An area law for 2D frustration free spin systems

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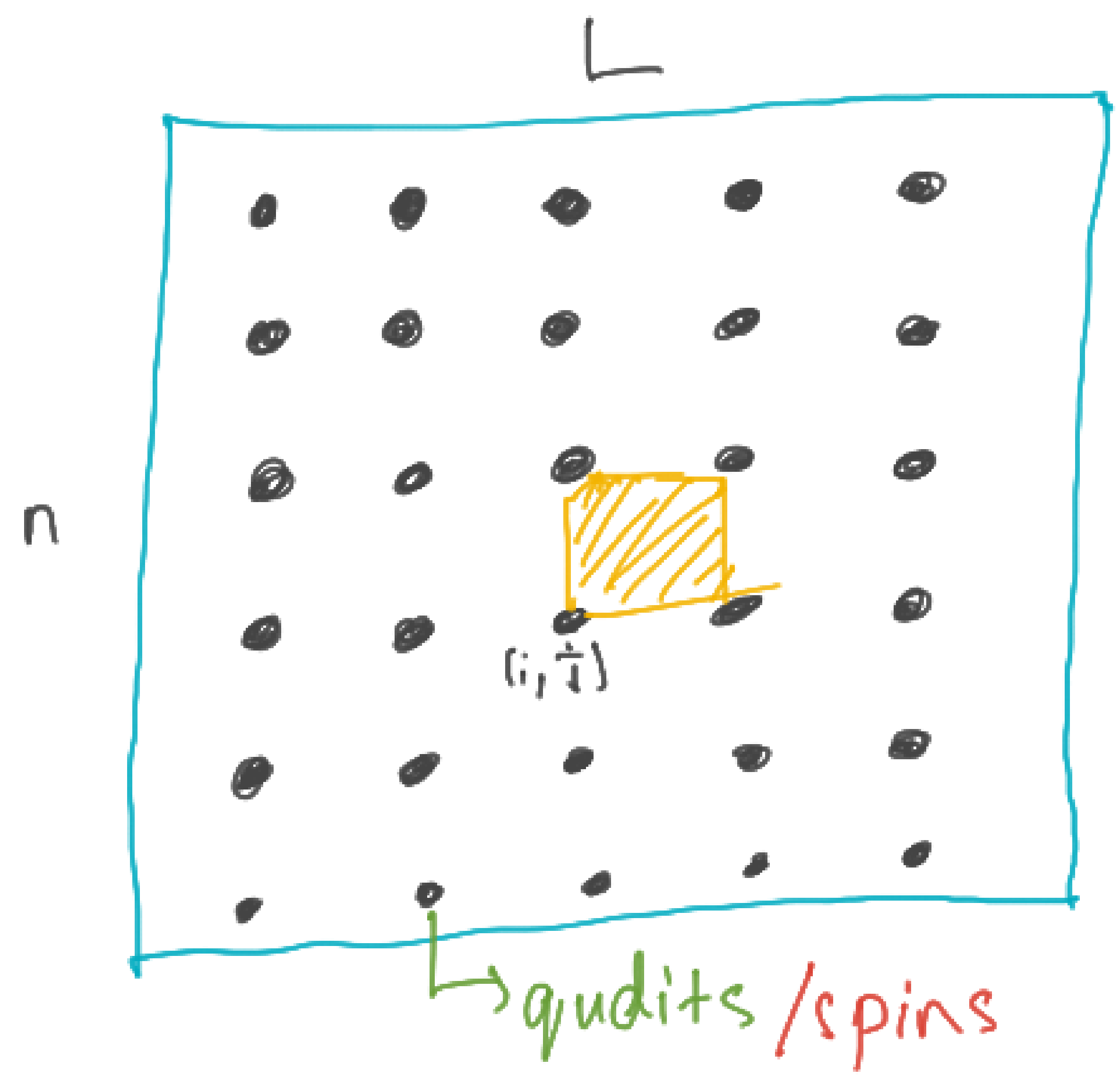
2D spin system



