Exponential modalities and complimentarity

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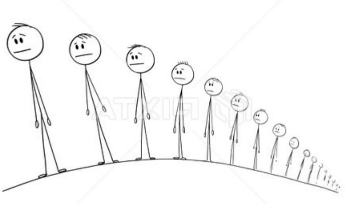
!**A** refers to an infinite supply of the resource A

?**A** represents the notion of infinite demand.



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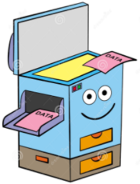

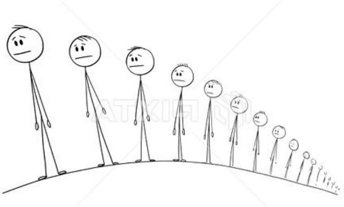

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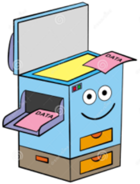

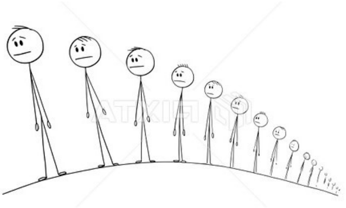

$!A$ can be duplicated and destroyed.

! is used a de facto structure to model arbitrary dimensional spaces such as Bosonic Fock spaces in Physics.

Complimentarity in quantum mechanics

A **quantum observable** refers to a measurable property of quantum system.

A pair of quantum obsevables are **complimentary** if measuring one observable increases uncertanity regarding the value of the other.

Example: position and momentum of an electron

Question

Is there a connection between exponential modalities of linear logic and complimentary observables of quantum mechanics?

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YES!!!

Mathematical framework

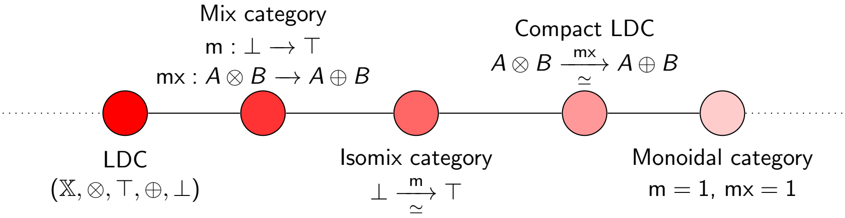
Categorical semantics of linear logic

Linearly distributive categories (LDC)[1]:

$$(\mathbb{X}, \otimes, \top, a_\otimes, u^L_\otimes, u^R_\otimes) \qquad (\mathbb{X}, \oplus, \bot, a_\oplus, u^L_\oplus, u^R_\oplus)$$

linked by linear distributors: $\partial_L : A \otimes (B \oplus C) \to (A \otimes B) \oplus C$

Monoidal categories: LDCs in which $\otimes = \oplus$



[1] Robin Cockett and Robert Seely (1997). Weakly distributive categories.

Categorical semantics of ! and ?

In a $(!, ?)$-LDC[2]

- ! is a monoidal coalgebra comodality
  - $(!, \delta : ! \Rightarrow !!, \varepsilon : ! \Rightarrow \mathbb{I})$ is a monoidal comonad
  - For each A, $(!A, \Delta_A, e_A)$ is a cocommutative comonoid

- ! is a comonoidal algebra modality
  - $(?, \mu : ?? \Rightarrow ?, \eta : \mathbb{I} \Rightarrow ?)$ is a comonoidal monad
  - For each A, $(?A, \nabla_A, u_A)$ is a commutative monoid

- $(!, ?)$ is a linear functor

- The pairs (δ, μ), (ε, η), (Δ, ∇) are linear transformations

Examples: Category of finiteness relations, category of finiteness matrices over a commutative rig

[2] Richard Blute, Robin Cockett, and Robert Seely (1996). "! and ? - Storage as tensorial strength."

Compact †-linear logic: †-monoidal categories

†-monoidal categories: Monoidal categories with $\dagger : \mathbb{X}^{op} \longrightarrow \mathbb{X}$ such that

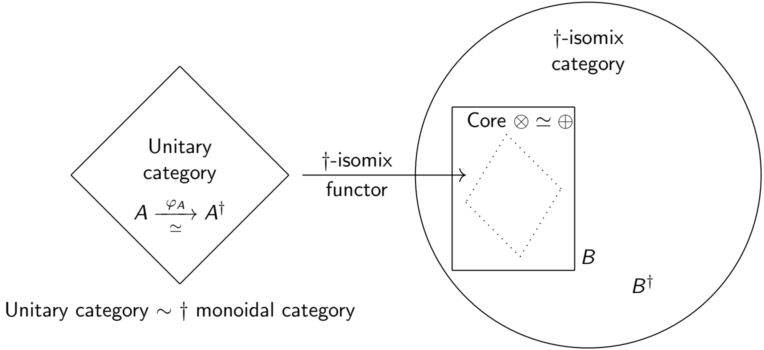
- $A^\dagger = A$
- $f^{\dagger\dagger} = f$
- $(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$
- All basic natural isomorphisms are unitary (i.e., $a_\otimes^\dagger = a_\otimes^{-1}$)

Examples: Category of Hilbert spaces and linear maps, category of sets and relations

Non-compact †-linear logic ...

