauer, M. Burke and M. Ching, *Tangent ∞-categories and Goodwillie calculus*. (2021)
- [Ch] M. Ching, *Dual tangent structures for ∞-toposes*. (2021)

Bibliography (2)

- [El] R. Eldred, *Cosimplicial models for the limit of the Goodwillie tower*. (2013)
- [GII] T.G. Goodwillie, *Calculus II: Analytic functors*. (1991/1992)
- [GIII] T.G. Goodwillie, *Calculus III: Taylor Series*. (2002/2003)
- [Joh] P. T. Johnstone, *Sketches of an elephant*. (2002)
- [KR] J. Klein and J. Rognes, *A chain rule in the calculus of homotopy functors*. (2003)
- [HTT] J. Lurie, *Higher topos theory*. (2009)
- [HA] J. Lurie, *Higher algebra*. (2017)
- [Re] C. Rezk, *A streamline proof of Goodwillie's n-excisive approximation*. (2013)
- [Sin] D. Sinha, *The topology of spaces of knots: cosimplicial models*. (2009).