$$H = \sum_{i,j} h_{ij} \otimes \mathbb{1}_{\text{everything except } (i,j)}$$

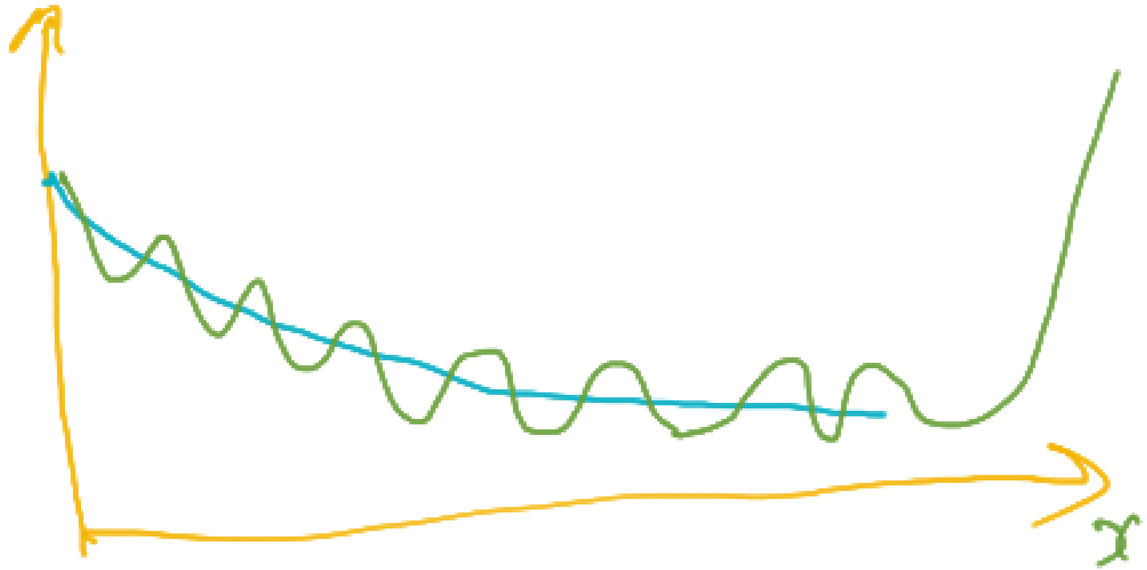
WLOG: $0 \leq h_{ij} \leq \mathbb{1}$

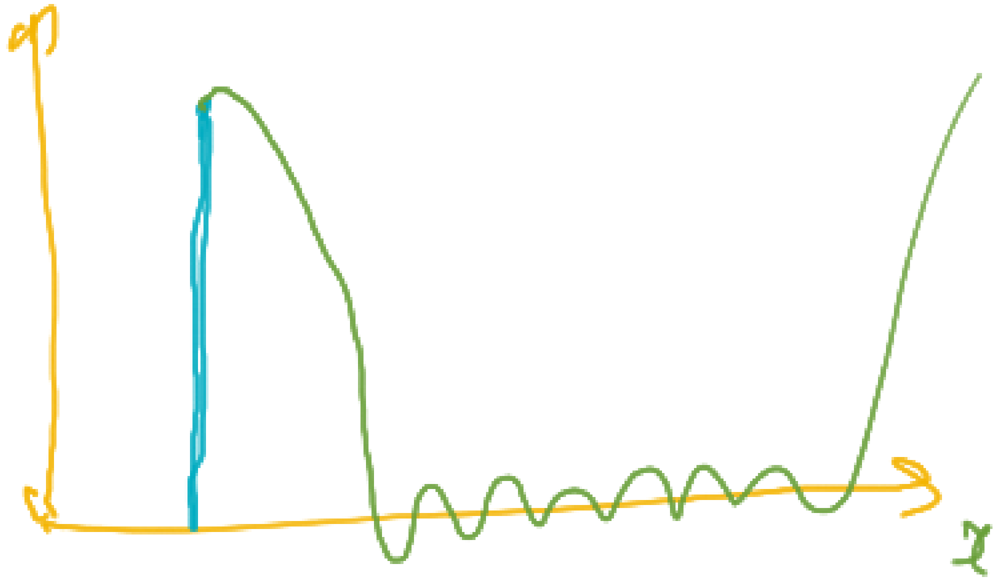
Ground states: $|\Omega\rangle$

Spectral gap: excited - ground

Physics questions: Correlation, Entanglement, Tensor network.