Non-compact †-linear logic: Mixed unitary categories

Mixed Unitary Category:



Examples:

- Complex finite dimensional matrices embedded into finiteness matrices
- Finite relations embedded into finiteness relations
- A'canonical' MUC can be constructed from any †-isomix category: the category of pre-unitary objects embed into the †-isomix category

! and ? in †-linear logic

In a $(!, ?)$-**dagger**-LDC

- ! is a monoidal coalgebra comodality
  - $(!, \delta : ! \Rightarrow !!, \varepsilon : ! \Rightarrow \mathbb{I})$ is a comonad
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A very rough plan

Step 1: Formulate measurements in MUCs

Step 2: Formulate complimentary systems in MUCs

Step 3: Prove the connection between exponential modalities and complimentary observables

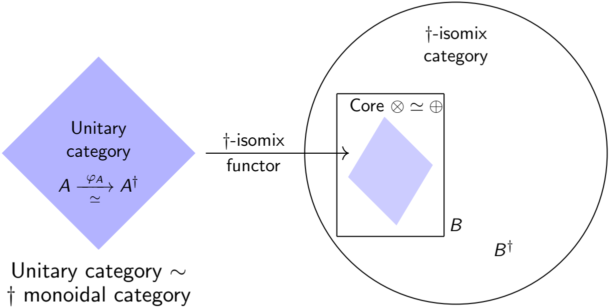
Step 1: Mesurement in MUCs

Demolition measurement

In a †-monoidal category, a **demolition measurement**[3] on an object A is retract from A to a special commutative †-Frobenius algebra, E.

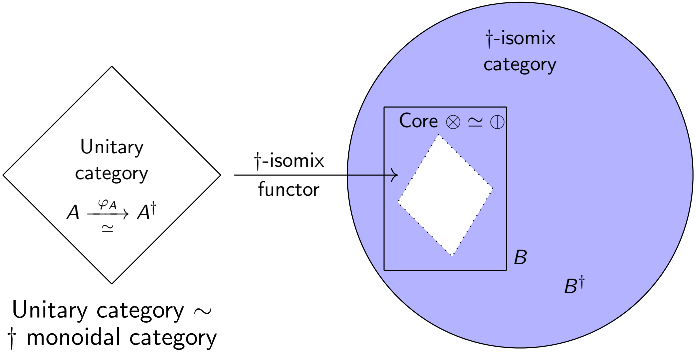
$$A \underset{r^\dagger}{\overset{r}{\rightleftarrows}} E \quad \text{such that } r^\dagger r = 1_E$$

E represents a quantum observable.

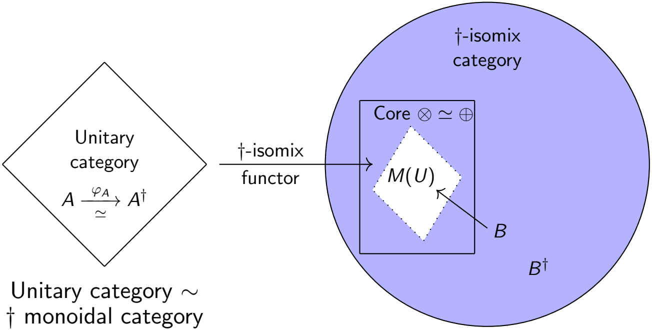


[3] Bob Coecke and Dusko Pavlovic (2006). "Quantum measurements without sums"

Compaction


Unitary
category
$A \xrightarrow[\simeq]{\varphi_A} A^\dagger$
†-isomix
functor
†-isomix
category
Core ⊗ ≃ ⊕
B
$B^\dagger$
Unitary category ∼
† monoidal category