Polynomial approximations (in a different world)





- Given $F(x)$, find $P(x)$: $|F(x) - p(x)| \leq \epsilon$
- Chebyshev approximation (usually optimal in degree)

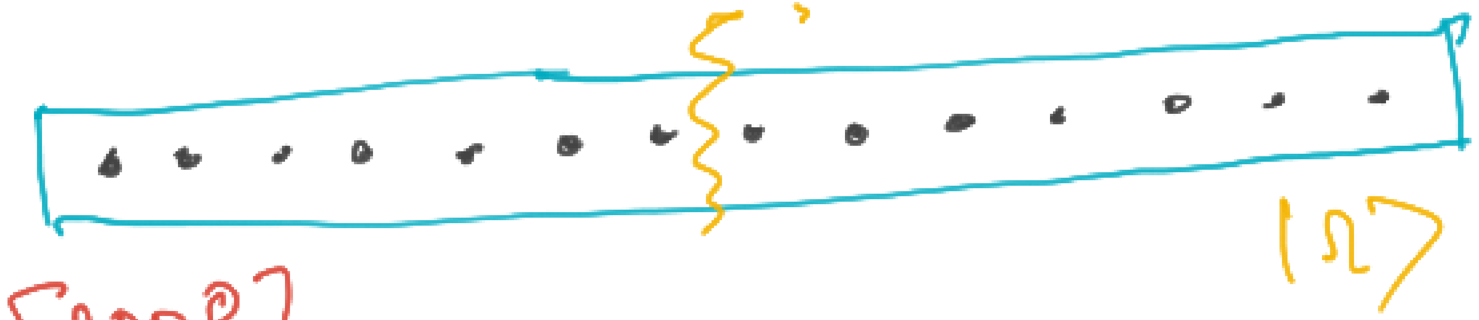
Polynomial approximations $\longleftrightarrow$ spin systems

- Power of Chebyshev approximation
- Correlation length in FF ($h_{ij}|\Omega\rangle = 0$)

  $1/\text{gap}$ $\rightarrow$ Hastings [2004], Nachtergaele-Sims [2006]

  $1/\sqrt{\text{gap}}$ $\rightarrow$ Gosset-Huang [2016]

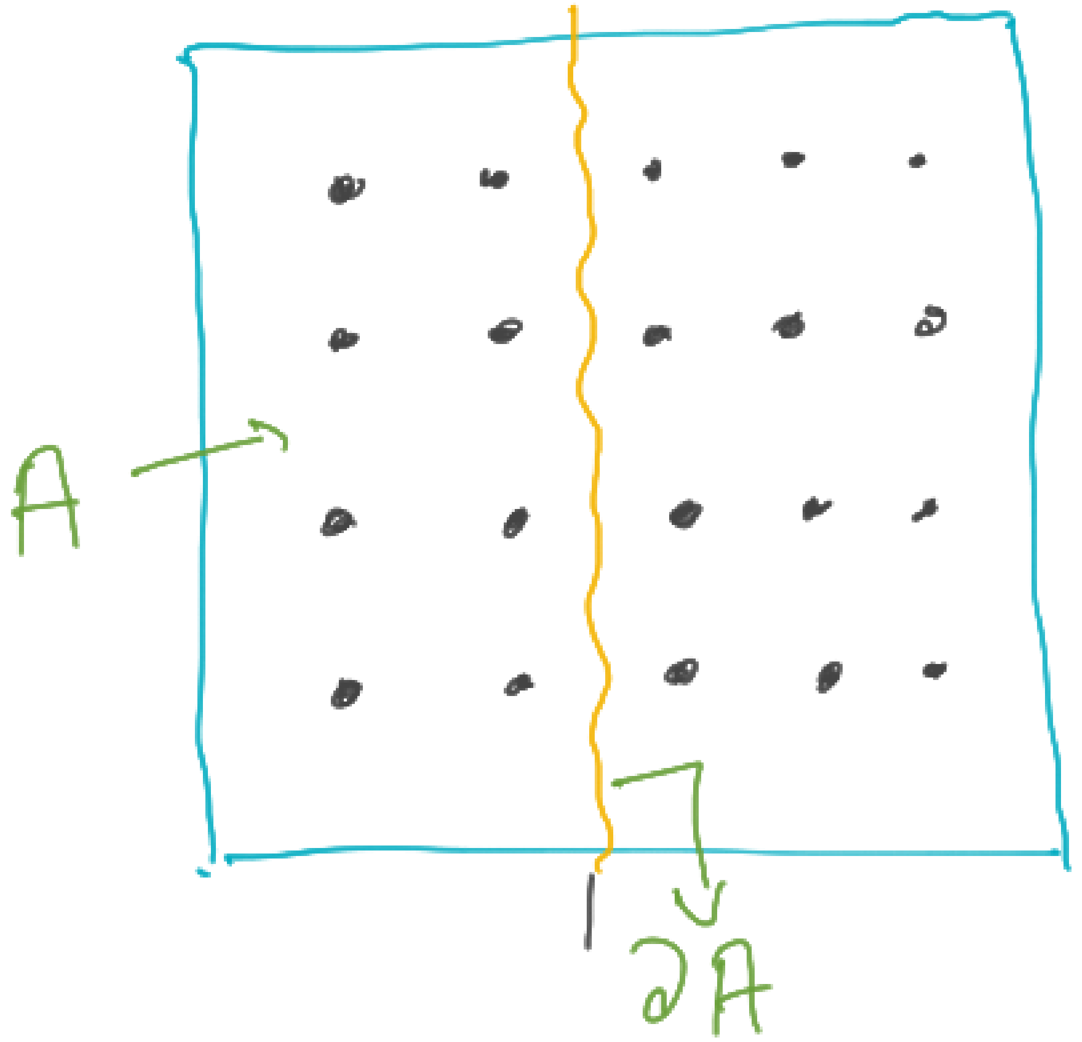
- Area laws in 1D



  $e^{1/\text{gap}}$ $\rightarrow$ Hastings [2008]

  $1/\text{gap}$ $\rightarrow$ Arad, Kitaev, Landau, Vazirani [2013]

Entanglement entropy in 2D



Given $|\psi\rangle$, how does

$$S(\psi_A) = \mathrm{Tr}\left(\log \frac{1}{\psi_A} \cdot \psi_A\right)$$

scale?

Random quantum states:

$$S(\psi_A) \sim |A| \quad (\text{Volume})$$

Unique gapped ground states:

Area law conjecture: $S(\psi_A) \sim |\partial A|$

Main result:

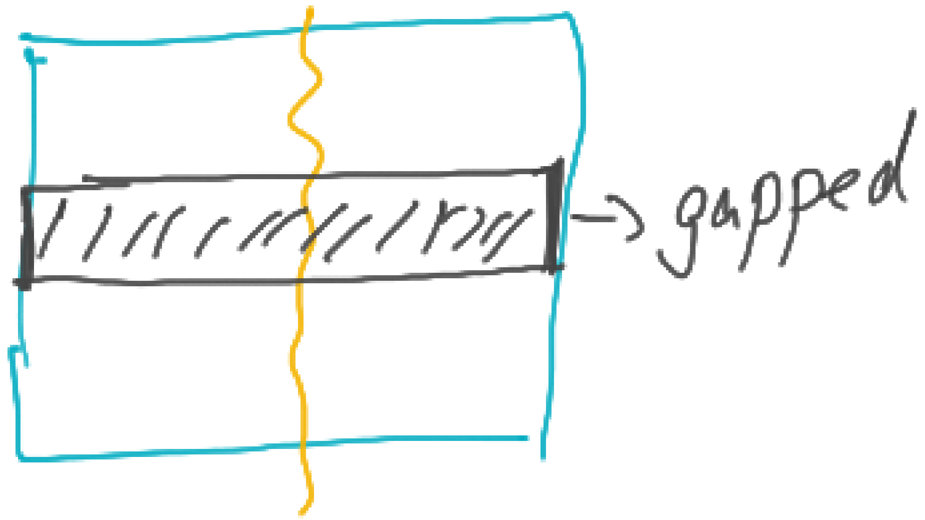
Unique g.s. $|\Omega\rangle$ of locally gapped FF hamiltonians in 2D satisfies across vertical cut:

$$S(\Omega_A) \leq |\partial A|^{1+\frac{\text{const.}}{\sqrt{\log|\partial A|}}}$$