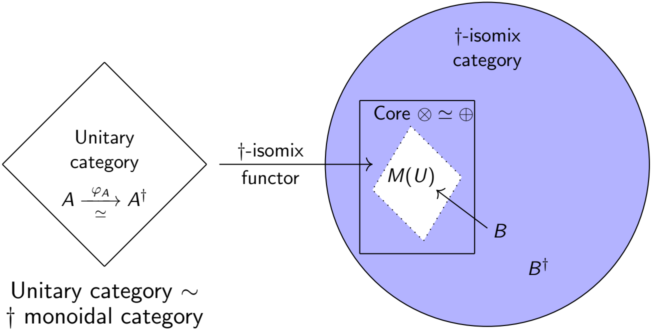
Compaction



$r : B \longrightarrow M(U)$

$s : M(U) \longrightarrow B$

$sr = 1_{M(U)}$



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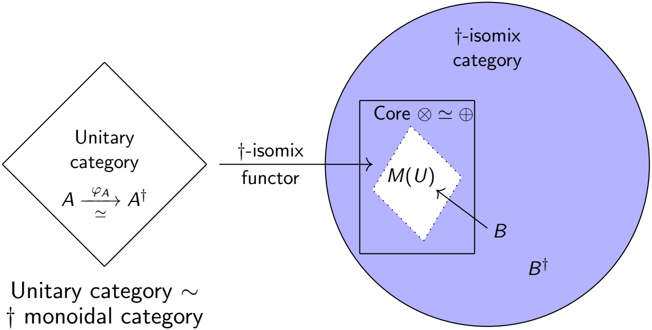
A **Compaction** in a MUC, $M : \mathbb{U} \longrightarrow \mathbb{C}$, is a retraction to an object in the unitary core $r : B \longrightarrow M(U)$.

Compaction



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MUC measurement = Compaction and Demolition measurement

Binary idempotents

$$\begin{array}{ccc} A & & A^{\dagger} \\ \downarrow{\scriptstyle r} & & \uparrow{\scriptstyle r^{\dagger}} \\ M(U)^{\dagger} & \xrightarrow[\alpha]{\simeq} & M(U)^{\dagger} \end{array} \qquad \begin{array}{ccc} A & & A^{\dagger} \\ \uparrow{\scriptstyle s} & & \downarrow{\scriptstyle s^{\dagger}} \\ M(U)^{\dagger} & \xrightarrow[\alpha]{\simeq} & M(U)^{\dagger} \end{array}$$

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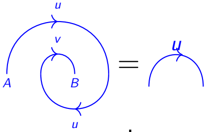
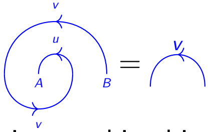
Binary idempotent (any category): $A \underset{v}{\overset{u}{\rightleftarrows}} B$ such that:

$uvu = u$





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splitting $e_A := uv$ and $e_B := vu$ gives isomorphic objects

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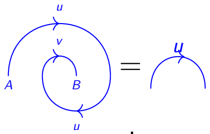

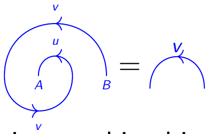

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$\dagger$-**binary idempotent**: **($\dagger$-LDC)** $A \underset{v}{\overset{u}{\rightleftarrows}} A^\dagger$ such that $iu^\dagger = u \quad vi = v^\dagger$

Observation: $(e_A)^\dagger = v^\dagger u^\dagger = = viu^\dagger = vu = e_{A^\dagger}$

Compaction = splitting coring †-binary idempotents

Theorem:

In a †-isomix category, U is the canonical compaction of an object A,

$$\Updownarrow$$

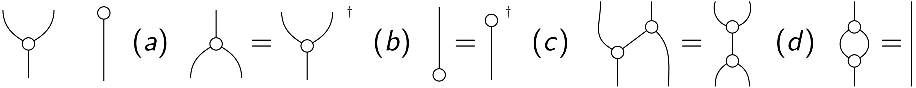
U is given by splitting a coring* †-binary idempotent on A.

* coring if and only if the idempotent split through the core

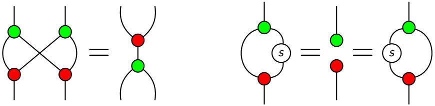
Step 2: Formulate complimentary systems in MUCs

Complimentary systems in †-monoidal categories

In a †-monoidal category, a quantum observable is given by a special commutative †-Frobenius algebra $(A, ,)$ also known as a **classical structure**[4].



Two †-Frobenius algebras $(A, ,)$, $(A, ,)$, on an object are complimentary[5] if they interact to produce two Hopf algebras.



[4] Bob Coecke, Dusko Pavlovic and Jamie Vlcary (2013). "A new description of orthonormal basis"

[5] Bob Coecke and Ross Duncan (2008). "Interacting quantum observables"

Linear monoids

Linear monoids generalize Frobenius algebras to LDCs.