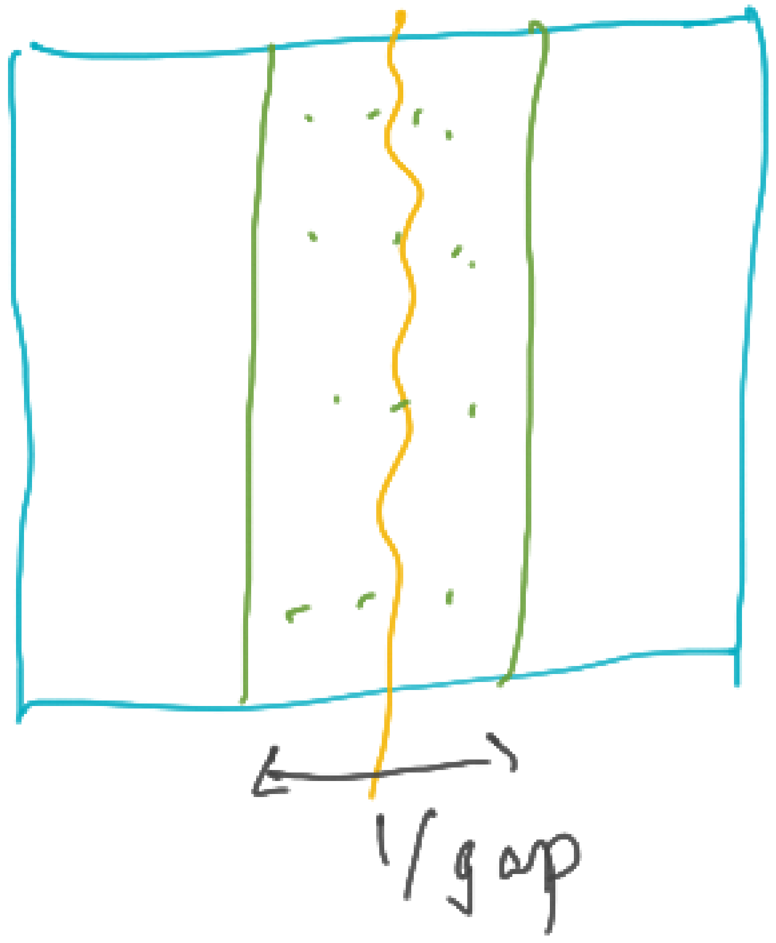
$2^{\sqrt{\log|\partial A|}}$

→ "Unique" not necessary.

+ log (dimension of ground space)



Outline of proof



Correlation of A ~ green region

$S(A) \sim |\text{green}| \sim |\partial A|$

→ Formally established in 1D
(Brandão – Horodecki [2013])

→ High level view: optimal poly-approximation "along ∂A"
⇒ area law for A

Optimal poly approx in 1D → "Lift" to 2D area law

Prior work in 2D

Area law under:

→ Sub exponential # of low energy eigenstates
Hastings [2007], Masanes [2009]

→ Spin $\frac{1}{2}$ lattice with NN interaction
Beaudrap, Osborne, Eisert [2010]

→ Adiabatic assumption Cho [2014]

→ Specific heat assumption Brandao, Cramer [2015]

False in high dimensions: Aharonov et. al. [2014].

How to bound entanglement

→ Schmidt rank of K ($SR[K]$):

$$K = \sum_{i=1}^{D} K_A^i \otimes L_{A^c}^i$$

→ Find K of small $SR[K]$ that approximates $|\Omega\rangle$

→ Approximation:

- ℓ_1: $\| K - |\Omega\rangle\langle\Omega| \|_1 \leq \epsilon$ (very hard)
- ℓ_∞: $\| K - |\Omega\rangle\langle\Omega| \|_\infty \leq \epsilon$ ✓

→ Example: $\left(1 - \frac{H}{nL}\right)^{\text{large power}}$

How to bound entanglement

→ Approximate ground state projector (AGSP): K with

$$\|K - |\Omega\rangle\langle\Omega|\|_\infty \le \epsilon \text{ and } SR[K] \le D$$

→ Would $\epsilon = \frac{1}{10}$ and $D = 100$ suffice?

→ No! [Aharonov et. al.]: $|EPR\rangle$

→ Theorem (Arad, Landau, Vazirani 2012):

If $D\epsilon < \frac{1}{2}$ then $S(\Omega_A) \le 2 \log D$

→ Intuition:

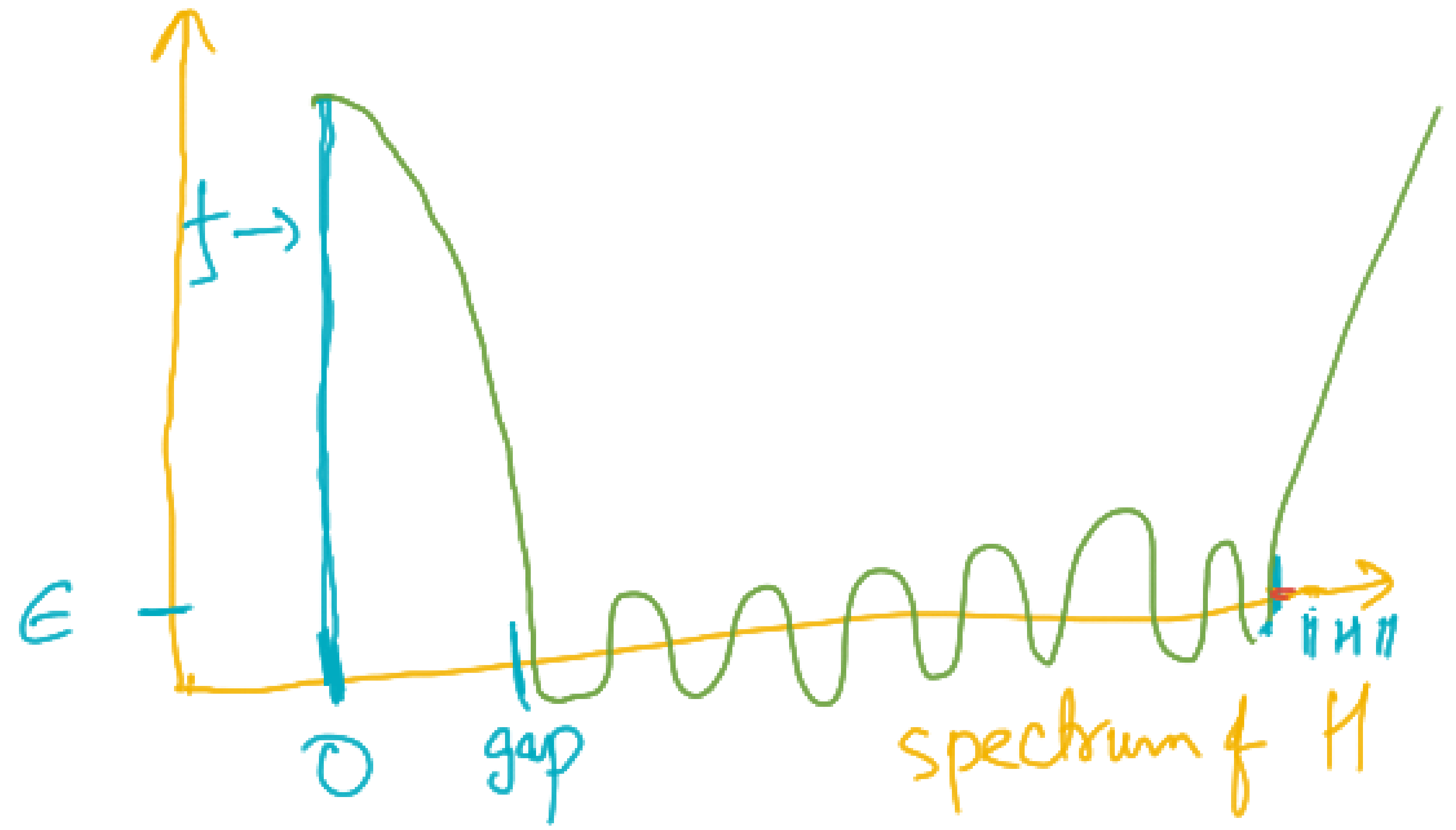
$\epsilon = 0$ $K = |\Omega\rangle\langle\Omega|$, $SR[K]$

Polynomial approximation to ground state

→ $\left(1 - \frac{H}{nL}\right)^{\text{large power}}$ is a polynomial.

→ Small degree ⇒ low SR. $\left(\sim e^{\text{degree}} \right)$

→ Chebyshev polynomials are optimal.