In a symmetric LDC, **a linear monoid**, $A \xrightarrow{\circ}\!\!\!\!\dashv\!\!\vdash B$, contains a:

- a monoid $(A, \curlyvee : A \otimes A \longrightarrow A,\ \circ\!\!\!\mid : \top \longrightarrow A)$

- a dual for A, $(\eta, \varepsilon) : A \dashv\!\!\vdash B$

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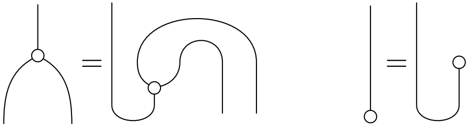
together producing a comonoid $(B, \curlywedge : B \rightarrow B \oplus B, \,\downarrow\, : B \rightarrow \bot)$



A **self** linear monoid is a linear monoid, $A \overset{\circ}{\multimap\!\!\shortmid} B$, with $A \simeq B$

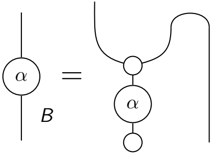
Linear monoids generalize Frobenius algebras

An object which is a Frobenius algebra is always a self-dual whereas a linear monoid has a monoid and a comonoid on distinct dual objects

A morphism of a Frobenius algebra is an isomorphism where as a morphism of linear monoid is more general.

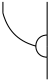
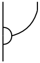
Proposition:

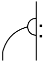
In a monoidal category, a Frobenius algebra is precisely a self linear monoid $A \overset{\circ}{-\!\!\!\dashv\!\mid} B$, ($\alpha : A \xrightarrow{\alpha} B$) satisfying the equation:



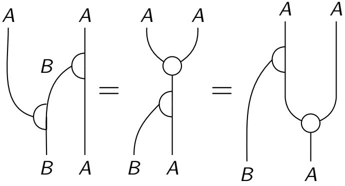
Alternate characterization of linear monoids

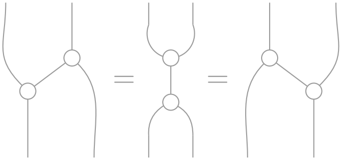
A linear monoid, A  B, consists of a $\otimes$-monoid, $(A,$ , $)$, and a $\oplus$-comonoid, $(B,$ , $)$ and:

- monoid actions:  : $A \otimes B \to B$;  : $B \otimes A \to A$

- comonoid coactions: : $B \to A \oplus B$;  : $B \to A \oplus B$

satisfying certain equations. The Frobenius equation is given as follows:





Linear bialgebras

Linear monoid

a $\underline{\otimes\text{-monoid}}$ and a dual:

$(A,\ \curlyvee : A \otimes A \longrightarrow A,\ ⫯ : \top \longrightarrow A)$

$(\eta, \varepsilon) : A \dashv\vdash B$

Linear comonoid

a $\underline{\otimes\text{-comonoid}}$ and a dual:

$(A,\ \curlywedge : A \longrightarrow A \otimes A,\ ⫰ : A \longrightarrow \bot)$

$(\eta, \varepsilon) : A \dashv\vdash B$

Linear bialgebras

- a linear monoid $(A, \curlyvee, ⫯)$; $(\eta, \varepsilon) : A \dashv\vdash B$

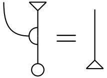
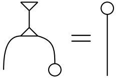
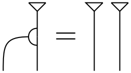
- a linear comonoid $(A, \curlywedge, ⫰)$; $(\eta', \varepsilon') : A \dashv\vdash B$

such that $(A, \curlyvee, ⫯, \curlywedge, ⫰)$ is a $\otimes$-bialgebra; $(B, \curlyvee, ⫯, \curlywedge, ⫰)$ is a $\oplus$-bialgebra

A **self-linear bialgebra** is a linear bialgebra where $A \simeq B$

Complimentary systems

A **complimentary system** in an isomix category a self-linear bialagebra, A, such that:

[comp.1]  **[comp.2]**  **[comp.3]** 

Lemma: If A is a complimentary system, then A is a $\otimes$-Hopf and $\oplus$-Hopf.

Step 3: Proving the connection between exponential modalities and complimentary systems

Main result

Theorem:

In a $(!, ?)$-isomix category with free exponential modalities, every complimentary system arises as a splitting of a binary idempotent on the linear bialgebra induced on the exponential modalities.

(The proof uses a series of results)

The structures and results discussed extend directly to †-linear bilagebras in †-LDCs with free exponential modalities due to the †-linearity of $(!, ?)$, (η, ε), (Δ, ∇), and (⫰, ⫯).

Future work

Examples in physics...

Acknowledgement:
Thank you JS for many useful discussions on the exponential modalities and examples!

Article available in arXiV

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