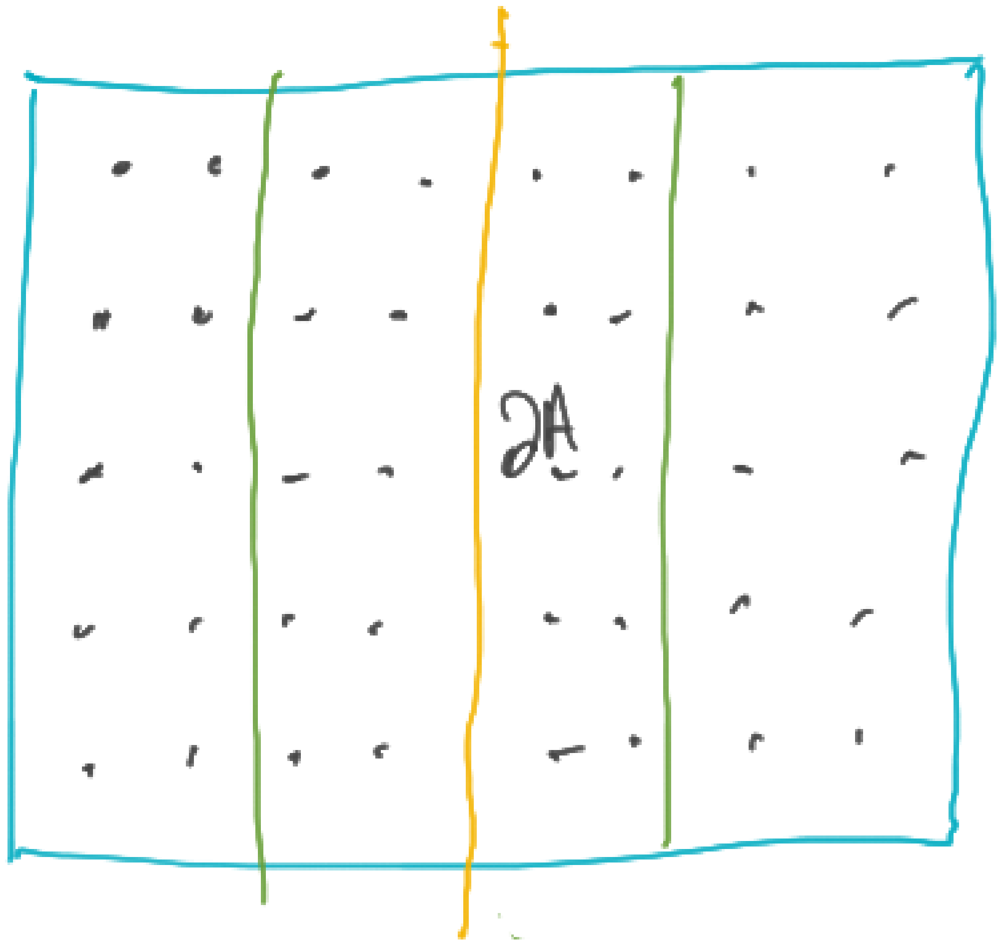
$f(H) = |\Omega\rangle\langle\Omega|$

$g(H)$ = polynomial

ϵ

degree: $\sqrt{\|H\|} \cdot \log\frac{1}{\epsilon}$

Trouble with polynomial approximation



- AGSP: K of degree d and error ϵ $\quad SR \sim e^{d}$
- Chebyshev achieves ϵ with

$$d \approx \sqrt{|\partial A|} \log \frac{1}{\epsilon}$$

↳ too high

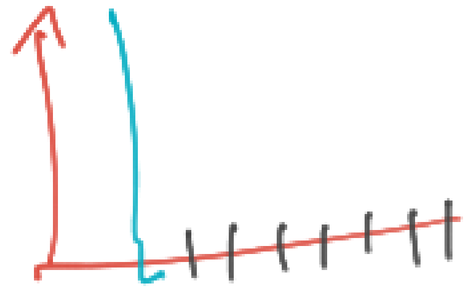
$$SR[K] \approx e^{d} \approx \left(\frac{1}{\epsilon}\right)^{\sqrt{|\partial A|}}$$

$\boxed{\text{Bottleneck: Chebyshev is optimal}}$

Observation : the commuting case

- $[h_{ij}, h_{kl}] = 0$

- Area law holds



- "Super-chebyshev" approximation (Optimal)

  - degree $\sqrt{|\partial A| \log \frac{1}{\epsilon}}$ for error ϵ

  (Kahn, Linial, Samorodintzky 1995), (Buhrman, de Wolf, Cleve. Zalka 1999)

- For $\epsilon = 2^{-\Omega(|\partial A|)}$, degree : $|\partial A|$, SR : $e^{|\partial A|}$

Main technical result

- Optimal polynomial approximation in 1D locally gapped FF hamiltonians

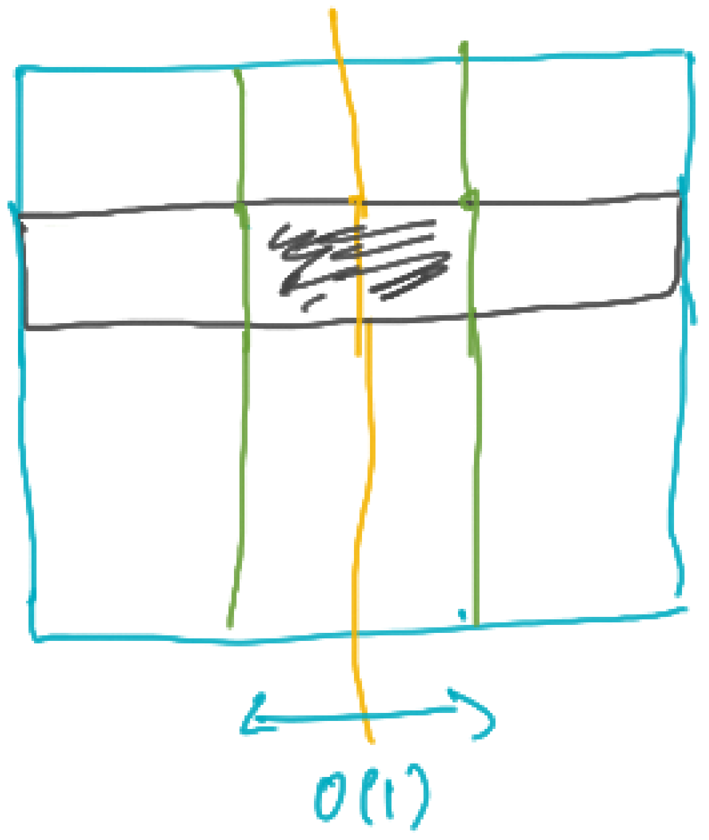
Thm: Given Π_{gr} as the ground space of 1D FF hamiltonian $H = \sum_{i=1}^{n} h_i$ with local gap $= O(1)$,
$\exists\ P_d[h_1, h_2, \dots]$ such that

$$\| P_d - \Pi_{gr} \|_\infty \leq \exp(-d^2/n)$$

$\boxed{\text{Chebyshev bottleneck?: Multivariate polynomials}}$

Lift to 2D

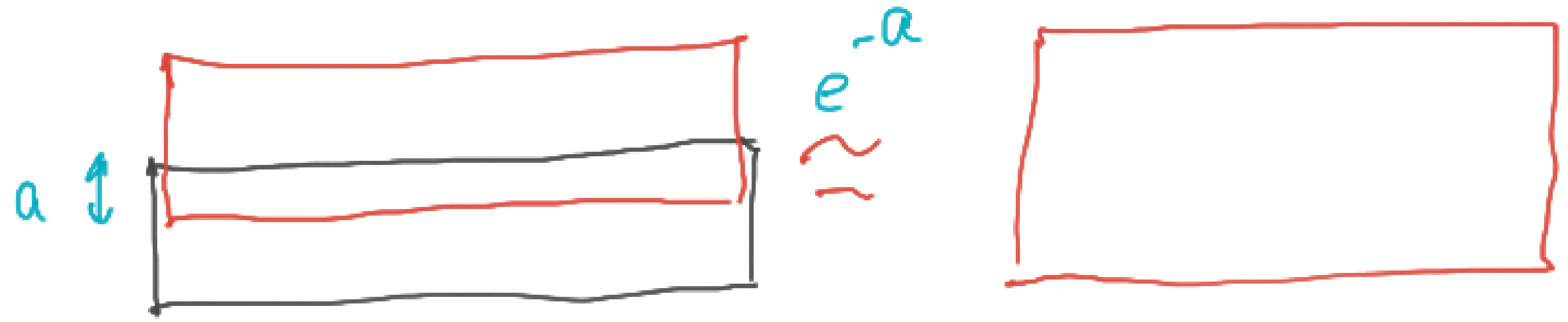
Realize the 1D polynomial around ∂A



$$H' = \sum_{i=1}^{|\partial A|} h'_i$$

h'_i = truncated version of $\left(\sum_J h_{iJ}\right)$

(Truncation introduced in Arad, Kitaev, Landau, Vazirani [2013])



(Merge)

Ensured by local gap

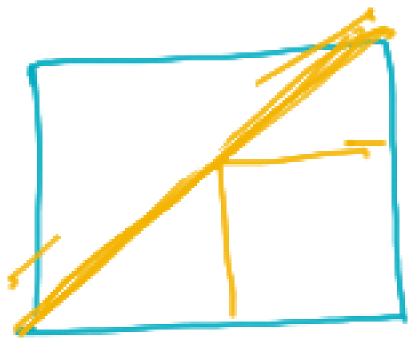
Open questions

- Area law in 3D? Requires optimal poly approximation in 2D. Current methods break.

- Frustrated area law?

- Local gap assumption $\Rightarrow$ gap assumption?

  violated in "edge states". (Bachmann, Hamza, Nachtergaele, Young [2015])



- PEPS representation?